ON CONJECTURES OF ARAKELYAN AND LITTLEWOOD

By

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To Walter Hayman

1. Introduction

In this paper, we first prove a conjecture of Littlewood [10]:

Theorem 1. *Let g be a polynomial of degree at most n. Then there exist absolute constants* $c, \beta > 0$ *such that*

$$\iint_{\Delta} \frac{|g'|}{1+|g|^2}\,dxdy \leqq cn^{1/2-\beta}, \qquad n = 1, 2, \ldots \tag{1.1}$$

where $\Delta = \{z : |z| < 1\}$.

The integrand in (1.1) is the spherical derivative of g, therefore

$$\iint_{\Delta} \frac{|g'|^2}{(1+|g|^2)^2}\,dxdy \leqq 4\pi n,$$

which gives the trivial estimate

$$\iint_{\Delta} \frac{|g'|}{1+|g|^2}\,dxdy \leqq 2\pi n^{1/2}.$$

Our strategy is to divide Δ into subsets in each of which we can do a little better than this.

Let (a) $\psi(n)$, (b) $\phi(n)$, denote the supremum of the left hand side of (1.1) over (a) rational functions of degree less than or equal to n, (b) polynomials of degree less than or equal to n. Hayman [8] proved that $\psi(n) \geqq cn^{1/2}$, $n = 1, 2, \ldots$, for some absolute constant c. Thus Theorem 2 is false for rational functions.

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Hayman [8] also showed that $\phi(n) \geqq c \log n$, $n = 2, 3, \ldots$. The best previous upper bound for polynomials was due to Eremenko and Sodin [4] who showed that $\phi(n) = o(n^{1/2})$ as $n \to \infty$.

Littlewood proved in [10] that his conjecture, if true, has a remarkable consequence, which can be stated roughly as follows: *let f be an entire function of finite nonzero order, then there is an infinitesimal portion S of the plane such that for almost every w, the roots of $f(z) = w$ lie, with negligible exceptions, in S.*

Next we note that N. U. Arakelyan [1] was the first to construct examples of entire functions of finite order with an infinite set of deficient values. He also conjectured (see [7, problem 1.6]) that

$$\sum_{k=1}^{\infty} [\log 1/\delta(a_k, f)]^{-1} < +\infty$$

when f has finite order and $a_1, a_2, \ldots$ are the deficient values of f.

In this paper, we prove a weak form of Arakelyan's conjecture:

Theorem 2. *Let f be an entire function of finite lower order μ with Nevanlinna deficient values $a_1, a_2, \ldots$. There exists an absolute constant γ, $0 < \gamma < 1/3$, such that*

$$\sum_k \delta(a_k, f)^{1/3-\gamma} < +\infty.$$

There is a distinction between entire functions and meromorphic functions. For meromorphic functions with $\mu < \infty$, Weitsman proved in [14] that this series converges when $\gamma = 0$; examples due to Hayman and Gol'dberg (cf. [6, p. 98)] show that $\Sigma\delta(a_k)^{1/3}$ can converge arbitrarily slowly. The hypothesis $\mu < \infty$ is necessary, as follows from examples of Fuchs and Hayman (cf. [6, p. 86]).

Our Theorem 2 is the first result that shows that Weitsman's theorem is not sharp for entire functions.

As for our proofs, the first author has known for some time that Weitsman's method in [14] can be used to reduce the proof of Theorem 2 to a problem on harmonic measure (see [3, 7.40, p. 564]). In a discussion with the first author, W. Hayman suggested a possible relation between Arakelyan's and Littlewood's conjectures. Using a technique of Bourgain [2], the second author has recently solved the harmonic measure problem (Theorem 3). The proof of Theorem 1 presented here bears some resemblance to that of Eremenko and Sodin [4] on $\phi(n) = o(n^{1/2})$; we have better estimates due to Theorem 3.

Theorem 3. *Let $\{Q_j\}_1^N$ be a sequence of closed dyadic squares of side length r contained in $\{z = x + iy\colon 0 \leqq x \leqq 1/2,\ 0 \leqq y \leqq 1/2\}$. Let $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, $U = \hat{\mathbf{C}} - \bigcup_1^N Q_j$, and let $\omega(z, F)$, $z \in U$, $F \subseteq \partial U$ be the harmonic measure at z of F relative to U. There exist absolute constants $r_0, \varepsilon_0 > 0$ so that if $0 < r < r_0$, then*

$$\omega(\infty, \partial Q_j) > r^{2+\varepsilon_0} \tag{1.3}$$

for at most $r^{-2+\varepsilon_0}$ *values of* j, $1 \leqq j \leqq N$.

Theorem 3 remains true in $\mathbf{R}^n$, $n \geqq 3$, when harmonic measures of cubes are evaluated at a finite fixed point, and ε_0 is properly modified to depend on n. The proof is essentially unchanged. We note from the proof of Theorem 3 that it suffices to take $\varepsilon_0 = 2^{-260}$.

In §2 we prove Theorem 1, assuming Theorem 3; and prove a conjecture of Eremenko and Sodin which also leads to Theorem 1. From the proof it follows that Theorem 1 holds for $0 < \beta \leqq (1/16)2^{-260}$. In §3 we make some preliminary reductions for Theorem 2. These will be used in §4 to show that if ε_0 is as in Theorem 3, $\varepsilon_1 = \varepsilon_0/40$, and

$$\Gamma(n) = \{k : 2^{-(n+1)} \leqq \delta(a_k, f)^{1/3} \leqq 2^{-n}\},$$

then Theorem 3 implies

$$\Gamma^{\#}(n) \leqq 2^{n(1-\varepsilon_1)}, \tag{1.4}$$

for $n \geqq n_0$, where $n_0 = n_0(r_0, \varepsilon_0, \mu, \delta(a_1, f))$. Here $G^{\#}$ is the number of elements in the set G. Clearly (1.4) implies Theorem 2 whenever $0 < \gamma < \varepsilon_1/3$. Thus Theorem 2 is valid when $0 < \gamma < (1/120)2^{-260}$. In §5 we prove Theorem 3.

We would like to thank J. Bourgain for a preliminary version of his paper; and D. Drasin for bringing Eremenko and Sodin's paper [4] to our attention, translating it, and suggesting that our methods could answer a conjecture in [4], and for his numerous comments on our original manuscript.

2. Proof of Theorem 1

Let g be a polynomial of degree at most n with zeros (a_i) and note that

$$|g'/g|(z) \leqq \sum_i |z - a_i|^{-1}.$$

Using this observation, we see that

$$\begin{aligned} \iint_{\Delta \cap \{|g| \geqq n^{3/4}\}} |g'|(1 + |g|^2)^{-1} dxdy &\leqq n^{-3/4} \iint_{\Delta} |g'|\,|g|^{-1} dxdy \\ &\leqq 100 n^{-3/4} \cdot n = 100 n^{1/4}. \end{aligned} \tag{2.1}$$

Let

$$\Omega = \{z : |g(z)| < n^{3/4}\} \cap \{z : |z| < 3/2\}$$

and let $\{Q_j\}$ be a sequence of Whitney squares, see [12], for Ω with

$$100l(Q_j) \geqq d(Q_j, \partial\Omega) \geqq 3l(Q_j), \qquad j = 1, 2, \ldots$$

where $l(Q_j)$ is the side length of Q_j and $d(Q_j, \partial\Omega)$ denotes the distance from Q_j to $\partial\Omega$. We note that

$$d\mu(z) = 4n^{-1} \frac{|g'|^2(z)}{(1 + |g|^2(z))^2} dxdy$$

is the Riesz measure associated with the nonnegative subharmonic function, $u = n^{-1} \log(1 + |g|^2)$. It follows from the Riesz representation formula for subharmonic functions that for $z_0 \in \mathbf{C}$ and $\rho > 0$,

$$\mu(\Delta(z_0, \rho)) \leqq (2\pi \log 2)^{-1} \left(\int_0^{2\pi} (u(z_0 + 2\rho e^{i\theta}) - u(z_0)) d\theta \right). \tag{2.2}$$

From (2.2) and the definition of Ω, we deduce for each j that

$$\mu(Q_j) \leqq 4 \log(1 + n)/n.$$

Let

$$\Phi = \{j : l(Q_j) \geqq n^{-1/4}\}.$$

Then from this bound on $\mu(Q_j)$ and Schwarz's inequality it follows for $j \in \Phi$ that

$$n^{-1/2} \iint_{Q_j} |g'|(1 + |g|^2)^{-1} dxdy \leqq \mu(Q_j)^{1/2} l(Q_j)$$

$$\leqq 2(\log(1 + n))^{1/2} n^{-1/2} l(Q_j) \leqq 2(\log(1 + n))^{1/2} n^{-1/4} l(Q_j)^2.$$

Summing over j, we get

$$\sum_{j \in \Phi} n^{-1/2} \iint_{Q_j} |g'|(1 + |g|^2)^{-1} dxdy \leqq 100[\log(1 + n)]^{1/2} n^{-1/4}. \tag{2.3}$$

Next suppose that k_0 is the largest positive integer such that $2^{-k_0} \geqq n^{-1/4}$. Let

$$\Psi(k) = \{j : 2^{-(k+1)} \leqq l(Q_j) \leqq 2^{-k}, Q_j \cap \Delta \neq \varnothing\}$$

for $k \geqq k_0$. Let $A(k) \subseteq \Psi(k)$ be the indices of all good squares Q_j for which

$$\mu(Q_j) M(2, u)^{-1} \leqq 10^{20} l(Q_j)^{2+\varepsilon_0}, \tag{2.4}$$

ε_0 as in Theorem 3. Here $M(2, u) = \max_{|z|=2} u(z)$. Assume that Theorem 3 is valid, we shall prove that if $B(k) = \Psi(k) \setminus A(k)$ and $2^{-k_0} \leqq r_0$, r_0 as in Theorem 3, then

$$B(k)^{\#} \leqq c_1 2^{k(2-\varepsilon_0)}, \tag{2.5}$$

for some absolute constant $c_1 > 0$.

Indeed, if $Q_j \cap \Delta \neq \varnothing$, there exists $\zeta_j \in \partial\Omega$ such that

$$u(\zeta_j) = n^{-1}\log(1 + n^{3/2}), \quad \text{and} \quad d(\zeta_j, Q_j)) \leqq 100 l(Q_j).$$

Hence from (2.2)

$$\mu(Q_j) \leqq (2\pi \log 2)^{-1} \int_0^{2\pi} [u(\zeta_j + 204 l(Q_j)e^{i\theta}) - u(\zeta_j)]d\theta$$

$$\leqq 2 \max_{z \in \Delta(\zeta_j, 204 l(Q_j))} (u(z) - u(\zeta_j))^+.$$

For each $j \in \Psi(k)$ choose a closed dyadic square $Q_j' \subseteq Q_j$ with common side length $l(Q_j') \geqq l(Q_j)/4$, and denote by z_j the center of Q_j'. Then

$$\mu(Q_j) \leqq 2 \max_{z \in \Delta(z_j, 800 l(Q_j))} [u(z) - u(\zeta_j)]^+.$$

Let

$$E = \{w : 8w \in Q_j', j \in \Psi(k)\},$$

and observe that $E \subseteq \{|z| < 3/16\}$. Let $U = \hat{\mathbf{C}} \setminus E$ and G be the Green function for U with pole at ∞. Extend G to a continuous subharmonic function in $\mathbf{C}$ by defining $G \equiv 0$ on E. Then there exist a positive Borel measure ν and a real number b, so that

$$G(w) = \int_E \log|w - \zeta|\, d\nu(\zeta) + b, \qquad w \in \mathbf{C},$$

where $\nu(F)$ is the harmonic measure $\omega(\infty, F)$ of F relative to U. In fact, if a is any point interior to E, then

$$G(z, w) = \log\frac{|z - a|}{|z - w|} - \int_{\zeta \in E} \log\frac{|\zeta - a|}{|\zeta - w|}\, d\omega(z, \zeta)$$

can be seen by regarding $G(z, w)$ as the Green function on U with pole at w, and comparing the singularities near w and the boundary values of both sides. As $z \to \infty$, the symmetry of the Green function yields this representation for $G(w)$. From the maximum principle, we observe that

$$G(w) \geqq \log|\tfrac{16}{3}w|, \qquad w \in U;$$

hence

$$G(w) \geqq \log \tfrac{4}{3} \qquad \text{for } |8w| = 2.$$

By the maximum principle again, we have

$$u(8w) - u(\zeta_j) \leqq M(2, u)(\log \tfrac{4}{3})^{-1} G(w)$$

when $|8w| \leqq 2$.

We assume, as we may, that $n > 10^{16}$. Therefore $\Delta(z_j, 1600l(Q_j)) \subseteq \{|z| < 2\}$ for $j \in \Psi(k)$. It follows from the above estimates that for $j \in \Psi(k)$

$$\mu(Q_j) \leqq 2M(2, u)(\log \tfrac{4}{3})^{-1} \max_{\Delta_j} G(z)$$

where $\Delta_j = \Delta(z_j/8, 100l(Q_j))$. Denote by

$$Q_j'' = \{w : 8w \in Q_j'\}, \qquad j \in \Psi(k),$$

and note that Q_j'' has center $z_j/8$ and side length $l(Q_j')/8 \geqq l(Q_j)/32$. Because Theorem 3 is translation invariant and $\bigcup Q_j'' \subseteq \Delta(0, \frac{3}{16})$, it follows from Theorem 3 that

$$\omega(\infty, Q_j'') < l(Q_j'')^{2+\varepsilon_0}$$

for all squares Q_j'', $j \in \Psi(k)$, with possibly $l(Q_j'')^{-2+\varepsilon_0}$ many exceptions. From the disjointness of the $\{Q_j''\}$, we deduce that there are at most $10^9 l(Q_j'')^{-2+\varepsilon_0}$ disks $\Delta(z_j/8, 200l(Q_j))$, $j \in \Psi(k)$, which meet these exceptional squares; and we note that $10^9 l(Q_j'')^{-2+\varepsilon_0}$ constitutes the bound in (2.5).

If a disk $\Delta(z_j/8, 200l(Q_j))$ does not meet any exceptional square, then the corresponding square Q_j is good in the sense of (2.4). Indeed, this $\Delta(z_j/8, 200l(Q_j))$ can meet at most 10^9 squares Q_l'', $l \in \Psi(k)$, and they must satisfy $\omega(\infty, Q_l'') < l(Q_l'')^{2+\varepsilon_0}$. By the Riesz representation theorem for subharmonic functions, and the fact that $\nu(\Delta(z_j/8, l(Q_j'')/2)) = 0$, we deduce that

$$\max_{\Delta_j} G(z) \leqq \frac{3}{2\pi} \int_{\partial\Delta(z_j/8, 200l(Q_j))} G(z) \frac{|dz|}{|z|} = 3 \int_{l(Q_j'')/2}^{200l(Q_j)} \nu(\Delta(z_j/8, t)) \frac{1}{t}\, dt$$

$$\leqq 3\nu(\Delta(z_j/8, 200l(Q_j))) \log 12800 \leqq 3 \cdot 10^9 l(Q_j'')^{2+\varepsilon_0} \log 12800.$$

Property (2.4) follows trivially from this inequality and $\mu(Q_j) \leqq 2M(2, u)(\log \frac{4}{3})^{-1} \max_{\Delta_j} G(z)$.

From (2.4) and Schwarz's inequality we get if $2^{-k_0} \leqq r_0$,

$$\sum_{j\in A(k)} n^{-1/2} \iint_{Q_j} |g'|(1+|g|^2)^{-1}dxdy \leqq \sum_{j\in A(k)} \mu(Q_j)^{1/2} l(Q_j)$$

$$\leqq 10^{10}M(2,u)^{1/2} \sum_{j\in A(k)} l(Q_j)^{2+\varepsilon_0/2} \leqq 10^{11}M(2,u)^{1/2} n^{-\varepsilon_0/8} \sum_{j\in A(k)} l(Q_j)^2,$$

for $k \geqq k_0$. If $A = \bigcup_{k=k_0}^{\infty} A(k)$, it follows from disjointness of $\{Q_j\}$ that

$$\sum_{j\in A} n^{-1/2} \iint_{Q_j} |g'|(1+|g|^2)^{-1}dxdy \leqq 10^{13}M(2,u)^{1/2}n^{-\varepsilon_0/8}. \tag{2.6}$$

Next we have

$$n^{-1/2} \iint_{\cup_{j\in B(k)} Q_j} |g'|(1+|g|^2)^{-1}dxdy \leqq \mu\left(\bigcup_{j\in B(k)} Q_j\right)^{1/2} \left|\bigcup_{j\in B(k)} Q_j\right|^{1/2}$$

where $|F|$ denotes the area of the set F. Using (2.5) we see that

$$\left|\bigcup_{j\in B(k)} Q_j\right| \leqq c_1 2^{-k\varepsilon_0}.$$

If $B = \bigcup_{k=k_0}^{\infty} B(k)$ we conclude that

$$n^{-1/2} \iint_{\cup_{j\in B} Q_j} |g'|(1+|g|^2)^{-1}dxdy \leqq 100\sqrt{c_1} \sum_{k=k_0}^{\infty} 2^{-k\varepsilon_0/2} \leqq c_2 n^{-\varepsilon_0/8}, \tag{2.7}$$

for some absolute constant c_2. Combining (2.7), (2.6), and (2.3) we find if $n^{-1/4} \leqq \frac{1}{2}r_0$, then

$$\begin{aligned} &n^{-1/2} \iint_{\Omega} |g'|(1+|g|^2)^{-1}dxdy \\ &\leqq c_2 n^{-\varepsilon_0/8} + 10^{13}M(2,u)^{1/2}n^{-\varepsilon_0/8} + 100(\log(1+n))^{1/2}n^{-1/4}. \end{aligned} \tag{2.8}$$

Finally we show that there exists $c_3 > 0$ and $n_0 > (2/r_0)^4$ such that either

$$M(2,u) \leqq n^{\varepsilon_0/8}, \qquad n \geqq n_0, \tag{2.9}$$

or

$$|\Omega| \leqq c_3 n^{-\varepsilon_0/8}, \qquad n \geqq n_0. \tag{2.10}$$

If (2.9) holds, note from (2.8) that

$$n^{-1/2} \iint_{\Omega} |g'|(1+|g|^2)^{-1} dxdy \leqq c_4 n^{-\varepsilon_0/16},$$

for some absolute $c_4 > 0$ and $n \geqq n_0$. If (2.10) holds the last inequality is also valid for c_4 large enough as follows easily from Schwarz's inequality. Using (2.1) we conclude that Theorem 1 is valid once we show that either (2.9) or (2.10) holds.

Suppose that $M(2, u) > n^{\varepsilon_0/8}$. We write $g(z) = P(z)Q(z)$, where

$$P(z) = \lambda \prod_{|a_i|>4} (z - a_i), \quad Q(z) = \prod_{|a_i| \leqq 4} (z - a_i).$$

If m is the degree of P, observe that

$$\tag{2.11} \begin{aligned} &\log\Big(|\lambda| \prod_{|a_i|>4} |a_i|\Big) - m\log 2 \\ &\leqq \log|P(z)| \leqq \log\Big(|\lambda| \prod_{|a_j|>4} |a_j|\Big) + m\log 2, \end{aligned}$$

when $|z| \leqq 2$. Since $|Q(z)| \leqq 6^{(n-m)}$ for $|z| \leqq 2$, it follows that if (2.9) is false then

$$\tag{2.12} \log\Big(|\lambda| \prod_{|a_i|>4} |a_i|\Big) \geqq \tfrac{1}{8} n^{(1+\varepsilon_0/8)}, \quad \text{for } n \geqq n_0,$$

provided n_0 is large enough. Since $u \leqq 1$ in Ω we deduce from (2.11) and (2.12) that for $z \in \Omega$,

$$\log|Q(z)| < -10^{-2} n^{(1+\varepsilon_0/8)},$$

when $n \geqq n_0$ large. Hence for $z \in \Omega$,

$$\prod_{|a_i| \leqq 4} |z - a_i| \leqq \exp[-10^{-2} n^{1+\varepsilon_0/8}].$$

Therefore, each $z \in \Omega$ is contained in the disk $\{z : |z - a_{i(z)}| \leqq \exp[-10^{-2} n^{\varepsilon_0/8}]\}$ for some $i(z)$. This implies that

$$|\Omega| \leqq n\pi \exp[-\tfrac{1}{50} n^{\varepsilon_0/8}] \leqq c_3 n^{-\varepsilon_0/8} \quad \text{when } n \geqq n_0.$$

This completes the proof of Theorem 1.

Eremenko and Sodin observed in [4] that a positive answer to the following conjecture would imply Theorem 1.

Let u, $0 \leqq u < 1$, be subharmonic in Δ with associated Riesz measure μ. Given $t > 0$, there exist sets $E = E(t)$, $F = F(t)$ so that $\{z : u(z) < t\} = E \cup F$ with $\max[\mu(E), |F|] < ct^{\gamma}$, where c and γ are positive absolute constants.

Theorem 3 can also be used to establish this conjecture with $\gamma = \varepsilon_0/8$. We shall outline the proof. Let $\{Q_j\}$ be a sequence of closed Whitney squares for $\{z : u(z) < t\}$ in Δ. Let E_1 be the union of all Q_j with $l(Q_j) \geqq t^{1/4}$. Then as in the proof of (2.3) it follows that $\mu(E_1) \leqq ct^{1/2}$. Let F_1 be the union of all Q_j which have a non-empty intersection with $\{z : 1 - t^\varepsilon \leqq |z| < 1\}$ and $l(Q_j) < t^{1/4}$. Here $\varepsilon = \varepsilon_0/8$ and ε_0 is as in Theorem 3. Clearly $|F_1| \leqq 100t^\varepsilon$. Let Q_j' be a dyadic square contained in Q_j of side length $\geqq \frac{1}{4}l(Q_j)$. If $2^{-k} \leqq t^{1/4}$, let

$$H(k) = \{w : 4w - 1 - i \in Q_j',\ Q_j \cap F_1 = \varnothing,\ 2^{-(k+1)} \leqq l(Q_j) \leqq 2^{-k}\}$$

and observe that

$$H(k) \subseteq \Delta\left(\frac{1+i}{4}, \tfrac{1}{4}(1 - t^\varepsilon)\right).$$

Let $V = \mathbf{C} - H(k)$, and let G be Green's function for V with pole at ∞. Using the maximum principle and the above inclusion, we deduce

$$(u - t)(4w - 1 - i) \leqq -G(w)/[\log(1 - t^\varepsilon)]$$

when $|w - \frac{1}{4} - \frac{1}{4}i| < \frac{1}{4}$. The conjecture now follows from this inequality and the argument from (2.4) on. The "good" squares are put in E while the "bad" squares are put in F.

Finally we remark that this argument remains valid if u, $0 \leqq u \leqq 1$, is subharmonic in the unit ball of $\mathbf{R}^n$, because as we remarked in §1, the analogue of Theorem 3 remains true in $\mathbf{R}^n$, $n \geqq 3$.

3. Preliminary reductions for Theorem 2

As usual we let

$$T(r, g) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |g(re^{i\theta})|\, d\theta, \qquad 0 < r < \infty,$$

when g is entire in $\mathbf{C}$. Here $a^+ = \max(a, 0)$. Let f be an entire function of finite lower order μ and let $(r_m)_1^\infty$ be a sequence of Pólya peaks of order μ for $T(r, f)$. In particular, $\lim_{m\to\infty} r_m = \infty$, and for any fixed $b > 1$,

$$T(r, f) \leqq (r/r_m)^\mu T(r_m, f)(1 + o(1)), \qquad m \to \infty, \tag{3.1}$$

for $r_m \leqq r \leqq br_m$.

We follow [14] and let $(\alpha_m)_1^\infty$ be a sequence of positive numbers satisfying

$$\sqrt{T(r_m, f)} \leqq \alpha_m \leqq \sqrt{T(r_m, f)} + \log 2.$$

Corresponding to a given $(\alpha_m)_1^\infty$, $(r_m)_1^\infty$, let

$$\Omega = \Omega[(\alpha_m), (r_m)] = \bigcup_{m=1}^{\infty} \{z = re^{i\theta} : r_m < r < 24r_m, \log|f'(re^{i\theta})| < -\alpha_m\}.$$

For the proof of Theorem 2, we assume, as we may, that f has at least one finite deficient value, and let $\delta(a_1, f) > 0$ be the largest deficiency, so that in particular

$$\delta(a_1, f) = \liminf_{r\to\infty} \frac{\left(\frac{1}{2r}\int_0^{2\pi} \log^+(1/|f(re^{i\theta}) - a_1|)d\theta\right)}{T(r, f)} > 0.$$

We also assume that the intervals $[r_m, 48r_m]$ are pairwise disjoint. Finally, K denotes a positive constant which may depend only on $\delta(a_1, f)$ and μ, not necessarily the same at each occurrence. The following lemmas can be found in Weitsman [14, §2].

Lemma A. *For f and $(r_m)_1^\infty$ as above,*

$$\delta(a_1, f)T(r, f)(1 + o(1)) \leqq T(r, f') \leqq 2T(r, f)(1 + o(1)) \tag{3.2}$$

as $r \to \infty$ through the intervals $r_m \leqq r \leqq 48r_m$.

Lemma B. *Let f and $(r_m)_1^\infty$ be as above. There exist $(\alpha_m)_1^\infty$, Ω, as above, and pairwise disjoint subsets Ω_k, $k = 1, 2, \ldots$, of Ω such that each Ω_k is the union of components of Ω, and if $\Omega_k(r) = \Omega_k \cap \{z : |z| = r\}$, then for $k = 1, 2, \ldots$, and $r_m < r < 24r_m$,*

$$-\frac{1}{2\pi}\int_{\Omega_k(r)} \log|f'(z)|\,|dz| \geqq (r/3)\delta(a_k, f)T(r, f'), \qquad m > m_0(k). \tag{3.3}$$

From (3.1), (3.2), and [6; p. 18] it is easily shown that there exists $m_1 = m_1(\mu, \delta(a_1, f))$, such that

$$\log M(24r_m, f) \leqq 3T(48r_m, f') \leqq KT(r_m, f'), \qquad m \geqq m_1. \tag{3.4}$$

Let $\Gamma(n)$ be as defined in §1 and note from Nevanlinna's second fundamental theorem that $\Gamma^*(n) \leqq 2^{3n+4}$. Hence we may choose $m_2(n)$, $n = 1, 2, \ldots$, such that $m_2(n) \geqq m_1$ and (3.3) holds for each $k \in \Gamma(n)$ and $m \geqq m_2(n)$. For later use we also choose $m_2(n)$ so large that for $m \geqq m_2(n)$, $n = 1, 2, \ldots$,

$$\alpha_m 2^{10n} \leqq T(r_m, f'). \tag{3.5}$$

For fixed, $n, m \geqq m_2(n)$, let σ_k, $k \in \Gamma(n)$, denote the number of zeros of f', counting multiplicities, in $\Omega_k \cap \{z : r_m \leqq |z| \leqq 6r_m\}$ and let A_k be the area of this set. Given ε, $0 < \varepsilon < 10^{-1}$, let

$$\Gamma_1(n) = \{k \in \Gamma(n) : \max[\sigma_k / T(r_m, f'), A_k r_m^{-2}] \leqq 2^{n(\varepsilon - 1)}\}.$$

We observe from a well-known corollary of Jensen's formula and (3.4) that

$$\begin{aligned} \{z : f'(z) = 0, |z| \leqq 6r_m\}^{\#} &\leqq (\log 2)^{-1} T(12r_m, f') + O(\log r_m) \\ &\leqq KT(r_m, f'), \end{aligned} \tag{3.6}$$

where multiplicities are counted. Using (3.6) it is easily seen that

$$[\Gamma(n) - \Gamma_1(n)]^{\#} \leqq K \cdot 2^{n(1-\varepsilon)}. \tag{3.7}$$

Next, let $d(z, B)$ denote the distance from z to the set B and let $B(r) = \{z : |z| = r\} \cap B$. For $l = 0, 1, \ldots, 2^n - 1$, and $k \in \Gamma(n)$, we set

$$\Omega_{k,l} = \{z \in \Omega_k : (3 + l2^{-n})r_m < |z| < (3 + (l+1)2^{-n})r_m\},$$

$$O_{k,l} = \{z \in \Omega_{k,l} : d(z, \partial\Omega_{k,l}) > 2^{-n(1+2\varepsilon)} r_m\},$$

$$\begin{aligned} H_{k,l} &= \{z \in \Omega_{k,l} : d(z, O_{k,l}) \leqq 2^{-n} r_m\}, && \text{if } O_{k,l} \neq \varnothing, \\ &= \varnothing, && \text{if } O_{k,l} = \varnothing. \end{aligned}$$

Since f is entire, we may choose $m_2(n)$ large enough so that each component of $\Omega_{k,l}$ is simply-connected. Let $A_{k,l}$ be the area of $\Omega_{k,l}$ and let $\sigma_{k,l}$ be the number of zeros of f', counting multiplicities, in $\Omega_{k,l}$. Given $k \in \Gamma(n)$ let

$$\Lambda_k = \{l : \max[\sigma_{k,l} / T(r_m, f'), A_{k,l} r_m^{-2}] \leqq 4 \cdot 2^{n(\varepsilon - 2)}\}.$$

From the definition of $\Gamma_1(n)$ we see for $k \in \Gamma_1(n)$ that

$$\Lambda_k^{\#} \geqq 2^{n-1}. \tag{3.8}$$

We shall need the following lemma.

Lemma 1. *If $m \geqq m_2(n)$, $k \in \Gamma(n)$, and $l \in \Lambda_k$, then there exists $n_1 = n_1(\varepsilon, \delta(a_1, f), \mu) \geqq 100$, such that for $n \geqq n_1$,*

$$-\frac{1}{2\pi} \int_{H_{k,l}(r)} \log|f'(z)|\,|dz| \geqq \frac{r}{6} \delta(a_k, f) T(r, f'), \tag{3.9}$$

whenever $(3 + (l + 1/4)2^{-n})r_m < r < (3 + (l + 3/4)2^{-n})r_m$.

Proof. Choose $n, m \geqq m_2(n)$, $k \in \Gamma(n)$, $l \in \Lambda_k$, and note that $\psi = -\log|f'| - \alpha_m$ is superharmonic in **C**, positive in $\Omega_{k,l}$ with $\psi = 0$ on

$$\partial\Omega_{k,l} \cap \{z : (3 + l2^{-n})r_m < |z| < (3 + (l+1)2^{-n})r_m\}.$$

From the Riesz representation formula for $\Omega_{k,l}$ we see that

$$\psi(z) = \sum_{\{a \in \Omega_{k,l} : f'(a) = 0\}} g(z, a) + h(z), \qquad z \in \Omega_{k,l},$$

where $g(\cdot, a)$ denotes Green's function for $\Omega_{k,l}$ with pole at $a \in \Omega_{k,l}$, h is the greatest harmonic minorant of ψ in $\Omega_{k,l}$ and multiplicities are counted in Σ.

Denote $d(\cdot)$ for $d(\cdot, \partial\Omega_{k,l})$ and let F be a univalent conformal mapping that maps $\Omega_{k,l}$ onto $|z| < 1$, and the point a to the origin. Then $g(z, a) = -\log|F(z)|$. From the Koebe distortion theorem, we see that

$$g(z, a) \leqq c + \log\frac{d(z)}{|z - a|}, \quad \text{for } |z - a| < \frac{d(z)}{2d(a)},$$

$$g(z, a) \leqq c(1 - |F(z)|) \leqq c|F'(z)|d(z), \quad \text{for } |z - a| \geqq \frac{d(z)}{2d(a)},$$

for some absolute constants $c > 0$. We claim that

$$\text{(3.10)} \qquad \int_{\Omega_{k,l}(r)} g(z, a)d(z)^{-1}|dz| < c < +\infty$$

for some absolute constant $c > 0$. In fact, the integral over the arc $\Omega_{k,l}(r) \cap \{z : |z - a| < d(z)/2d(a)\}$ is bounded above by an absolute constant, as follows from the above estimate of $g(z, a)$; the integral over the remaining part is bounded above by $c\int_{\Omega_{k,l}(r)}|F'(z)|\,|dz|$, which is bounded above uniformly in view of a theorem of Hayman–Wu [8] and Garnett–Gehring–Jones [5] on the lengths of level curves.

Using (3.10) it follows that if

$$\rho = 2^{-n(1+2\varepsilon)}r_m,$$

$I_{k,l} = \Omega_{k,l} - H_{k,l}$ and $(3 + l2^{-n})r_m < r < (3 + (l+1)2^{-n})r_m$, then

$$\text{(3.11)} \qquad \sum_{\{a \in \Omega_{k,l} : f'(a) = 0\}} \int_{I_{k,l}(r)} g(z, a)|dz| \leqq c\rho\sigma_{k,l} \leqq 4c2^{-n(3+\varepsilon)}r_m T(r_m, f'),$$

where we have used the fact that $l \in \Lambda_k$ and that $I_{k,l} \cap O_{k,l} = \varnothing$.

To estimate h, let $M(t) = M(t, h) = \sup_{z \in \Omega_{k,l}(t)} h(z)$ and let s_1, s_2 satisfy

$$(3 + l2^{-n})r_m < s_1 < (3 + (l + 1/8)2^{-n})r_m,$$

$$(3 + (l + 7/8)2^{-n})r_m < s_2 < (3 + (l+1)2^{-n})r_m.$$

If $\beta = \max[M(s_1), M(s_2)]$ we claim that for $n \geqq 5/\varepsilon$, (3.12) $$h(z) \leqq \beta \exp[-c_1 2^{2n\varepsilon}], \qquad z \in I_{k,l}(r),$$

where $c_1 > 0$ is an absolute constant and

$$(3 + (l + 1/4)2^{-n})r_m < r < (3 + (l + 3/4)2^{-n})r_m.$$

To prove (3.12) we argue as follows. Fix $z \in I_{k,l}(r)$ where r is as in (3.12). Then $d(w) \leqq \rho$ whenever $w \in \Omega_{k,l}$ and $|w - z| \leqq 2^{-n}r_m$. Since each component of $\Omega_{k,l}$ is simply connected, from the theorem of Milloux–Schmidt, we see that for some α, $0 < \alpha < 1$, $h(w) \leqq \alpha\beta$ whenever $w \in \Omega_{k,l}$, $|z - w| \leqq 2^{-n}r_m$ and $s_1 + 2\rho < |w| < s_2 - 2\rho$. Using this inequality and the theorem of Milloux–Schmidt again it follows that $h(w) \leqq \alpha^2\beta$ whenever $w \in \Omega_{k,l}$, $|z - w| \leqq 2^{-n}r_m - 2\rho$ and $s_1 + 4\rho < |w| < s_2 - 4\rho$. Continuing this process ν times we obtain

(3.13) $$h(w) \leqq \alpha^\nu\beta,$$

when $w \in \Omega_{k,l}$, $|z - w| \leqq 2^{-n}r_m - 2(\nu - 1)\rho$, and $s_1 + 2y\rho < |w| < s_2 - 2\nu\rho$. Let

$$\nu = [2^{-(n+5)}r_m/\rho] = [2^{2n\varepsilon - 5}],$$

where $[\cdot]$ denotes the greatest integer function. We conclude first that (3.13) holds with w replaced by z and thereupon that (3.12) holds with $c_1 = -(\log \alpha)/64$.

Next we use the known inequality (see [11, p. 25]),

(3.14) $$r^{-1} \int_1^r \log M(t, 1/f')dt \leqq c_2 T(2r, f'),$$

where c_2 is an absolute constant. From (3.14), and the fact that $h \leqq \log(1/|f'|)$ we deduce that there exist s_1, s_2 in the intervals described at the beginning of the last paragraph, with

(3.15) $$\beta \leqq c_2 T(8r_m, f')2^{(n+6)}.$$

Using (3.15), (3.12), and (3.11), we find that

$$-\frac{1}{2\pi} \int_{I_{k,l}(r)} \log|f'(z)|\,|dz|$$

$$\leqq c2^{-n(3+\varepsilon)}r_m T(r_m, f') + c_2 r_m T(8r_m, f')2^{(n+8)}\exp[-c_1 2^{2n\varepsilon}] + 4\alpha_m r_m.$$

From this inequality, (3.4), and (3.5) we see there exists $n_1 = n_1(\varepsilon, \delta(a_1, f), \mu)$ such that if $n \geqq n_1$, then

$$-\frac{1}{2\pi}\int_{I_{k,l}(r)} \log|f'(z)|\,|dz| \leqq 2^{-(3n+6)}rT(r,f') \leqq \tfrac{1}{8}\delta(a_k,f)rT(r,f'),$$

where we have used the fact that $k\in\Gamma(n)$. Lemma 1 is now an easy consequence of the above inequality and (3.3) of Lemma B.

Before proceeding further we note that a more elementary proof of (3.11) could be given arguing as in (3.12). Indeed as in the proof of (3.12) it can be shown that

$$g(z,a)\leqq c_3\exp\left(-c_4\frac{|z-a|}{\rho}\right),\quad \text{for } \tfrac{1}{2}d(a)\leqq|z-a|\leqq 2^{-n}r_m,$$

and

$$g(z,a)\leqq c_3\exp[-c_4 2^{2n\varepsilon}]\quad\text{when } |z-a|>2^{-n}r_m,$$

where c_3, c_4 are absolute constants, $z\in I_{k,l}(r)$, and $a\in\Omega_{k,l}$. The proof again uses the Milloux–Schmidt inequality and the fact that $g(z,a)\leqq$ constant when $|z-a|=d(a)/2$. These inequalities clearly imply (3.11).

To continue the preliminary reductions let $\Delta(\zeta,s)=\{z:|z-\zeta|<s\}$ and recall that $\rho=2^{-(n+2\varepsilon)}r_m$. For fixed ε, $n\geqq n_1$, $m\geqq m_2(n)$, $k\in\Gamma(n)$, and $l\in\Lambda_k$ we now apply a well-known covering lemma (see [12, §1.6]) to obtain $p=p(k,l)$ a positive integer and $z_j=z_j(k,l)\in O_{k,l}$, $1\leqq j\leqq p$, such that

$$O_{k,l}\subseteq\bigcup_{j=1}^{p}\Delta(z_j,\rho)\subseteq\Omega_{k,l},\tag{3.16}$$

$$\Delta(z_j,\tfrac14\rho)\cap\Delta(z_i,\tfrac14\rho)=\varnothing,\qquad i\neq j,\tag{3.17}$$

$$H_{k,l}\subseteq\bigcup_{j=1}^{p}\Delta(z_j,2^{1-n}r_m).\tag{3.18}$$

Lemma 2. *Under the above notations, we have*

$$2^{-(n+5)}r_m^2\delta(a_k,f)T(r_m,f')\leqq\sum_{j=1}^{p}\iint_{\Delta(z_j,4\tau)}\log^+|f'(z)|\,dxdy\tag{3.19}$$

for $n\geqq n_1\geqq 100$, $m\geqq m_2(n)$, $k\in\Gamma(n)$, $l\in\Lambda_k$ *and* $\tau=2^{n(3\varepsilon-1)}r_m$.

To prove Lemma 2, we deduce from (3.17) and $A_{k,l}\leqq 4\cdot 2^{n(\varepsilon-2)}r_m^2$ that

$$1\leqq p\leqq\frac{64}{\pi}2^{n(\varepsilon-2)}r_m^2\rho^{-2}\leqq\frac{64}{\pi}2^{5n\varepsilon}.\tag{3.20}$$

Also, integrating (3.9) of Lemma 1 with respect to r, $(3+(l+1/4)2^{-n})r_m < r < (3+(l+3/4)2^{-n})r_m$ and using (3.18), we get

$$
\begin{aligned}
2^{-(n+4)} r_m^2 \delta(a_k, f) T(r_m, f') &\leqq \iint_{H_{k,l}} \log^+ \frac{1}{|f'(z)|}\, dxdy \\
&\leqq \sum_{j=1}^{p} \iint_{\Delta(z_j, 2^{1-n} r_m)} \log^+ \frac{1}{|f'(z)|}\, dxdy.
\end{aligned}
\tag{3.21}
$$

Now from (3.16) and the fact that $l \in \Lambda_k$ we see that

$$\Delta(z_j, \tau) \cap \partial\Omega_k \neq \varnothing, \qquad 1 \leqq j \leqq p.$$

Hence, $-\log|f'(\zeta_j)| = \alpha_m$ for some $\zeta_j \in \Delta(z_j, \tau)$. From the mean value property for subharmonic functions applied to $\log|f'|$, it follows that

$$
\begin{aligned}
\iint_{\Delta(\zeta_j, 2\tau)} \log^+ \frac{1}{|f'(z)|}\, dxdy &\leqq \iint_{\Delta(\zeta_j, 2\tau)} \log^+ |f'(z)|\, dxdy + \alpha_m 4\pi\tau^2 \\
&\leqq \alpha_m 4\pi\tau^2 + \iint_{\Delta(z_j, 4\tau)} \log^+ |f'(z)|\, dxdy.
\end{aligned}
\tag{3.22}
$$

We note from (3.20) and (3.5) that

$$\alpha_m 4\pi\tau^2 p \leqq 2^{-(n+5)} r_m^2 \delta(a_k, f) T(r_m, f').$$

Using this fact, (3.21), and (3.22) we get (3.19).

4. Proof of Theorem 2

Let $p = p(k, l)$, $z_j = z_j(k, l)$ $(1 \leqq j \leqq p)$, $\rho = 2^{-n(1+2\varepsilon)} r_m$, and $\tau = 2^{-n(1-3\varepsilon)} r_m$ be defined as in §3. Let $\rho_1 = \rho/(192 r_m)$, $\tau_1 = \tau/(6 r_m)$ and

$$w_j = (z_j/48 r_m) + \tfrac{1}{4} + \tfrac{1}{4}i, \qquad i = \sqrt{-1},$$

when $1 \leqq j \leqq p$, $k \in \Gamma(n)$, $l \in \Lambda_k$. We observe that if $L(z) = z/48 r_m + \frac{1}{4} + \frac{1}{4}i$, then

$$L[\Delta(z_j, \rho/4)] = \Delta(w_j, \rho_1), \quad L[\Delta(z_j, 8\tau)] = \Delta(w_j, \tau_1), \tag{4.1}$$

for $1 \leqq j \leqq p$. From (4.1) we see that

$$\Delta(w_j, \tau_1) \subseteq \Delta(\tfrac{1}{4} + \tfrac{1}{4}i, \tfrac{1}{4})$$

and that there exists for each $k \in \Gamma(n)$, $l \in \Lambda_k$, and $1 \leqq j \leqq p(k, l)$, a closed dyadic square $Q_{k,l,j} \subseteq \Delta(w_j, \rho_1)$ with common side length $r \geqq \rho_1/4$. Put

$$E = \bigcup_{k,l,j} Q_{k,l,j}$$

where the union is taken over all $k \in \Gamma(n)$, $l \in \Lambda_k$, and j with $1 \leqq j \leqq p(k, l)$. Let N be the number of the above squares and let G be Green's function for $U = \hat{\mathbf{C}} - E$ with pole at ∞. Then as in §2 there exists a positive Borel measure μ and a real number b for which

$$G(w) = \int_E \log|w - \zeta|\, d\mu(\zeta) + b, \qquad w \in \mathbf{C},$$

and

$$\text{(4.2)} \qquad \mu(F) = \omega(\infty, F), \qquad F \text{ Borel} \subseteq E,$$

where $\omega(\cdot, F)$ is the harmonic measure of F relative to U. From the maximum principle for harmonic functions we observe that

$$G(w) \geqq \log(4|w - \tfrac{1}{4} - \tfrac{1}{4}i|), \qquad w \in U.$$

Hence,

$$\text{(4.3)} \qquad G(w) \geqq \log 2 \quad \text{for } |w - \tfrac{1}{4} - \tfrac{1}{4}i| = \tfrac{1}{2}.$$

We assume that Theorem 3 is valid, and let ε_0, r_0 be the constants in that theorem, and $n_2 = \max\{n_1, 2\log(1/r_0)\}$. Consider only $n \geqq n_2$, thus $r \leqq 2\rho_1 < r_0$. We shall say $Q' \in \{Q_{k,l,j}\}$ is a bad square if

$$\omega(\infty, \partial Q') = \mu(Q') > r^{2+\varepsilon_0}.$$

Otherwise $Q' \in \{Q_{k,l,j}\}$ is said to be a good square. Theorem 3 implies that there are at most $(1/r)^{2-\varepsilon_0}$ bad squares.

We claim that if $k \in \Gamma(n)$, $l \in \Lambda_k$, and $1 \leqq j \leqq p(k, l)$, then the total number N^* of disks $\Delta(w_j, \tau_1)$ which have points in common with some bad square Q' satisfies

$$\text{(4.4)} \qquad N^* \leqq 16(\tau_1/\rho_1)^2 \left(\frac{1}{r}\right)^{2-\varepsilon_0} \leqq 10^8 2^{\alpha n},$$

where

$$\alpha = 10\varepsilon + (1 + 2\varepsilon)(2 - \varepsilon_0).$$

In fact, if $Q' \subseteq \Delta(w_i(k_0, l_0), \rho_1]$ and $Q' \cap \Delta(w_j, \tau_1) \neq \varnothing$, then

$$\Delta(w_j, \tau_1) \subseteq \Delta[w_i(k_0, l_0), 4\tau_1].$$

The left-hand inequality in (4.4) is an easy consequence of this inclusion, Theorem 3 and the fact that the disks

$$\{\Delta(w_j, \rho_1)\}, \quad k \in \Gamma(n), \quad l \in \Lambda_k, \quad 1 \leqq j \leqq p(k, l),$$

are pairwise disjoint. The right-hand inequality follows from the left-hand inequality, the definition of ρ_1, τ_1 and the inequality $\rho_1/4 \leqq r < 2\rho_1$.

Next suppose that ζ_j, $1 \leqq j \leqq p(k, l)$, $k \in \Gamma$, $l \in \Lambda_k$, is the center of $Q_{k,l,j} \subseteq \Delta(w_j, \rho_1)$. From the Riesz representation formula we deduce that

$$\max_{|w - w_j| \leqq \tau_1/2} G(w) \leqq \max_{|w - \zeta_j| \leqq 5\tau_1/8} G(w) \leqq \frac{11}{2\pi} \int_0^{2\pi} G(\zeta_j + \tfrac{3}{4}\tau_1 e^{i\theta}) d\theta$$

$$= 11 \int_{\rho_1/8}^{3\tau_1/4} \mu[\Delta(\zeta_j, t)] t^{-1} dt \leqq 11\mu[\Delta(w_j, \tau_1)] \log(6\tau_1/\rho_1).$$

Again if $\Delta(w_j, \tau_1)$ has no points in common with any bad square, then since the disks $\{\Delta(w_i, \rho_1)\}$ are pairwise disjoint, this disk can have a nontrivial intersection with at most $16(\tau_1/\rho_1)^2$ good squares. Using this fact, (4.2), and the above inequality we deduce that

$$\max_{|w - w_j| \leqq \tau_1/2} G(w) \leqq 10^3 (\tau_1/\rho_1)^2 r^{2+\varepsilon_0} \log(6\tau_1/\rho_1) \leqq n\, 10^5 2^{\beta n}, \tag{4.5}$$

where

$$\beta = 10\varepsilon - (1 + 2\varepsilon)(2 + \varepsilon_0),$$

and we have used the definition of τ_1, ρ_1, r.

We now prove (1.4) under the assumption that Theorem 3 is true. Suppose

$$\Gamma(n)^* \geqq 2^{n(1-\varepsilon_1)}, \tag{4.6}$$

where $\varepsilon_1 = \varepsilon_0/40$. Let $\varepsilon = 2\varepsilon_1 = \varepsilon_0/20$ and observe from (3.7), (3.8), that there exists $n_3(\varepsilon_0, r_0, \mu, \delta(a_1, f)) \geqq n_2$ such that

$$\{(k, l) : k \in \Gamma_1(n), l \in \Lambda_k\}^* \geqq 2^{n(2-\varepsilon)}, \quad \text{for } n \geqq n_3. \tag{4.7}$$

Retracing our steps we see from (4.4) that since

$$\alpha = 10\varepsilon + (1 + 2\varepsilon)(2 - \varepsilon_0) < 2 - \varepsilon_0/5 = 2 - 4\varepsilon,$$

there exists $n_4 \geqq n_3$, depending on ε_0, r_0, μ, and $\delta(a_1, f)$ such that

$$N^* < 2^{n(2-\varepsilon)}, \quad \text{for } n \geqq n_4.$$

This inequality and (4.7) imply there exist (k, l) such that each $\Delta(w_j, \tau_1)$, $1 \leqq j \leqq p(k, l)$, has an empty intersection with any bad square. Hence (4.5) holds for $1 \leqq j \leqq p(k, l)$. Let L be as in (4.1). Using (4.1), (4.3) and the maximum principle for subharmonic functions we see for $|z| \leqq 24r_m$ that

$$\log^+|f'(z)| \leqq \log M(24r_m, f')(\log 2)^{-1}G(L(z)). \tag{4.8}$$

With (k, l) fixed as above we deduce from (4.8), (4.5), and (3.4) that

$$\log^+|f'(z)| \leqq nKT(r_m, f')2^{\beta n} \tag{4.9}$$

when $z \in \Delta(z_j, 4\tau)$, $1 \leqq j \leqq p(k, l)$. From (4.9), (3.19), and (3.20), it follows that

$$2^{-(n+5)}r_m^2\delta(a_k, f)T(r_m, f') \leqq \sum_{j=1}^{p} \iint_{\Delta(z_j, 4\tau)} \log^+|f'(z)|\,dxdy$$

$$\leqq nKT(r_m, f')2^{\beta n}p\tau^2 \leqq nKT(r_m, f')r_m^2 2^{\theta n},$$

where

$$\theta = 21\varepsilon - (1 + 2\varepsilon)(2 + \varepsilon_0) - 2 < -\tfrac{1}{10}\varepsilon_0 - 4.$$

Since $\delta(a_k, f) \geqq 2^{-3(n+1)}$, the above inequality can only hold for $n \leqq n_5$, provided $n_5 = n_5(\varepsilon_0, \mu, \delta(a_1, f))$ is large enough. Putting $n_0 = \max(n_4, n_5)$ we get (1.4); (1.4) implies Theorem 2 for $0 < \gamma < \varepsilon_1/3$, as mentioned in §1.

5. Proof of Theorem 3

The method of proof is a reformulation of that of Bourgain in [2], in which it is proved that in $\mathbf{R}^n$, $n \geqq 3$, harmonic measures are supported on sets of Hausdorff dimension at most $n - \varepsilon(n)$. The technique is adapted from $\mathbf{R}^3$ to $\mathbf{R}^2$, with a closer account of the constants.

Given a square Q, denote by $l(Q)$ its side length, $|Q|$ its area, cQ the square concentric to Q with $l(cQ) = cl(Q)$ and $Q_* = \frac{1}{10}Q$. Given a set A, the ρ-dim dyadic content of A is

$$h_\rho(A) = \inf\left\{\sum_j l(I_j)^\rho : I_j \text{ closed dyadic squares}, A \subseteq \bigcup I_j\right\}. \tag{5.1}$$

Denote by $\omega(z, F, \Omega)$ the harmonic measure of $F \subseteq \partial\Omega$ at $z \in \Omega$ with respect to the domain Ω, and $\omega(F) = \omega(F, \Omega)$ the harmonic measure at ∞.

Lemma 3. *Let J be a square and E be a closed set in $\{0 \leqq x \leqq 1/2, 0 \leqq y \leqq 1/2\}$, and $1 < \rho \leqq 2$. Then at least one of the following holds:*

$$\omega(z, J \cap E, J \setminus E) \geqq \tau > 0 \quad \textit{for } z \in J_*, \tag{5.2}$$

$$h_\rho(E \cap J_*) \leqq 576 \cdot 2^{-1/\tau}h_\rho(J). \tag{5.3}$$

Proof. It suffices to show that when $l(J)=1$, either (5.2) or

$$h_\rho(E \cap J_*) \leqq 36 \cdot 2^{-1/\tau} \tag{5.3$'$}$$

holds. The extra factor "16" occurs in (5.3), because each non-dyadic square is contained in a dyadic square of side length at most four times that of the original.

It follows from the proof of Frostman's theorem as presented in [13; p. 64] that there exists a non-negative measure μ on $E \cap J_*$ of total mass $m \geqq \frac{1}{36} h_\rho(E \cap J_*)$ so that $\mu(\{|z-a|<b\}) \leqq b^\rho$ for any $a \in \mathbf{C}$ and $b>0$. Let

$$u(z) = \int_E \log \frac{1}{|z-\zeta|}\, d\mu(\zeta).$$

It is easy to check that

$$u \leqq \int_0^{m^{1/\rho}} \log \frac{1}{t}\, dt^\rho \leqq \frac{m}{\rho}\left(1+\log\frac{1}{m}\right) \leqq m\left(1+\log\frac{1}{m}\right);$$

on ∂J, $u \leqq m \log \frac{20}{9}$; and on J_*, $u \geqq m \log(10/\sqrt{2})$. Hence in J,

$$\begin{aligned}\omega(z) &\equiv \omega(z, J\cap E, J\setminus E)\\ &\geqq \left(u(z) - m\log\frac{20}{9}\right)\left(m\left(1+\log\frac{1}{m}\right) - m\log\frac{20}{9}\right)^{-1}.\end{aligned}$$

Thus on J_*,

$$\omega(z) \geqq \left(\log\frac{9}{2\sqrt{2}}\right)\left(1+\log\frac{1}{m} - \log\frac{20}{9}\right)^{-1},$$

which implies

$$m \leqq \frac{9e}{20}\left(\frac{9}{2\sqrt{2}}\right)^{-1/\omega(z)} \leqq 2^{-1/\omega(z)}.$$

Therefore either (5.2) or (5.3)$'$ must hold, and the lemma is proved.

Given a dyadic $a \in (0,1)$, let S_j be the collection of closed dyadic squares of side length a^j, and $S = \bigcup_{j=0}^\infty S_j$. Let

$$m_\rho(A, \varepsilon) = \inf\left\{\sum_j l(I_j)^\rho : I_j \in S,\ l(I_j) \leqq \varepsilon \text{ and } A \subseteq \bigcup I_j\right\}. \tag{5.4}$$

Thus $m_\rho(A,\varepsilon) \geqq h_\rho(A)$ for any $\varepsilon > 0$.

Lemma 4. *There exist $p \in (1, 2)$, dyadic $a \in (0, 1)$, so that if E is a closed set in $\{0 \leqq x \leqq 1/2, 0 \leqq y \leqq 1/2\}$ and I is a square in S_j, $j \geqq 0$, then at least one of the following holds:*

(5.5) $$m_p(E \cap I, a^{j+1}) \leqq a^{pj};$$

(5.6) $$\sum_{\substack{J \in S_{j+1} \\ J \subseteq I}} \omega(E \cap J)^{1/2} |J|^{1/2} \leqq \tfrac{1}{8} \omega(E \cap I)^{1/2} |I|^{1/2};$$

where ω is the harmonic measure on the boundary of $\hat{\mathbf{C}} \setminus E$ evaluated at ∞.

The lemma becomes trivial if a^{pj} is replaced by $2a^{pj}$ in (5.5), or $\frac{1}{8}$ is replaced by $\sqrt{2}$ in (5.6).

Proof. Given $\tau \in (0, 2^{-6}]$, choose $p \in (1, 2)$, and dyadic $a \in (0, 1)$ so that

(5.7) $$2^{-1/(2\tau)} < a < 2^{-15}\tau(15 - \log \tau)^{-1}$$

and

(5.8) $$2 - p \leqq a^5.$$

These choices readily guarantee that $a < 2^{-25}$, $a^{p-2} \leqq 2$, $2 - p < 2^{-125}$ and, using the right side of (5.7), that

(5.9) $$\left(1 - \frac{\tau}{16}\right)^{-1 + 2^{-11}a^{-1}} < 2^{-14}\tau.$$

Suppose that there is a subsquare $J \in S_{j+1}$ of I satisfying (5.3). Because each dyadic square can be covered by at most a^{-2} squares in S, the fact that $a^{p-2} \leqq 2$, (5.1), (5.3) and (5.4) now yield that

$$m_p(E \cap J_*, a^{j+1}) \leqq 2h_p(E \cap J_*) \leqq 1152 \cdot 2^{-1/\tau} h_p(J).$$

Since $a < 2^{-25}$, J_* contains at least 2^{42} squares in S_{j+2}. Hence

$$\begin{aligned} m_p(E \cap I, a^{j+1}) &\leqq m_p(I \setminus J, a^{j+1}) + m_p(J \setminus J_*, a^{j+1}) + m_p(E \cap J_*, a^{j+1}) \\ &\leqq a^{(j+1)p}(a^{-2} - 1) + a^{(j+2)p}(a^{-2} - 2^{42}) + 1152 \cdot 2^{-1/\tau} a^{(j+1)p} \\ &= a^{pj}\{a^{p-2} - a^p + a^{2p-2} - a^{2p}2^{42} + 1152 \cdot 2^{-1/\tau} a^p\}. \end{aligned}$$

From (5.8) and the fact that $a < 2^{-25}$, it follows that

$$(2 - p)\log \frac{1}{a} < a^5 \log \frac{1}{a} < a^{2+p} < \log\left(1 + \frac{2^{40}a^{2+p}}{1 + a^p}\right),$$

which is equivalent to $a^{p-2} - a^p + a^{2p-2} - 2^{40}a^{p+2} < 1$. Thus (5.5) follows from the left side of (5.7).

Suppose next that every subsquare $J \in S_{j+1}$ of I satisfies (5.2). Let

$$I_k = (1 - 2ka)I, \quad \text{for } 1 \leqq k \leqq K \equiv a^{-1}2^{-11}.$$

We will show in a moment that

$$\omega(I_K \cap E) \leqq 2^{-10}\omega(I \cap E). \tag{5.10}$$

Points in I, except lattice points, are contained in at most two subsquares $J \in S_{j+1}$ of I. Therefore, it follows from (5.10) and Schwarz's inequality that

$$\begin{aligned}\sum_{\substack{J \subseteq I \\ J \in S_{j+1}}} \omega(J \cap E)^{1/2} |J|^{1/2} &= \sum_{J \subseteq I_K} + \sum_{J \not\subseteq I_K} \\ &\leqq (2\omega(I_K \cap E))^{1/2} |I_K|^{1/2} + (2\omega(I \cap E))^{1/2} |I \setminus I_K|^{1/2} \\ &\leqq 2^{-9/2}\omega(I \cap E)^{1/2} |I|^{1/2} + \omega(I \cap E)^{1/2} 2^{-4} |I|^{1/2} \\ &< 2^{-3}\omega(I \cap E)^{1/2} |I|^{1/2}\end{aligned}$$

We obtain (5.6) in this case.

To prove (5.10), we observe, by the maximum principle, that

$$\begin{aligned}\omega(I_K \cap E) &\leqq \omega(I_K, \hat{\mathbf{C}} \setminus (E \cup I_K)) \\ &\leqq \omega(I_1, \hat{\mathbf{C}} \setminus (E \cup I_1)) \cdot \sup_{z \in \partial I_1} \omega(z, I_K, \hat{\mathbf{C}} \setminus (E \cup I_K)),\end{aligned} \tag{5.11}$$

and that

$$\omega(I \cap E) \geqq \omega(I_1, \hat{\mathbf{C}} \setminus (E \cup I_1)) \cdot \inf_{z \in \partial I_1} \omega(z, I \cap E). \tag{5.12}$$

Because every subsquare $J \in S_{j+1}$ of I satisfies (5.2), we have, for any $z \in \partial I_1$,

$$\omega(z, I \cap E) \geqq \omega(z, I \cap E, I \setminus E) \geqq \tau\omega\Bigg(z, \bigcup_{\substack{J \in S_{j+1} \\ J \subseteq \overline{I_1 \setminus I_2}}} J_*, I \setminus \bigcup_{\substack{J \in S_{j+1} \\ J \subseteq \overline{I_1 \setminus I_2}}} J_*\Bigg). \tag{5.13}$$

For a fixed $z \in \partial I_1$, the harmonic measure on the right side of (5.13) is bounded below by the harmonic measure of the square J_* that is nearest to z relative to $I \setminus J_*$. From the maximum principle and a change of scale, it is bounded below by $\omega(\sqrt{2}/2, \{|z| = \frac{1}{20}\}, \{\frac{1}{20} < |z| < \frac{3}{2}\}) > \frac{1}{16}$. By (5.12) and (5.13), we obtain

$$\omega(I \cap E) \geqq \frac{\tau}{16}\omega(I_1, \hat{\mathbf{C}} \setminus (E \cup I_1)). \tag{5.14}$$

For $z \in \partial I_1$, by reasons similar to the above,

$$\omega(z, I_2, \hat{\mathbf{C}} \setminus (E \cup I_2)) \leqq 1 - \omega(z, E \setminus I_2, \hat{\mathbf{C}} \setminus (E \cup I_2)) \leqq 1 - \tau/16,$$

and this holds with 1 replaced by k, $2 \leqq k \leqq K-1$. We apply the maximum principle to $\partial I_1, \partial I_2, \ldots, \partial I_{K-1}$ successively to deduce that

$$
\begin{aligned}
\sup_{z \in \partial I_1} \omega(z, I_K, \hat{\mathbf{C}} \setminus (E \cup I_K)) &\leqq \prod_{k=1}^{K-1} \sup_{z \in \partial I_k} \omega(z, I_{k+1}, \hat{\mathbf{C}} \setminus (E \cup I_{k+1})) \\
&\leqq (1 - \tau/16)^{K-1}. \qquad (5.15)
\end{aligned}
$$

It follows from (5.9), (5.11), (5.14) and (5.15) that

$$
\omega(I_K \cap E) \leqq \frac{16}{\tau}\left(1 - \frac{\tau}{16}\right)^{(a^{-1}2^{-11}-1)} \omega(I \cap E) \leqq 2^{-10}\omega(I \cap E).
$$

This completes the proof.

To prove the theorem, we let $E = \bigcup_{j=1}^{N} Q_j$, retain the choices of τ, a and ρ from Lemma 4, and recall that $l(Q_j) = r$ and that S_j is the collection of the closed dyadic squares of side length a^j. Choose r_0, ε_0:

$$
0 < \varepsilon_0 \leqq (2-\rho)^2 \qquad (5.16)
$$

and

$$
0 < r_0 \leqq 2^{-1/\varepsilon_0}; \qquad (5.17)
$$

and let $0 < r < r_0$. Let M and M_0 be the integers that satisfy

$$
a^{M+1} \leqq r < a^M
$$

and

$$
M_0 = \left[(1+M)\frac{2-\rho}{4\log 2}\log\frac{1}{a}\right] + 1, \qquad (5.18)
$$

where [ ] is the greatest integer function. Clearly, $1 \leqq 2^{M_0}(a^{1+M})^{(2-\rho)/4} \leqq 2$; and $M_0 \geqq 1/(8-4\rho)$ because of (5.16) and (5.17). The significance in choosing M_0 as a small fraction of M will become apparent after (5.20) and (5.22) are proved.

Suppose that $I \in S_j$, $0 \leqq j \leqq M$ and $I \cap E \neq \varnothing$.

If I satisfies (5.5), we attach to I a family of squares $\{J_\alpha\} \subseteq \bigcup_{j+1}^{\infty} S_k$, with mutually disjoint interiors, $J_\alpha \subseteq I$, $\varnothing \neq E \cap I \subseteq \cup J_\alpha$, and $\Sigma l(J_\alpha)^\rho \leqq l(I)^\rho$, and call them immediate descendants of I. We require, as we may, $l(J_\alpha) \geqq a^{M+1}$, since E is composed of dyadic squares of side length $r \geqq a^{M+1}$.

If I satisfies (5.6) but not (5.5), we call all $J \in S_{j+1}$, $J \subseteq I$, immediate descendants of I.

Starting with $I_0 = \{0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$, we build a family tree following the previous procedure of assigning descendants, with the following exception: if a square $I \in S_j, j \leqq M$, has exactly M_0 ancestors satisfying (5.6), we stop, and do not assign to it any descendants.

A terminal square I either has side length a^{M+1} (call this class C_1), or has side length at least a^M but has exactly M_0 ancestors satisfying (5.6) (call this class C_2). It is clear that $E \subseteq \bigcup\{I : I \in C_1 \cup C_2\}$; $a^{M+1} \leqq l(I) \leqq a^{M_0}$ for $I \in C_1 \cup C_2$; $C_1 \cap C_2 = \varnothing$; and that the squares in $C_1 \cup C_2$ have mutually disjoint interiors.

For any $I \in S$, let $\mathscr{F}(I)$ denote the family of immediate descendants of I. Then

$$\sum_{J \in \mathscr{F}(I)} l(J)^\rho \leqq l(I)^\rho, \quad \text{if } I \text{ satisfies (5.5);}$$

and, since $a^{\rho-2} \leqq 2$,

$$\sum_{j \in \mathscr{F}(I)} l(J)^\rho = a^{-2} a^{(j+1)\rho} \leqq 2 l(I)^\rho, \quad \text{if } I \text{ satisfies (5.6).} \tag{5.19}$$

Each I in $C_1 \cup C_2$ has at most M_0 ancestors which satisfy (5.6); thus it follows from (5.19) that

$$\sum_{C_1 \cup C_2} l(I)^\rho \leqq 2^{M_0} l(I_0)^\rho = 2^{M_0}. \tag{5.20}$$

Let $\mathscr{S}_m$ be the class of squares in the family tree which has exactly m ancestors that satisfy (5.6); thus $C_2 \subseteq \mathscr{S}_{M_0}$. Using (5.6), we obtain that

$$\begin{aligned}\sum_{I \in C_2} \omega(I \cap E)^{1/2} |I|^{1/2} &\leqq \sum_{I \in \mathscr{S}_{M_0}} \omega(I \cap E)^{1/2} |I|^{1/2} \\ &\leqq \frac{1}{8} \sum_{Q \in \mathscr{A}} \omega(Q \cap E)^{1/2} |Q|^{1/2},\end{aligned} \tag{5.21}$$

where $\mathscr{A} \equiv \{Q : \bigcup \mathscr{F}(Q) = \mathscr{S}_{M_0}\}$. Each $Q \in \mathscr{A}$ is either in $\mathscr{S}_{M_0-1}$ or contained in some $J \in \mathscr{S}_{M_0-1}$; different squares in $\mathscr{A}$ are mutually disjoint except possibly on the boundaries. Fix $J \in \mathscr{S}_{M_0-1}$; points in J, with the exception of finitely many lattice points, are covered by at most two distinct squares from $\mathscr{A}$. Therefore by the Schwarz inequality, we have

$$\sum_{\substack{Q \in \mathscr{A} \\ Q \subseteq J}} \omega(Q \cap E)^{1/2} |Q|^{1/2} \leqq (2\omega(J \cap E))^{1/2} |J|^{1/2}.$$

It follows from the second inequality in (5.21) and the above estimates that

$$\sum_{I \in \mathscr{S}_{M_0}} \omega(I \cap E)^{1/2} |I|^{1/2} < \frac{1}{4} \sum_{J \in \mathscr{S}_{M_0-1}} \omega(J \cap E)^{1/2} |J|^{1/2}.$$

Applying (5.6) and Schwarz's inequality alternatively to $\mathscr{S}_{M_0-1}, \mathscr{S}_{M_0-2}, \ldots$, we obtain that

$$\sum_{I \in \mathscr{S}_{M_0}} \omega(I \cap E)^{1/2} |I|^{1/2} \leqq 4^{-M_0}.$$

Combining this with (5.21), we conclude that

$$\sum_{I \in C_2} \omega(I \cap E)^{1/2} |I|^{1/2} \leqq 4^{-M_0}. \tag{5.22}$$

Let $C_3 = \{I \in C_2 : \omega(I \cap E) > 8^{M_0} |I|\}$, $C_4 = C_2 \setminus C_3$, and $\gamma = (2-\rho)3/4$. We note from (5.18) that $2^{-M_0} \leqq a^{(2-\rho)(M+1)/4}$. Because $a^{M+1} \leqq |I|$, we have

$$\omega(I \cap E) \geqq |I|^{1-\gamma}, \quad \text{if } I \in C_3.$$

Hence

$$\sum_{I \in C_3} l(I)^{3\rho/2 - 1} \leqq 1. \tag{5.23}$$

From (5.16), (5.18), (5.22) and $2 - \rho \leqq 2^{-125}$, it follows that

$$\begin{aligned} \sum_{I \in C_4} \omega(I \cap E) &\leqq \sum_{I \in C_4} \omega(I \cap E)^{1/2} 8^{M_0/2} |I|^{1/2} \leqq 8^{M_0/2} 4^{-M_0} \\ &\leqq 2^{-M_0/2} \leqq r^{(2-\rho)/8} \leqq r^{2^{122} \varepsilon_0}. \end{aligned} \tag{5.24}$$

Let N_k be the number of our given Q_j's that are contained in $\bigcup\{I : I \in C_k\}$, $k = 1, 3$. Since each Q_j contains either none or exactly $(r/a^{M+1})^2$ squares in C_1, it follows from (5.16), (5.18), (5.20) and $2 - \rho \leqq 2^{-125}$ that

$$\begin{aligned} N_1 &\leqq 2^{M_0} a^{-(M+1)\rho} a^{2(M+1)} r^{-2} = 2a^{-(M+1)(2-\rho)/4 + (M+1)(2-\rho)} r^{-2} \\ &\leqq 2r^{-2+3(2-\rho)/4} < 2r^{-2+2^{120}\varepsilon_0}. \end{aligned}$$

Because $a^M \leqq l(I) \leqq a^{M_0}$ for $I \in C_3$, we obtain from (5.16), (5.17), (5.18), (5.23) and $a^{\rho-2} \leqq 2$ that

$$\begin{aligned} N_3 &\leqq (a^{M_0})^{1-3\rho/2} \left(\frac{a^{M_0}}{r}\right)^2 \leqq r^{-2} (a^{M+1})^{(M_0/(M+1))3(2-\rho)/2} \\ &\leqq r^{-2+(3(2-\rho)^2/8 \log 2)\log(1/a)} \leqq r^{-2+9\varepsilon_0}. \end{aligned}$$

In view of (5.24), there are at most $r^{-2+2^{120}\varepsilon_0}$ squares Q_j that are contained in $\bigcup\{I : I \in C_4\}$ satisfying $\omega(Q_j) > r^{2+\varepsilon_0}$. Because $r_0 < 2^{-1/\varepsilon_0}$ and $E \subset \bigcup\{I : I \in C_1 \cup C_3 \cup C_4\}$, there are at most $2r^{-2+2^{120}\varepsilon_0} + r^{-2+9\varepsilon_0} + r^{-2+2^{120}\varepsilon_0} < r^{-2+\varepsilon_0}$ squares Q_j with $\omega(Q_j) > r^{2+\varepsilon_0}$. This proves Theorem 3.

Throughout the paper, we have not attempted to make constants nearly the best possible. Properties (5.7), (5.8), (5.16) and (5.17) are all satisfied if $\tau = 2^{-6}$, $a = 2^{-26}$, $\rho = 2 - 2^{-130}$, $\varepsilon_0 = 2^{-260}$ and $r_0 = 2^{-2^{260}}$.

Added in proof. Details of [4] have appeared in Izv. Akad. Nauk SSSR, Ser. Mat. **51** (1987), 421–428.

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