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The IGL logo was designed by Grace Work.
Message from the Director, Illinois Geometry Lab

Now entering its seventh year, the Illinois Geometry Lab (IGL) continues to spearhead department efforts to promote undergraduate research and to engage with partners across campus and in our community. Within our fall research portfolio, you’ll find groups working on projects ranging from the dynamics of Hamiltonian flows to properties of automata which recognize space-filling curves, and from optimal betting strategies for the Nash-Shapley poker model to feature detection by the YOLO image recognition system. The national profile of the IGL continues to rise as our students present and publish their research. Members of several past IGL groups attended this past summer’s second Geometry Labs United (GLU) conference in Seattle to showcase their work. We’ve also added summer research projects to the mix; information about this past summer’s research can be found later in this brochure.

On the outreach side, the big event of the fall was the Four Color Fest, commemorating and celebrating the 40th anniversary of Kenneth Appel and Wolfgang Haken’s signature achievement in conquering the Four Color Problem. The IGL contributed to the festivities by hosting a Four Color-themed Open House for community members and school children. A brisk November Saturday morning found over a dozen IGL volunteers staffing tables in the Alice Campbell Alumni Center, where numerous parents and kids gathered to learn about map coloring, graphs, algorithms, and the history of the Four Color Theorem through hands-on activities and computer demonstrations. Our students’ passion for communicating mathematics to all audiences never ceases to amaze me.

As always, we are extremely grateful for the support which we receive from many committed partners. Funding for the IGL comes from the Department of Mathematics, the National Science Foundation, the Mathematical Association of America, and a generous gift from a private donor.

—Jeremy Tyson
Director, Illinois Geometry Lab
IGL Members, Fall 2017

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Zhi Zeng
Jianxiong (Eric) Zhao
Alan Zhou
Jiamin Zhong
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Consider the function $P(z) = \prod_{j=1}^{n} (z - a_j)$, where $a_j = x_j + iy_j$ is complex for each $j$. Then, let $P^*(z)$ be the product of $z - \overline{a_j}$. For $F(z) = \frac{(P/P^*)(z)}{F(P^*)(z)}$, it is conjectured that $\frac{F}{2n} F'(z)$ will have at most $2n - 1$ real zeros. This is related to a conjecture of Maxwell on static electrical equilibria. Taking $x_j \in \mathbb{R}$, $y_j \in \mathbb{R}^+$, and $n = 3$, we generated the Fatou sets of $\frac{F}{2n} F'(z)$ which allow us to observe the critical points of $F(z)$.

Our overarching objective is to study the location of the zeros of a function and its derivatives. In order to do this, we utilized MATLAB programs to plot the Fatou and Julia sets of the Newton’s method function, $N_f(z) = z - f(z)/f'(z)$.
Data Science and Traffic Patterns: Applying Low-Rank Approximation to Parking Data

Faculty Mentor: Richard B. Sowers
Team Leaders: Sathwik Tejaswi Madhusudhan, Vipul Satone
Scholars: Pranav Bhardwaj, Eden Brewer, Kexin Fang, Anthony Fontana, Ziying Wang, Wenting Xu, Jiahao Zhu

The purpose of low-rank approximation is to take a large data set, and create a compressed representation of the data. Our goal is to create dynamic visualizations that show how parking demand changes over time. The image below is a visualization of Seattle parking activity at 6:30 AM. We see that many people are parking around Lake Union Park early in the morning, perhaps because this is an ideal place to exercise early in the morning.

Seattle, Washington January 2, 2015
Persistent homology is a robust way of looking at large dynamic data and analyzing topological properties across time. We use persistent homology to analyze the level of congestion mergence in New York City during regular days compared to days when special events occurred. Below is an image comparison displaying the congestion mergence in Manhattan on separate days. The congestion mergence based on severity level is significantly different between the day of the New York City Marathon (left) and the following day (right) at 10 am.
Virtual Reality and Movement Disorders

Faculty Mentors: Richard Sowers, Manuel Hernandez
Team Leaders: Daan Michiels, Rachneet Kaur
Scholars: Yizhen Ding, Yiyang (Angela) Kong, Alex Layton, Sean Lin, Linchen Wang, Jiaying Wu, Pengzheng Zhang, Martin Bantchev

Virtual Reality provides a safe space for therapy dealing with physical disabilities. A treadmill and harness can be used to protect the subject, while a non-lab environment is simulated. Our project combines this with modern electroencephalography (EEG) techniques. We created VR environments designed to either create an anxious reaction or soothe the subject. Subjects used EEG while engaging in the simulated environment and have their brain activity collected. From the data an anxiety level is created, from which the environment modifies itself to be correspondingly more or less soothing, which can help train to manage the mental aspects of dealing with a movement disorder.
Surfaces Moving under Hamiltonian Flows

Faculty Mentor: Ely Kerman

Team Leaders: Stefan Klajbor Goderich, Joel Villatoro

Scholars: Eben Arnold, Nathaniel Bleier, Erik Hernandez, Zhenqin Yuan

A function $F$ of the three dimensional sphere defines a surface $\Sigma_F$ in four dimensional space. If $F$ does not depend on an angular coordinate $\theta$ on the sphere, that is the function $\frac{\partial F}{\partial \theta}$ vanishes, then $\Sigma_F$ has a rotational symmetry. Such functions also determine Hamiltonian vector fields on four dimensional space. In this project we are studying what happens when the surface $\Sigma_F$ moves under the Hamiltonian vector field of the function $\frac{\partial F}{\partial \theta}$. If $\Sigma_F$ already has the rotational symmetry, then it doesn’t move. Otherwise, its motion is described by a nonlinear partial differential equation whose form suggests that it converges to a surface with the desired symmetry. We have developed several numerical models to test this conjectured convergence to symmetry. All of our current evidence suggests that it is true.
An interesting question related to both automata theory and real analysis is the following: given a real function $f$ and its domain $D$, does there exist a Büchi automaton that recognizes the set $\{ (x, f(x)) \mid x \in D \}$? If the answer is yes, then the related function is called regular. We are interested in the properties of regular continuous functions and regular space-filling curves.

An example of a regular space-filling curve is the Hilbert curve. The pictures below show the first 6 iterations of the Hilbert curve, and a Büchi automaton that recognizes it with $x$ coded in base 4 and $f(x)$ coded in base $2 \times base 2$. 

![Hilbert curve iterations and Büchi automaton graphs](image-url)
Poncelet Theorem and Complex Analysis

Faculty Mentor: Jeremy Tyson
Team Leaders: Michelle Leung, Hadrian Quan
Scholars: Ashley Hill, Mirna Ljevar, Hanyu Lu, Zhi Zeng

For our project, we are exploring the interesting connection between Poncelet's Theorem and complex analysis. Poncelet's Theorem states that given two conics, one inside the other, there exists an infinite number of circumscribed polygons with all points lying on the outer circle. If a Poncelet triangle for an ellipse exists, all Poncelet triangles can be obtained from the associated Blaschke product, an expression in complex analysis:

\[ B_{a,b}(z) = \frac{z - a}{1 - \bar{a}z} \frac{z - b}{1 - \bar{b}z} \]

Using a Blaschke product applet, we have collected statistics of Poncelet triangles for degree three Blaschke products. In addition, we are working on an elementary geometric proof of a general case where the inner conic is a Steiner inellipse.
Our group continued to study the mathematics of poker. Specifically, we investigated the Nash-Shapley poker model for three players, one of the only mathematical poker models involving more than two players. Nash and Shapley derived optimal betting strategies for each player that use both bluffing and sandbagging.

In our research we investigated extensions and variations of the Nash-Shapley model. We also developed Mathematica code to reproduce, and extend, the findings in Nash and Shapley’s paper and to create visualizations and simulations of their model.
Chaotic maps, Exotic Number Systems, and Arithmetic Coding Schemes

Faculty Mentor: A.J. Hildebrand and KenStolarsky
Team Leader: Junxian Li
Scholars: Mark Cao, Weiru Chen, Xinxin Chen, Stephen Fan, Jared Krandel

A Beatty sequence is a sequence of integers in the form \([an]\) where \(a\) is some irrational number greater than 1. These sequences are at the heart of a remarkable result called Beatty’s Theorem, which states that if two irrational numbers \(a, b\) satisfy the equation \(\frac{1}{a} + \frac{1}{b} = 1\), then the Beatty sequences associated with \(a\) and \(b\) partition the integers. The image on the left shows the Beatty partition when \(a = \Phi\) and \(b = \Phi^2\), where \(\Phi\) is the Golden Ratio.

A theorem of Uspensky says that Beatty’s Theorem does not extend directly to partitions into three or more sets. In our project we are trying to circumvent this limitation by using “Beatty-like” sequences. The image on the right shows the construction of such a “Beatty-like” partition into three parts.
The Four Color Theorem: Archival Documentation and Outreach

Faculty Mentor: Jeremy Tyson
Team Leader: Ruth Luo
Scholars: Alan Zhou, Anna Chlopecki, Cora McVey, Tejo Nantalapati, Xiaohan Liu

The Four Color Theorem, proved in 1976 by Kenneth Appel and Wolfgang Haken, states that any map can be colored with no more than four colors: a coloring is considered proper if no two adjacent states share the same color. Though the statement of this theorem is clear and concise, its proof contains much detail and structure which cannot be understood with just a single glance. This IGL project entails the analyzing and archiving of the Appel Papers, donated by Carole Appel, to the University Archives, as well as creating outreach material to present to a variety of different audiences within the community. This project enabled us to understand the Four Color Theorem and its proof, as well as examine and handle first hand material, such as notes, computer punch cards, and correspondence in relation to its proof.
Noncommutative Geometry

Faculty Mentor: Marius Junge
Team Leader: Li Gao
Scholars: Ben Norton, Daniel Klovsky, Eric Zhao

First and foremost, we are understanding the theory behind a low dimensional trilinear form:

$$\psi(a_0, a_1, a_2) = tr(\gamma a_0 [D, a_1][D, a_2])$$

Which induces a homotopy-invariant index of a projection $e$:

$$\text{Index}_{\psi}(e) = \psi(e, e, e)$$

- $D^2 = \Delta$ is the Laplacian of a directed graph.
- The bracket $[A, B] = AB - BA$ is the commutator of $A$ and $B$.
- $\gamma$ commutes with every $a$ in the given algebra, which means $\gamma a = a \gamma$, and anticommutes with $D$, which means $D \gamma = -\gamma D$.

An interesting application of this formula is to directed graphs. Given a directed graph $G = (V, E)$, some subset $S \subseteq V$ of vertices, and the indicator $1_S$ on $S$:

$$\psi(1_S, 1_S, 1_S) = \#\text{Edges coming in to } S - \#\text{Edges going out of } S$$
Video as a Sensor

Faculty Mentor: Richard Sowers (Mathematics and ISE), Dan Work (CEE)

Team Leaders: Sai Nori Tanushree, Deborshi Goswami, Fangyu Wu

Scholars: Kyle Begovich, YiChia Huang, Jianfeng Xia, Mayank Kathuria, Lucas Gong, Yanbing Wang, Dingyang Chen, Dongjun Seung, Yuxuan Ren

This semester, we set out to develop a system that maintained the real-time detection robustness of YOLO (our underlying system) while also adding on new features. Some of these features are demonstrated below, as we can determine whether a bicyclist is wearing a helmet or not now, utilizing the previous work from over the Summer. We’ve also added on some privacy preserving software systems. We’ve included face recognition software and licence plate recognition software to locate and appropriately remove them from the video.

Additionally, we’ve been working to improve the performance of the system in both speed and accuracy, which includes some technical work on embedded computational systems.
IGL Members, Summer 2017

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Rich Sowers  
Dan Work

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Sai Aishwarya Korukanti  
Mateo Muro  
Bowen Song  
Fan Wu  
Lang Yin  
Yuan Zhang  
Ruoyu Zhu
A Mathematical Game and its Solution

Faculty Mentor: Sheldon Katz
Team Leader: Matthew Mastroeni
Scholars: Paul Genelly, Advith Govindarajan, Lang Yin, and Yuan Zhang

Consider the following game played with a Noetherian commutative ring $R$. The first player chooses a nonzero element $x \in R$ and passes the quotient ring $R/xR$ to the second player. The second player then chooses a nonzero element of the quotient ring and passes the quotient of $R/xR$ by that element back to the first player and so on until no more moves are possible. The player who makes the last move loses. Since $R$ is Noetherian, the game must finish after a finite number of moves.

We seek classes of rings for which the first player has a winning strategy. Although this game has been part of mathematical folklore for decades, the most general ring for which a winning strategy is known is $k[x, y]$ with $k$ a field. We were able to extend known results to some Gorenstein rings of small dimension as well as to various double points in the plane.

\[ y = x^2(x+1) \]

A winning move in $k[x, y]/(x(x-1))$ is a losing position $\mathbb{C}[x, y]/(y^2 - x^2(x+1))$
Our group focused on skill levels of poker players based on John von Neumann’s classical poker model and its subsequent generalizations. We classified players according to their skill levels and sought to quantify the relative effects and profit returns of skill and chance to determine which factor dominates in each poker model. Using game theoretic methods and Mathematica simulations and visualizations, we determined the long-time behavior of the profits for different levels of skill and different poker models.
Poker games are perfect natural experiments showing how people compete in a zero-sum game with imperfect information, gather information about their opponent’s strategy, and adjust their own strategy accordingly. We mainly focused on two specific Poker models: the McAdams “World’s Simplest Poker” model and the Alon Model. Both models shed light on two-player Poker games and assume that both players act simultaneously.

We derived a reduced formula to express expected payout of a game given strategies from both sides, and utilized the classical minimax method to find the optimal strategy corresponding to maximal expected payout for each player given their opponent’s game strategy. Then we came up with a generalized function that indicates the Best Response Strategy with the opponent’s strategy as input, and iterated this Best Response Strategy over multiple rounds of game. It turned out that in McAdams Model, both players will be trapped in a cycle after several rounds of game, while in Alon Model, strategies of both players will converge to the very conservative “Never Bet” strategy.
Graph Theory and Statistical Quantum Mechanics

Faculty Mentor: Ivan Contreras

Scholars: Sai Aishwarya Korukanti, Andrew Eberlein, Mateo Muro

The statistical approach of quantum mechanics provides useful information about measurements and evolution of systems at the quantum level. In this project we consider a discrete model of quantum mechanics, based on the combinatorics of a finite graph, which models the configuration space of a quantum particle. We have used the incidence and adjacency matrices to compute the graph Laplacian operator for several graphs, and we have computed certain graph observables, such as the expected value of the number of walks between two vertices, the Jaccard index, among others.

By using this approach we intend to understand the structure of written texts (left figure) and approximations of solutions of the Schrodinger equation for a free particle (right figure).
June 21, 1976: Kenneth Appel and Wolfgang Haken announce their solution to the celebrated Four Color Problem. A 124-year old conjecture finally falls to a combination of clever reasoning, detailed and patient hand analysis, and 1200 hours of processing time on three high-performance computers.

The Four Color Problem was originally posed in 1852 by Francis Guthrie, a South African mathematician while coloring a map of the counties of England. Guthrie wondered whether only four colors could always be used to distinguish different countries on a map, without countries sharing the same color appearing directly next to each other. Years later, University of Illinois mathematicians Haken and Appel would answer Guthrie’s question with their proof of the Four Color Theorem. This was the first substantial mathematical proof to make essential use of computers.

In November, the Department of Mathematics at the University of Illinois held a Four Color Fest to commemorate the 40th anniversary of Haken and Appel’s achievement. Activities during the Four Color Fest included public lectures, a musical concert and an open house hosted by the Illinois Geometry Lab.

The IGL open house had four color-related activities for K-12 students and the community. Various guided and unguided activities led participants to explore topics such as map coloring, planarity of graphs, map coloring on orientable and nonorientable surfaces, and algorithmic methods. Other lessons explained some of the ideas and concepts in Appel and Haken’s proof. The University Archives displayed an exhibit of the Kenneth Appel papers, organized and curated by the Fall 2017 IGL project “The Four Color Theorem: Archival Documentation and Outreach”. The exhibit contained original source material such as computer printouts and Fortran punchcards used in the proof, as well as correspondence received by Haken and Appel in the months following their announcement.

The Four Color Fest public lectures were given by Robin Wilson and Andrew Appel. Robin Wilson, president of the British Society for the History of Mathematics, and the author of the 2002 book Four Colours Suffice: How the Map Problem Was Solved, gave a lecture surveying the history of the Four Color Theorem as well as the University of Illinois’ role in its solution.
Andrew Appel, son of Kenneth Appel and the Eugene Higgins Professor of Computer Science at Princeton University, spoke about how ideas and algorithms arising from the various attempts to solve the Four Color Problem have subsequently been used in elsewhere in computer science, including the design of software architecture and memory allocation.

The celebration concluded with the musical concert “A Portrait in Four Colors”. The featured instrument of the evening was the Haken Continuum, designed by Lippold Haken, son of Wolfgang Haken and faculty member in the Department of Electrical and Computer Engineering at Illinois. The Continuum was played by internationally renowned musician Rob Schwimmer, traveling from New York City for the event. Schwimmer was joined by a group of Illinois faculty musicians including Rudolf Haken, Professor of Viola.

At the Four Color Fest Open House hosted by the Illinois Geometry Lab, graduate student Melinda Lanius helps visitors wearing colored ponchos figure out how to color a map with only four colors.
Mentoring an undergraduate research project provides valuable experience for graduate students and postdoctoral faculty beginning their academic career. Here are updates on the current positions of a few of our former IGL mentors.

**Francesco Cellarosi** (postdoc mentor for Fall 2014 IGL project Random Discrete Sets): currently Assistant Professor at Queen’s University, Canada.

**Noel DeJarnette** (IGL Outreach Manager 2012-2013): currently Assistant Director of the Math/Science Support Center at the University of Cincinnati.

**Michelle Delcourt** (graduate student mentor for Fall 2013 IGL project Percolation Theory and Fall 2016 IGL project Polyhedral Geometry for Analyzing Phylogenetic Methods and IGL Leadership Team member 2013-2016): currently Research Fellow at the University of Birmingham, UK.

**Spencer Dowdall** (postdoc mentor for Fall 2013/Spring 2014 IGL project Visibility in Random Forests and Fall 2014 IGL project Exploring the entropy of curves on surfaces): currently Assistant Professor at Vanderbilt University.

**Neha Gupta** (graduate student mentor for Spring 2015/Fall 2015 IGL project Quantifying residual finiteness): currently Preceptor at Harvard University.

**Bill Karr** (IGL Research Manager 2014-2016): currently Data Scientist for ZIO.

**Anton Lukyanenko** (cofounder of the Illinois Geometry Lab and graduate student mentor for Fall 2012/Spring 2013 IGL project Lithium batteries: structure and efficiency and Fall 2013/Spring 2014 IGL project Geometric model of the visual cortex): currently Assistant Professor at George Mason University.

**Cary Malkiewich** (postdoc mentor for Spring 2016 IGL project Interactive Tools for Linear Algebra): currently Assistant Professor at the State University of New York at Binghamton.

**Stefan Müller** (postdoc mentor for Spring 2015 IGL project Dynamics of vector fields in 2D and 3D): currently Assistant Professor at Georgia Southern University.

continued ➔
Former IGL Mentors, continued

**Jenya Sapir** (postdoc mentor for Spring 2016 IGL project Interactive Tools for Linear Algebra): currently Assistant Professor at the State University of New York at Binghamton.

**Laura Schaposnik** (postdoc mentor for Spring 2014 IGL project Symmetries in Nature): currently Assistant Professor at the University of Illinois at Chicago.

**Armin Straub** (postdoc mentor for Fall 2014 IGL project p-adic properties of sequences and finite state automata): currently Assistant Professor at the University of South Alabama.

**Joe Vandehey** (graduate student mentor for Spring 2012 IGL project Mathematics and Music): currently Assistant Professor at the Ohio State University.

**Vyron Vellis** (graduate student mentor for Spring 2014 IGL project Tiling Harmonic Functions): currently Research Assistant Professor at the University of Connecticut.

**Grace Work** (graduate student mentor for Fall 2012/Spring 2013 IGL project Lithium batteries: structure and efficiency and IGL Leadership Team member Fall 2012-Spring 2016): currently Postdoc at Vanderbilt University.
About the Illinois Geometry Lab

The Illinois Geometry Lab is a facility in the Department of Mathematics at the University of Illinois focusing on mathematical visualization and community engagement.

At the lab undergraduate students work closely with graduate students and postdocs on visualization projects set forth by faculty members. In the community engagement component of the lab, IGL members bring mathematics to the community through school visits and other activities.

The IGL is affiliated with Geometry Labs United (GLU), the parent organization of a group of research labs hosted in mathematics departments around the country. The mission of GLU is to promote undergraduate mathematics research and public and community engagement. Other labs currently affiliated with GLU include the Experimental Geometry Lab (EGL) at the University of Maryland, the Mason Experimental Geometry Lab (MEGL) at George Mason University, the Experimental Algebra and Geometry Lab (EAGL) at the University of Texas Pan-American, the Washington Experimental Mathematics Lab (WXML) at the University of Washington, the Mathematical Computing Laboratory (MCL) at the University of Illinois at Chicago, and the Center for the Integration of Undergraduate, Graduate and Postdoctoral Research (iCenter) at Kansas State. To schedule an IGL activity with your group, contact the IGL outreach manager at igl@math.uiuc.edu.

Joining the Illinois Geometry Lab

Participation in the Illinois Geometry Lab is open to all undergraduate students who have completed Math 241. Some familiarity with programming is also recommended. Prerequisites vary by project.

For additional information on joining the lab, visit www.math.illinois.edu/igl/join.htm.

Schedule an outreach activity

For information on how to schedule an IGL outreach activity for your school or organization, please contact igl@math.uiuc.edu.