About the Illinois Geometry Lab

The Illinois Geometry Lab is a new facility at the Department of Mathematics at the University of Illinois focusing on mathematical visualization and community engagement.

At the lab, undergraduate students work closely with graduate students and postdocs on visualization projects set forth by faculty members. In the community engagement component of the lab, IGL members bring mathematics to the community through school visits and other activities.

Joining the Illinois Geometry Lab

IGL projects scheduled for the spring semester are listed inside the brochure. Undergraduate applications are due on December 31, and projects start on January 17 (first Thursday of the spring semester). All undergraduate students are invited to apply.

For additional information on joining the lab, visit:

http://www.math.illinois.edu/igl/join.htm

102 Altgeld Hall, Urbana, IL
E-mail: igl@math.illinois.edu
www.math.illinois.edu/igl
The following projects have been proposed for the spring semester:

- Minkowski space with Stephanie Alexander
- Apollonian circle packing density with Jayadev Athreya
- Lithium batteries: structure and efficiency with Jayadev Athreya
- Properties of random knots with Nathan Dunfield
- Stability of quasicrystals with George Francis
- Documentation of mathematical models with Wendy Harris
- Number-theoretic random walks with A. J. Hildebrand
- Adventures with n-dimensional integrals with A. J. Hildebrand
- Creative blockage II with Bruce Reznick
- In search of SL(3, C) character equivalence with Ilya Kapovich
- Creative blockage III with Bruce Reznick
- Local fields with Jonathan Manton
- Properties of random knots with Nathan Dunfield
- Adventures with n-dimensional integrals with A. J. Hildebrand
- Number-theoretic random walks with A. J. Hildebrand
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For details on the projects, visit:
http://www.math.illinois.edu/igl/projects-spring2013.htm

Applications will be available until December 31 at:
http://www.math.illinois.edu/igl/join.htm

Projects run from January 17 to May 2. Full IGL meetings will be held on the following Thursdays at 5 p.m.: January 17, March 14, and April 18.

Prerequisites

We invite all undergraduate students to apply. Some familiarity with programming and completion of multivariable calculus are recommended, but prerequisites vary by project.

IGL Members

Director
Jayadev Athreya
Manager
Anton Lukyanenko
Technical Director
Jonathan Manton
Associate Manager
Grace Work
Outreach Manager
Noel DeJarnette
Additional Leadership
George Francis

Faculty Mentors
Stephanie Alexander
Jayadev Athreya
Yuliy Baryshnikov
Steve Bradlow
Wendy Harris
Yujing Huang
Jayadev Athreya

Team Leaders
Darlayne Addabbo
Ilkyoo Choi
Bill Karr
Anton Lukyanenko
Nate Orlow
M.Tip Phaovibul
Caglar Uyanik
Joseph VanderWaarden

Scholars
David Chatman
Yiwang Chen
Yijun Cheng
Jeremy DeJournett
Brian Freidin
Hiroshi Fujii
Jon Graven
Rob Halliday
Jason Hempstead
Kaiyue Hou
Lingyi Kong
Moon Lee
Nishant Nangia
Rachel Poe
J.D. Quigley
Keshav Regmi
Hengzhi Shao

Undergraduate Members
Allison Rogala
Maggie Witkowski
Technological Innovation
Technical Director: Jonathan Manton

The IGL experimented with new technologies this semester:

- **3D Printing:** We purchased a *Makerbot Replicator*, a portable 3D printer. We successfully printed mathematical objects for both educational and research purposes, including:
  - An algebraic geometry example by Tom Nevins.
  - Heisenberg geometry models by Anton Lukyanenko.
  - Demonstration of conic sections by Steven Bradlow and Hiroshi Fujii.
  - Models of Altgeld Hall and Illini Hall after the upcoming renovations.

  In addition, the 3D printer was demonstrated at IGL outreach activities.

- **3D Scanning:** Combining a Microsoft Kinect camera with specialized software, we successfully scanned household objects and recreated them using the 3D printer. We anticipate that 3D scanning technology will be useful for the conservation of the Altgeld Hall mathematical model collection.

- **Video Presentations:** The XSplit Broadcaster software allows real-time editing and broadcasting of multimedia presentations. IGL scholars uploaded to YouTube a short video showcasing the “Lithium batteries: structure and efficiency” project, and will share their experience with other IGL teams in the spring semester.

Community Involvement
Outreach Manager: Noel DeJarnette

The IGL’s community engagement activities have grown under the direction of outreach manager Noel DeJarnette.

In October, the IGL represented the “Science at the Market” at the Urbana Farmer’s Market with demonstrations of student projects and 3D printing technology.

We continued our tradition of hosting visits by local schools with a field trip from Rantoul High School. Twenty students, ranging from freshmen to seniors, worked with lab members on a hands-on geometry activity and toured both the Altgeld library and bell tower. Reaching out to area high schools, Professor Hildebrand’s team “$n$-dimensional volumes” adapted their project for a high school presentation, visiting Urbana High School to talk about their work.

In November, the IGL strengthened our partnership with the Tap In Leadership Academy. Lab members Allison Rogala and Maggie Witkowski worked with Tap In to develop and execute a four week enrichment series for Tap In’s after-school program, providing middle school students with an exciting tour of polyhedral geometry.

In the meantime, Professor Bradlow’s team developed tools to supplement his upcoming course at the African Institute for Mathematical Sciences. The short course will focus on planar curves, and will now be augmented by both interactive computer tools and 3D-printed models of conic sections.

The IGL’s outreach work has been recognized by the University of Illinois through a Public Engagement Grant. The $15,000 grant will provide the first step in expanding the IGL’s engagement activities and bringing research mathematics closer to the university community.
Let $P$ be a closed polygon with edges $v_j = v_j v_{j+1}$, $1 \leq j \leq N$, and $v_{N+1} = v_1$ and, at first at least, assume the polygon is convex and has the usual counterclockwise orientation. Erect squares on each edge and connect the nearby corners of the outer edges of the squares. In this way we create a polygon with $2N$ vertices and $2N$ edges: half the edges of course are the edges of the original polygon.

If we talk about the edges only, and identify the directed edge with the complex number it represents in the plane, then the edges of the new figure can be expressed in terms of the edges of the old figure, if you view the (directed) edge as a complex number. If $v$ and $w$ are consecutive edges in the original, then $v, i(v-w), w$ are consecutive edges in the new one. Thus the way to think about this is to write the pair $(v, w)$ as a column vector, and multiply it by the matrices

\[
\begin{bmatrix} 1 & 0 \\ i & -1 \end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix} i & -1 \\ 0 & 1 \end{bmatrix}
\]

This procedure makes it possible to continue iterating the construction, even when it becomes non-convex (as it must no later than the third stage). It also happens that the group generated by these two matrices has order $96$. Thus the dimensions of the $n$th iterate grow at most linearly in $n$, whereas the number of sides grows exponentially. Thus the dimension of the $n$th iterate grows at most linearly in $n$, whereas the number of sides grows exponentially. By the sixth or seventh step, its pretty iconoclastic. By the sixth or seventh step, it's pretty iconoclastical.

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In Search of $SL(3, \mathbb{C})$ Character-Equivalent Words

Faculty Mentor: Ilya Kapovich
Team Leader: Caglar Uyanick
Scholars: Yijun Cheng, Rachel Poe, J.D. Quigley

A word $u \in F_2$ can be represented as a string of letters from the set $\{a, b, a^{-1}, b^{-1}\}$ such that no letter is followed by its inverse. The inverse of a word is the word written backwards with each letter written as its inverse. Two words $u, v \in F_2$ are conjugate if there exists a word $w \in F_2$ such that $u = w^{-1}vw$. A matrix representation is a homomorphism $\phi : F_2 \to SL(3, \mathbb{C})$ which can be thought of as mapping each letter of a word $u \in F_2$ to corresponding matrices $A, B, A^{-1}, B^{-1} \in SL(3, \mathbb{C})$. Two words $u, v \in F_2$ are character equivalent if for all $\phi$, $\text{trace}(\phi(u)) = \text{trace}(\phi(v))$.

It has been shown that by taking a word $u$ and its reverse $u^R$ ($u$ written backwards), one can generate a pair of words which are not conjugate in $F_2$ but which are character equivalent in $SL(2, \mathbb{C})$ (Horowitz, 1972). However, no non-trivial examples of $SL(3, \mathbb{C})$ words are known to exist. The goal of this project was to find a pair of such words by creating a genetic algorithm in Sage to generate these words based on pairing a word with its reverse. We were able to finish the genetic algorithm, but as yet have not run it enough or tested extensively enough to generate any suitable words. We hope to continue the project next semester and finish the algorithm, and then hope to generate a formal proof that such pairs exist. We also hope to explore representations to $SL(2, \mathbb{C})$ and representations from $F_n$.

Apollonian Circle Packing Density

Faculty Mentor: Jayadev Athreya
Team Leader: Joseph Vandehey
Scholars: Jason Hempstead, Kaiyue Hou, Danni Sun

An Apollonian Circle Packing is a collection of disjoint circles in which each set of three mutually tangent circles are all mutually tangent to two more circles. Instead of looking at an arbitrary packing, we like to study the Farey-Ford packing, which consists of circles with centers at $\left(\frac{p}{q}, \frac{1}{2q^2}\right)$ and radii $\frac{1}{2q^2}$, which are all tangent to the $x$-axis. (The $x$-axis can be thought of as a circle with infinitely large radius.)

Suppose we cut through the Farey-Ford packing with a horizontal line and color the intersections with the circles red (as in the picture). Let $F(h)$ measure the length of the red segments at height $h$ on the interval $[0, 1]$. As $h$ approaches 0, $F(h)$ approaches $3/\pi$.

We studied the error in this limit as $h$ approaches 0: the second picture shows how the plot of $|F(1/x^2) - 3/\pi|$ is well-approximated by $1/x^{3/2}$.

We have begun working on how this problem is different in other Apollonian Circle Packings and how it might be generalized to a three-dimensional sphere packing.
This project is part of a broader program to seek out and explore interesting new or little known applications of \(n\)-dimensional integrals. In Fall 2012 we focused on two such applications, each motivated by a well-known classical problem. The first of these problems is the volume of the "Steinmetz solid", the region of intersection of three pairwise perpendicular cylinders of unit radius depicted in the figure below. We introduced an appropriate notion of an \(n\)-dimensional Steinmetz solid, and we obtained exact formulas for its volume for dimensions up to 5, and numerical values for dimensions up to 9.

The second problem is the "Broken Stick Problem", which first appeared in a 19th century examination at Cambridge University. The problem asks for the probability that the three pieces obtained by breaking up a stick at two randomly chosen points can form a triangle. In our project we considered analogous questions for broken sticks with \(n\) pieces, and we obtained both theoretical and numerical results on these questions.

Our team also created animations and interactive modules to help visualize the geometric aspects of these problems, and we gave a presentation on this work to the Math Club Team at Urbana High School.
Quadratic Residue Random Walks
Faculty Mentor: A.J. Hildebrand
Team Leader: M.Tip Phaovibul
Scholars: Yiwang Chen, Mateusz Wala, Feng Yusheng

The sequence of quadratic residues modulo a prime $p$ is a sequence of $p - 1$ symbols 1 or $-1$ that encodes the solubility of quadratic congruences modulo $p$ and that behaves in many respects like a random binary sequence of length $p - 1$. In this project we studied a certain “random walk” in the plane formed with this sequence, the “Quadratic Residue Random Walk” (QRRW). The QRRW modulo $p$ is a finite walk in the plane consisting of $p - 1$ steps of unit length and starting at the origin.

A famous result of Gauss predicts the end point of a QRRW, but what happens along the way is rather mysterious and has not been unexplored in the literature. A cursory examination of QRRW graphs shows random-like features such as sudden turns, sharp cusps, and ragged edges, but also some unexpected symmetries, though no obvious patterns.

The goal of this project was to unravel some of these mysteries. We identified six distinct shapes for a QRRW, and we correlated these shapes with congruence classes of $p$ modulo small primes. We developed efficient algorithms and C code to facilitate large scale computations of QRRWs, and we used the campus computing cluster to carry out these computations. We computed a variety of quantities associated with a QRRW, such as the maximal distance to the origin and the amount of time spent in each quadrant. We also created animations showing the evolution of a QRRW.

A Holographic Alternative to jpeg
Faculty Mentor: Yuliy Baryshnikov
Team Leader: Darlayne Addabbo
Scholars: Jon Graven, Moon Lee, Nishant Nangia

A holograph is a recording of the interference patterns formed between two beams of coherent light coming from a laser on a light-sensitive media such as photographic film. The light beam coming from a laser is broken up into two beams by a beamsplitter. One beam is directed onto a 3D object, and the other beam goes to the photographic plate. Two sets of waves, from the object and the laser, form an interference pattern on the plate, and form a hologram. Since these images are not flat and contain highly oscillating patterns, the JPEG algorithm is highly inefficient when trying to compress holographic images.

Our project was to develop methods to compress holographic images and to determine which of these methods are most efficient. Given an image, we can extract the image data consisting of ordered triples $\{R, G, B\}$ representing the red, green, and blue color channels and put the data into a matrix. Since it is hard to generate a true holographic image, we take the Fast Fourier transform (FFT) of our $N \times N$ matrix to act as our approximate hologram. Next, we split our approximate hologram into square submatrices of an arbitrary size and take the Discrete Cosine transform (DCT) of each submatrix, a process borrowed from the JPEG compression algorithm. Now we approximate the compression of the image by zeroing out certain elements in the submatrices. This corresponds to storing fewer bits in our image. We then take the inverse DCT, combine the submatrices and take the inverse FFT to retrieve a compressed version of the original image.
The Department of Mathematics at Illinois holds one of the world's largest collections of mathematical models dating to the late 19th and early 20th century. These date to a time when models were the best way to illustrate mathematical formulae and concepts, given that powerful computers and software would not exist for another 70-90 years. Although many of the models have been photographed, the identification process has started. Already, we have identified that Ludwig Brill (Mathematical Institute of the Royal Polytechnicum, Munich), Martin Schilling (Halle and Leipzig), and Arnold Emch (University of Illinois) published most of our models.

Later in the project, we anticipate the development of a website which would include images of the models, their history, and the mathematicians who created them. The website would also include a virtual version created using the 3D printer available at the Illinois Geometry Lab.

As of the end of Fall 2012, the full collection of 358 plaster cast, string, metal, glass, plastic, and paper models have been photographed. The identification of the collection has started. Already, we have started to document the models for current and future generations.

Our goal was to make interactive demonstrations for students. By giving each group of students their own copy of the models, we aimed to help them understand the key concepts and their significance. In the past, students have been able to use this applet to explore the conic sections and their properties. Through the creation of virtual versions of the models, students can explore the relationship between the formulae and the visual representations of the conic sections.

Another project involved constructing an applet in Sage (a Python-like programming language) which allowed students to interact with the equation for the conic sections. By computing how to represent the conic section as the solution to a matrix equation and how to find the eigenvectors, students can explore the connection between the axes of the conic section and the eigenvectors of the matrix.

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