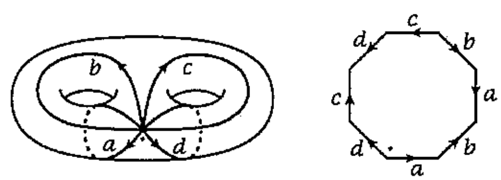
HOMOLOGICAL METHODS IN GROUP THEORY

Igor Mineyev. Math 595 HM, Fall 2016, MWF 3:00 pm, 441 Altgeld Hall.
www.math.uiuc.edu/~mineyev/class/16f/595hom-gr/

Historically groups have been studied by first splitting them into classes according to some property (free, nilpotent, solvable, groups of matrices, orderable, etc), and trying to establish relations among the classes. This is the algebraic approach; using group rings and group algebras is part of it. Later the use of *group presentations* allowed topological methods to be used: every group is the fundamental group of some cell complex, and every group acts freely on a simply connected cell complex. Thus one can often say something about the group if one knows something about (one of) its presentations. This area of mathematics has been called **combinatorial group theory**. In the last few decades, *geometric* properties of groups have been extensively used, giving rise to **geometric group theory**. Nowadays the terms "combinatorial group theory" and "geometric group theory" have become indistinguishable and should be used as synonyms.

The general principle of geometric group theory: a lot can be said about a group G *if* one can find a nice action of G on some nice metric space or a cell complex. As an example, let G be the fundamental group of an orientable surface S of genus 2. (The picture from Hatcher's book on the right. Exercise: Compute its homology.) Knowing that S has a hyperbolic manifold structure implies that G acts by isometries on the standard hyperbolic plane $\mathbb{H}^2$.



The main point of this course is to indicate how often a group action on a cell complex/metric space leads to some **homology/cohomology theory** that reflects both the geometry of the space and algebraic properties of the group. This is also the place where **functional analysis** comes in and is very useful. We will discuss various examples; here is a tentative list of topics.

- The (usual) homology/cohomology of a group with coefficients in a module.
- Free groups. Cohomological dimension. Stallings' cohomological characterization of free groups.
- Various other kinds of (co)homology of groups: ℓ^2-(co)homology, ℓ^1-homology, ℓ^∞-cohomology, bounded cohomology.
- Hilbert modules, Murray-von Neumann dimension, ℓ^2-Betti numbers of groups and of group actions.
- Subgroups of free groups. How to restate the strengthened Hanna Neumann conjecture (SHNC) in terms of ℓ^2-Betti numbers. Systems of graphs, systems of complexes. Submultiplicativity: how to generalize the *statement* of SHNC to other groups.
- Characterizations of hyperbolic groups by ℓ^∞-cohomology after Gersten, and by ℓ^1-homology after Allcock-Gersten.
- Homological bicombings. The characterization of hyperbolic groups by bounded cohomology (after myself).
- If time allows, a review of Gromov's simplicial volume, the Eilenberg-Ganea conjecture, the Whitehead conjecture, the Bestvina-Brady construction.