DEFORMATION AND QUASIREGULAR EXTENSION OF CUBICAL ALEXANDER MAPS

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Seppo Rickman (1935–2017) in memoriam

Abstract. In this article we develop a dimension-free topological deformation theory for cubical Alexander maps, and introduce a geometrical (quasiregular) extension procedure with distortion independent of degree.

We apply the topological theory to establish a Hopf degree theorem for a class of Alexander maps between spheres, and a Berstein–Edmonds type extension theorem for branched covering maps. We apply the geometrical theory to extend Rickman’s large local index theorem for quasiregular maps to all dimensions \( n \geq 4 \). We also construct, in dimension \( n = 4 \), a version of a wildly branching quasiregular map of Heinonen and Rickman, and a uniformly quasiregular map of arbitrarily large degree whose Julia set is a wild Cantor set.

The existence of a wildly branching quasiregular map yields an example of a metric 4-sphere \((S^4, d)\), which is not bilipschitz equivalent to the Euclidean 4-sphere \( S^4 \) but which admits a BLD-map to \( S^4 \).

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1. INTRODUCTION

In 1920, J. W. Alexander [2] showed that every closed oriented PL (piecewise linear) $n$-manifold $M$ can be triangulated to obtain an orientation-preserving simplicial branched covering map $M \to \mathbb{S}^n$, which maps neighboring $n$-simplices to the upper and the lower hemispheres of $\mathbb{S}^n$ respectively. We call any such map from an $n$-manifold (possibly having boundary) to $\mathbb{S}^n$ an Alexander map.

In this article we develop a dimension-free topological deformation theory for a class of Alexander maps, and introduce a geometrical (quasiregular) extension procedure with distortion independent of degree.

We apply the topological theory to establish a Hopf degree theorem for a class of Alexander maps between spheres, and a Berstein–Edmonds type extension theorem for branched covering maps. We apply the geometrical theory to extend Rickman’s large local index theorem for quasiregular maps to all dimensions $n \geq 4$. We also construct, in dimension $n = 4$, a version of a wildly branching quasiregular map of Heinonen and Rickman, and a uniformly quasiregular map of arbitrarily large degree whose Julia set is a wild Cantor set.

Recall that a continuous mapping $f : M \to N$ between oriented Riemannian $n$-manifolds is $K$-quasiregular for $K \geq 1$, if $f$ belongs to the Sobolev space $W^{1,n}_{\text{loc}}(M; N)$ and satisfies the distortion inequality

$$\|Df\|^n \leq KJ_f \quad \text{a.e. } M,$$

where $\|Df\|$ is the norm of the weak differential $Df$ of $f$ and $J_f$ is the Jacobian determinant of $f$.

History. A fundamental theorem in the Euclidean theory of quasiregular mappings is Rickman’s Picard theorem [45]: For each $n \geq 2$ and $K \geq 1$ there exists a constant $q = q(n, K) \geq 1$ with the property that each non-constant $K$-quasiregular map $\mathbb{R}^n \to \mathbb{S}^n$ omits at most $q$ points.

Rickman, in his 1985 paper [48], on the sharpness of the Picard theorem in dimension 3, introduced powerful piece-wise linear methods to construct
quasiregular mappings in $\mathbb{R}^3$. Using these methods, he proved that given a finite set in $S^3$ there exists a quasiregular mapping from $\mathbb{R}^3$ omitting exactly that set.

Also by applying the methods from \[48\], Rickman proved in \[47\] that for spatial quasiregular mappings, large local degree, even on a Cantor set, does not lead to large distortion. In 1995, he showed in \[50\] the sharpness of the quasiregular version of the Ahlfors defect relation in dimension 3 (see also \[46\]). In 1998, Heinonen and Rickman constructed in \[24\], a quasiregular mapping $S^3 \to S^3$ whose branch set contains a wild Cantor set.

In retrospect, we may distill three fundamental ideas introduced in \[48\]. Two of them, caving and sheets, are independent of dimension, at least for dimensions $n \geq 3$. In dimension 2, these methods are not available and they are not needed. In fact, every quasiregular map $\mathbb{R}^2 \to \mathbb{R}^2$ is the composition of an entire function with a quasiconformal map.

Rickman’s third method, deformation, is a branched cover homotopy theorem which converts 2-dimensional Alexander maps to a normal form. This, together with some geometric constraints, yields a flexible homotopy procedure, which can be applied to the extension problems. The reason for the dimension restriction for deformation is that the local combinatorics of the triangulations for Alexander maps are fully understood only in dimension 2. The general theory of deformation of $n$-dimensional Alexander maps remains open for $n \geq 3$.

In this article, we exhibit a deformation theory for shellable cubical Alexander maps in all dimensions $n \geq 3$. Deformation for this special class of maps is sufficiently useful in demonstrating the subtle properties of higher dimensional quasiregular maps. This deformation also yields another proof for the extension part of the Picard construction in \[14\].

Outline. The rest of this introduction is organized as follows. We first discuss Alexander maps, a higher dimensional deformation theory, a normal form of Alexander maps, and a Hopf theorem for Alexander maps. In the second topic, we explain the method of weaving, which is an expansion of Rickman’s sheet construction, for gluing branched covering maps, and an application of weaving in the topological construction of branched covers. The third topic is a method of extending Alexander maps to quasiregular maps with distortion independent of degree. The first two topics are topological in nature, while the third one is geometrical. After these, we discuss applications. At the end of the introduction, we state some open questions related to quasiregular mappings.

1.1. Alexander maps. J. W. Alexander showed in \[2\] that every closed orientable PL (piecewise linear) $n$-manifold $M$ can be realized as a branched cover of the standard $n$-sphere $S^n$.

Assume that $S^n$ is equipped with a PL structure which consists of exactly two $n$-simplices $S_+$ and $S_-$, upper and lower hemispheres, whose intersection is their pairwise common boundary $\partial S_+ = \partial S_- = S^{n-1}$.

Alexander proved that such a manifold $M$ has a triangulation for which neighboring $n$-simplices can be paired to obtain an orientation-preserving simplicial branched covering map $f: M \to S^n$. As already mentioned, we
call any such map from an \( n \)-manifold (possibly having boundary) \( M \) to \( S^n \) an Alexander map. Here a branched covering map refers to a simplicial map which is orientation preserving and preserves the dimension of simplices. More generally, we say that a continuous map is a branched covering map if it is discrete and open. For simplicial maps between manifolds, these two notions agree.

We refer to Hilden \[26\], Hirsch \[27\], and Montesinos \[36\] for results in dimension 3 on optimal degrees of branched covering maps \( M \to S^n \).

1.2. Rickman’s 2-dimensional deformation. In \[48\] Rickman studied the deformation of 2-dimensional Alexander maps, and called the triangulations, which carry Alexander maps, mapping complexes. Since Alexander maps are uniquely determined by the underlying complex and the orientation, Rickman reduced the deformation of an Alexander map to the issue of deforming its mapping complex.

We say that a branched cover \( F: M \times [0, 1] \to N \times [0, 1] \), where \( M \) and \( N \) are \( n \)-manifolds, is a branched cover homotopy if each map \( x \mapsto F(x, t) \), \( t \in [0, 1] \), is a level-preserving branched covering map \( M \times \{t\} \to N \times \{t\} \).

The following result, although not explicitly stated in \[48\], encapsulates Rickman’s deformation method in \[48\] Section 5.

Theorem (Rickman, 1985). Suppose that \( K \) is a 2-dimensional mapping complex with the surface \( \Sigma \) as its space, \( G \) is a subcomplex whose space is a 2-cell \( B \), and \( f: \Sigma \to S^2 \) is an Alexander map. Then there exists a branched cover homotopy \( F: \Sigma \times [0, 1] \to S^2 \times [0, 1] \) for which \( F|_{\Sigma \times \{0\}} = f \) and \( F|_{\Sigma \times \{1\}} \) is an Alexander map \( f': \Sigma \to S^2 \), associated to a mapping complex \( K' \) on \( \Sigma \) obtained by collapsing the 2-complex \( G \) to a 1-complex \( G' \) on a 1-cell, expanded by simple covers.

In this theorem, the mappings \( x \mapsto F(x, t) \) in the intermediate stages are merely branched covers and not Alexander maps.

We make the terms, such as simple covers, expansion, and collapsing, precise in Part I. Heuristically, the action of expansion by a simple cover increases the multiplicity of a given branched cover \( M \to S^n \) by one. The expansion is obtained by modifying a given map locally, in a neighborhood of a tame codimension one disk in \( M \), by inserting an \( n \)-cell which is mapped onto the entire target sphere; see Figure 1.2 for an illustration.

An interpretation of Rickman’s theorem is that an Alexander map of a 2-cell may be deformed to a normal form by collapsing the underlying complex and inserting a collection of simple covers.

In a 2-dimensional mapping complex, the configuration for two adjacent triangles, that is, 2-simplices meeting in an edge, falls into three combinatorial patterns; see Figure 1.2. This limited number of possible configurations and the planar topology are crucial in the proof of Rickman’s theorem.

1.3. Higher dimensional deformation. In dimensions \( n \geq 3 \), the combinatorial configurations of adjacent \( n \)-simplices are abundant. This and other topological problems make it difficult to keep track of the reduction steps in the deformation. Therefore, instead of asking for a unique normal form for all Alexander maps, we consider deformations of cubical Alexander
maps on shellable cubical complexes. We discuss briefly these terminologies, and refer to Sections 6.1, 6.2, and 6.3 for the exact definitions.

A cubical $n$-complex $K$ is a complex, analogous to a simplicial complex, with elements which are $k$-cubes for $k = 0, \ldots, n$. A cubical complex $K$ admits a canonical triangulation $K^\Delta$ which is a simplicial complex. A map $f : |K| \to S^n$ is a cubical Alexander map if $f$ is an Alexander map with respect to the simplicial complex $K^\Delta$; here $|K|$ is the space of the complex $K$.

A finite cubical complex $K$, for which $|K|$ is an $n$-cell, is shellable if there exists an order $Q_1, \ldots, Q_m$ for the $n$-cubes in $K$ for which the intersection $(Q_1 \cup \cdots \cup Q_{k-1}) \cap Q_k$ is an $(n-1)$-cell for each $k = 1, \ldots, m$. This is the cubical counterpart to the simplicial shellability in the literature. See Figure 3 for an example of a shellable cubical complex $K$ and its canonical triangulation $K^\Delta$. Not all cubical $n$-complexes of cells are shellable; see Remark 6.8.

The canonical triangulation $K^\Delta$ of a cubical complex $K$ provides the stability needed for deformation; it may be considered as a local regularity condition for the Alexander maps. On the other hand, the shellability is
a global condition which allows the underlying complexes to be reduced inductively.

Given a cubical complex $K$ on an $n$-cell, we denote by $K^*$ the (combinatorially) unique simplicial complex which is the star of a vertex in the interior of $|K|$ and agrees with $K^\Delta$ on the boundary $\partial|K|$; see Figure 3 for an example. We denote $(K^\Delta)^{(k)}$ the collection of $k$-simplices in the simplicial complex $K^\Delta$.

Our version of higher dimensional deformation theorem for cubical Alexander maps may be stated as follows.

**Theorem 1.1** (Cubical deformation). Let $n \geq 2$, and let $K$ be a shellable cubical $n$-complex, $K^\Delta$ a canonical triangulation of $K$, and $K^*$ a star-replacement of $K^\Delta$. Let $f : |K| \to S^n$ be a $K^\Delta$-Alexander map and $f^* : |K| \to S^n$ a $K^*$-Alexander map. Let

$$m = \left( \#(K^\Delta)^{(n)} - \#(K^*)^{(n)} \right) / 2,$$

and let $\tilde{f} : |K| \to S^n$ be a branched covering map obtained from $f^*$ by an expansion with $m$ mutually essentially disjoint free simple covers. Then $f$ and $\tilde{f}$ are branched cover homotopic rel $\partial|K|$.

1.4. **Hopf degree theorem for Alexander maps.** The Cubical Deformation Theorem leads to several versions of the Hopf degree theorem for cubical Alexander maps $S^n \to S^n$.

The classical Hopf degree theorem states that for a closed, connected, and oriented $n$-manifold $M$, two maps $M \to S^n$ are homotopic if and only if they have the same degree. In particular, the Hopf theorem shows that the homotopy class of a continuous map $M \to S^n$ is classified by a single integer.

Our first version of the Hopf theorem for cubical Alexander maps $S^n \to S^n$ reads as follows. We say a cubical complex $K$ on the sphere $S^n$ is shellable if there exists an $n$-cube $Q \in K$ for which $K \setminus \{Q\}$ is a shellable complex on the $n$-cell $S^n \setminus \text{int}Q$.

**Theorem 1.2** (First Hopf theorem for cubical Alexander maps). Let $K_1$ and $K_2$ be two shellable cubical complexes on $S^n$ having the same number of $n$-cubes. Then a $K_1^\Delta$-Alexander map and a $K_2^\Delta$-Alexander map, with the same orientation, are branched cover homotopic.

We record also two normal forms for cubical Alexander maps $S^n \to S^n$.

The first is the reduction to the identity, and the second to a winding map.
Theorem 1.3 (Second Hopf theorem for cubical Alexander maps). Let $K$ be a shellable cubical complex on $S^n$. Then an orientation preserving $K^\Delta$-Alexander map is branched cover homotopic to

1. the identity map $S^n \to S^n$ expanded by free simple covers, and to
2. a winding map $S^n \to S^n$.

Finally, we summarize these statements in a stabilized version.

Corollary 1.4. Let $n \geq 1$, and $f_0 : S^n \to S^n$ and $f_1 : S^n \to S^n$ be two cubically shellable Alexander maps of degrees $\deg f_0 > \deg f_1$. Then there exists a branched covering map $F : S^n \times [0,1] \to S^n \times [0,1]$ for which $F|_{S^n \times \{0\}} = f_0$ and $F|_{S^n \times \{1\}}$ is the Alexander map $f_1$ expanded by free simple covers.

1.5. Weaving. We now discuss the method of weaving Alexander maps, which is a variant of Rickman’s sheet construction [48, Section 7]. See also [14, Section 7] for another version. As an application, we obtain topological results on the existence and the extension of branched covers between manifolds, in the spirit of Hirsch and Berstein–Edmonds.

In contrast to Rickman’s original method, the setup for weaving is as follows. We are given an $n$-manifold $M$ (possibly with boundary), an essential manifold partition $(M_1, \ldots, M_m)$ of $M$, and an essential partition of the $n$-sphere $S^n$ into $n$-cells $(E_1, \ldots, E_p)$, with $2 \leq p \leq m$, arranged in a cyclic order. Given are also a surjection $c : \{1, \ldots, m\} \to \{1, \ldots, p\}$, and $m$ orientation-preserving branched covering maps $f_i : M_i \to E_{c(i)}$ whose boundary maps are compatible Alexander maps, with respect to a so-called CW$_\Delta$-structure on the total boundary $\bigcup_{i=1}^m \partial M_i$. Here, the compatibility requires mappings $f_i$ and $f_j$ to agree on the $(n-2)$-skeleton in their common boundary $M_i \cap M_j$, but not necessarily to agree on the entire $M_i \cap M_j$. 

Figure 4. A cubical Alexander map, a winding map, and an identity map expanded by simple covers in two-dimension, as in Theorem 1.3.
The objective of weaving is to glue together these individual branched covers $f_1, \ldots, f_m$ into a branched cover $M \to S^n$. To do this, we need to address the continuity, the discreteness and the openness – three essential criteria for branched covers – and the possibility that in the initial data, multiple elements in the partition $(M_1, \ldots, M_m)$ of $M$ are mapped to the same element in $(E_1, \ldots, E_p)$ on the target side.

As for the procedure, we first refine the manifold partition $(M_1, \ldots, M_m)$, in a neighborhood of the pairwise common boundaries $\bigcup_{i \neq j} (M_i \cap M_j)$, by a method we call sphericalization, and readjust the maps $f_1, \ldots, f_m$ on the refined partition accordingly. We then reconnect the elements in the refinement to form a new partition $(M'_1, \ldots, M'_m)$ with the property that the elements in $(M'_1, \ldots, M'_m)$ and in $(M_1, \ldots, M_m)$ are pairwise bilipschitz equivalent. We again readjust the maps. Associated to the newly adjusted branched covering maps $f'_i: M'_i \to E_{c(i)}$, the map $f: M \to S^n$ given by $f|_{M'_i} = f'_i$ for each $i = 1, \ldots, m$, is a well-defined branched cover. We refer to Theorem 10.5 for the details.

The topology of the boundary of the manifold $M$ has no role in weaving. This feature is particularly useful in extending maps from the boundary to the interior.

1.6. Berstein–Edmonds-type branched covers – topological construction. Weaving provides a powerful tool for constructing branched covers, in the spirit of Hirsch and Berstein–Edmonds, on $n$-manifolds for all $n \geq 3$. In this section, we consider the topological construction, where the distortion of the map is bounded but not controlled. In the next section, we move to the geometric construction, where the distortion is controlled in terms of the data.

Berstein and Edmonds proved in [5, Theorem 6.2] another version of the Hopf theorem in dimension 3. In its full generality their theorem is an extension theorem for branched covers over compact 3-manifolds with two boundary components.

**Theorem** (Berstein-Edmonds, 1979). Given a compact and oriented 3-manifold $M$ with boundary components $\Sigma_1$ and $\Sigma_2$ and PL branched covers $f_k: \Sigma_k \to S^2 \times \{k\}$, for $k = 0, 1$, of the same degree $\deg f_0 = \deg f_1 \geq 3$, there exists a PL branched cover homotopy $F: M \to S^2 \times [0, 1]$ extending $f_0$ and $f_1$.

Heinonen and Rickman [25] generalized the Berstein-Edmonds theorem to 3-manifolds having $p \geq 2$ boundary components and for maps of degree $d = 3i \geq d_0$, where $d_0 \geq 1$ is a constant depending only on the manifolds. A short proof of this theorem for all degrees $d \geq 3$ was later given in [37].

Recently, Piergallini and Zuddas proved in [42] that every compact connected oriented PL 4-manifold $M$ with $p \geq 0$ boundary components can be represented by a 'simple' branched covering map $M \to S^4 \setminus \mathrm{int}(B_1 \cup \cdots \cup B_p)$, where $B_i$’s are pairwise disjoint Euclidean balls, for which the degree may be chosen to be either 4 or 5, and the image of the branch set satisfies a certain flatness property. In the same paper, they also established an extension theorem for the so-called ribbon fillable simple branched covers on the boundary; see Theorems 1.2 and 1.8 in [42].
In the topological construction of branched covers, we have on the domain side an $n$-manifold with $m$ boundary components and on the target side an $n$-sphere with $p$ Euclidean balls removed, for $n \geq 3$ and $m \geq p \geq 2$. Since $m$ is allowed to be strictly larger than $p$, multiple boundary components in the domain may be mapped onto the same boundary component in the target. Furthermore, in each branched cover constructed, the image of the branch set is an $(n-2)$-sphere.

**Theorem 1.5** (Branched cover realization of boundary assignments). Let $n \geq 3$, $m \geq p \geq 2$, and let $c: \{1, \ldots, m\} \to \{1, \ldots, p\}$ be a surjection. Suppose that $M$ is an oriented compact PL $n$-manifold having boundary components $\Sigma_1, \ldots, \Sigma_m$, and $N = S \setminus \text{int} (B_1 \cup \cdots \cup B_p)$ is a PL $n$-sphere $S$ with $p$ pairwise disjoint closed $n$-cells $B_1, \ldots, B_p$ removed.

Then there exist a simplicial complex $K$ on $\partial M$ and a branched covering map $f: M \to N$ for which the image $f(B_f)$ of the branch set $B_f$ is an $(n-2)$-sphere, and each restriction $f|_{\Sigma_i}: \Sigma_i \to \partial B_{c(i)}$ is an Alexander map expanded by free simple covers.

The proof of Berstein and Edmonds in [5, Theorem 6.2] is based, in an essential way, on the fact that any 2-dimensional branched covering map may be approximated by 'simple' branched covering maps; simplicity here is defined in terms of branching index. In our proof of Theorem 1.5, weaving replaces the 3-dimensional handle decomposition employed by Berstein and Edmonds. Our construction of the branched cover works from the interior towards the boundary which is different from most other extension procedures.

For manifolds with a cubical structure, the method of Theorem 1.5 yields an extension theorem for cubical Alexander maps.

**Theorem 1.6** (Branched cover extension for stabilized Alexander maps). Let $n \geq 3$, $m \geq p \geq 2$, and let $c: \{1, \ldots, m\} \to \{1, \ldots, p\}$ be a surjection. Let $K$ be a cubical complex for which $M = |K|$ is an $n$-manifold with boundary components $\Sigma_1, \ldots, \Sigma_m$, and let $N = S \setminus \text{int} (B_1 \cup \cdots \cup B_p)$ be a PL $n$-sphere $S$ with $p$ mutually disjoint $n$-cells $B_1, \ldots, B_p$ removed.

For each $i = 1, \ldots, m$, let $g_i: \Sigma_i \to \partial B_{c(i)}$ be an orientation preserving $(K|_{\Sigma_i})^{\Delta}$-Alexander map. Then there exists a branched covering map $g: M \to N$ for which the image $g(B_g)$ of the branch set $B_g$ is an $(n-2)$-sphere, and each restriction $g|_{\Sigma_i}: \Sigma_i \to \partial B_{c(i)}$ is the map $g_i$ expanded by free simple covers.

Heuristically, this theorem asserts that after a suitable collection of simple covers being added, the given cubical Alexander maps on $\partial M$ may be extended to a branched cover on the entire manifold $M$.

1.7. Berstein–Edmonds-type branched covers – quasiregular construction. Theorem 1.5 is purely topological, in the sense that there is no a priori control of distortion. In particular, the distortion of the constructed map depends not just on the complex $K$, but also on the modifications and the refinements throughout the process.

To control the distortion of a mapping that is constructed by an infinite process, an extension theorem which provides distortion estimates depending
only on the initial data is indispensable. We now state such a theorem, which plays a key role in the forthcoming higher dimensional generalizations (Theorem 1.8 and Theorem 1.9) of Rickman’s large local index theorem and the Heinonen-Rickman theorem on wild branching. The word ‘mixing’ in the title of the next theorem refers to repeated trading of pieces of adjacent regions near the boundary.

**Theorem 1.7 (Mixing).** Let $n \geq 3$ and $m \geq 2$. Suppose that $K$ is a cubical complex on an $n$-manifold with boundary components $\Sigma_1, \ldots, \Sigma_m$ and that $K$ has a separating complex.

Then there exists a constant $K = K(n, K) \geq 1$ for the following. For any $k' \in \mathbb{N}$, there exist $k \geq k'$ and a $K$-quasiregular map $f : |K| \to \mathbb{S}^n \setminus \text{int}(B_1 \cup \cdots \cup B_p)$, where $B_1, \ldots, B_p$ are pairwise disjoint Euclidean balls, such that each restriction $f|_{\Sigma_i} : \Sigma_i \to \partial B_i$ is a $(\text{Ref}_{k'}(K)|_{\Sigma_i})^{\Delta}$-Alexander map expanded by free simple covers.

The complex $\text{Ref}_{k'}(K)$ in the statement is the $(1/3^k)$-refinement of $K$ derived by subdividing each $n$-cube in $K$ into $3^{nk}$ equal subcubes; see Definition 12.2.

Heuristically, a separating complex $Z$ in a cubical complex $K$ is a codimension one subcomplex for which the components of $|K| \setminus |Z|$ are open collars of the boundary components of $|K|$. Although not formally defined there, the notion of separating complex has its origin in [24]. While the existence of a separating complex is crucial to the proof, it is a mild condition on the cubical complexes (Proposition 16.2).

Theorem 1.7 produces mappings of arbitrarily large degree with uniformly bounded distortion. More precisely, the distortion $K$ of the branched cover $f$ in Theorem 1.7 depends only on the dimension $n$, the initial data $K$, and the separating complex $Z$. The degree of $f$, on the other hand, has magnitude $c_0^{nk}$, where $k$ is the refinement index and $c_0$ is an absolute constant. In other words, the distortion $K$ and the degree $\text{deg} f$ have no correlation.

The proof Theorem 1.7 employs the higher dimensional deformation, the weaving, and a rearrangement argument for cubical essential partitions introduced in [14].

1.8. **Large local index for quasiregular maps.** Rickman’s local index theorem provides a counterexample, in dimension 3, to a conjecture of Martio, Rickman, and Väisälä in [31] which states that, for $n \geq 3$ and $K \geq 1$, there exists a constant $c = c(n, K) \geq 1$ having the property that, for a quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, the set $E_f = \{ x \in \mathbb{R}^n : i(x, f) \geq c \}$ does not have accumulation points; here $i(x, f)$ is the local index of the map $f$ at a point $x \in \mathbb{R}^n$. In the counterexample, Rickman shows that there exist a constant $K \geq 1$ and $K$-quasiregular maps $\mathbb{S}^3 \to \mathbb{S}^3$ of arbitrarily large degree whose branch sets contain Cantor sets with the aforementioned property.

We construct, by Theorem 1.7, counterexamples for all dimensions.
Theorem 1.8 (Large local index). Let \( n \geq 3 \). Then there exists a constant \( K = K(n) \geq 1 \) having the property that for each \( c > 0 \) there exists a \( K \)-quasiregular mapping \( F: S^n \to S^n \) of degree at least \( c \) for which
\[
E_F = \{ x \in S^n : i(x, F) = \deg(F) \}
\]
is a Cantor set.

The expected tension between the distortion \( K \) and the local index \( i(\cdot, F) \) behind the original conjecture of Martio, Rickman, and Väisälä stems from the observation that, for a quasiregular map \( f: M \to N \) between manifolds, a large local index \( i(x, f) \) at a point \( x \in M \) has the effect of shrinking small neighborhoods of \( x \) severely. Since the branch set \( B_f \) of \( f \), i.e., where \( f \) is not a local homeomorphism, is either empty or has an image \( f(B_f) \) of positive \( (n - 2) \)-Hausdorff measure, this makes excessive shrinking impossible. A quantitative statement to this effect was obtained by Martio in [29]. Rickman and Srebro [52] have shown that a \( K \)-quasiregular does not maintain a high local index on a large set of evenly distributed points in some quantitative sense.

1.9. Wildly branching quasiregular maps. The second application of Theorem 1.7 is the existence of wildly branching quasiregular mappings in dimension 4.

Heinonen and Rickman proved in [24] that there exists a quasiregular mapping \( S^3 \to S^3 \) whose branch set contains a wild Cantor set, namely an Antoine’s necklace. We refer to Definition 19.1 for the definition of wild Cantor set.

From the metric point of view, this wild construction provide an example of a metric 3-sphere \( (S^3, d) \) which is indistinguishable from the standard sphere \( S^3 \), in the sense advocated by Semmes [56], yet does not admit bilipschitz parametrizations. It also gives an example of a quasiregular mapping \( S^3 \to S^3 \) whose Jacobian is not comparable to the Jacobian of any quasiconformal mapping \( S^3 \to S^3 \).

Using Theorem 1.7 together with a wild quasi-self-similar Cantor set in \( \mathbb{R}^4 \) constructed in the Appendix, we prove a Heinonen-Rickman type theorem for dimension 4. We combine the 4-dimensional result with that of Heinonen and Rickman in one statement.

Theorem 1.9 (Wildly branching quasiregular map). Let \( n \) be either 3 or 4. Then there exist a wild Cantor set \( X \subset \mathbb{R}^n \) and constants \( K \geq 1, c_0 \geq 1, \) and \( m_0 \geq 1 \) for the following. For each \( c \geq c_0, \) there exist \( c' \geq c \) and a \( K \)-quasiregular mapping \( F: S^n \to S^n \) for which \( i(x, F) = c' \) for each \( x \in X, \) and \( i(x, F) \leq m_0 \) for each \( x \in S^n \setminus X. \) Furthermore, given \( s_0 \geq 1, \) we may choose the mapping \( F \) to have the property: there exists \( s \geq s_0 \) for which
\[
\frac{1}{C} \text{dist}(x, X)^s \leq J_F(x) \leq C \text{dist}(x, X)^s.
\]
for almost every \( x \in S^n. \)

1.10. Bilipschitz and BLD parametrizations of metric spheres. For each \( n \geq 3 \) and \( n \neq 4, \) there exists a topological sphere \( (S, d) \), nearly indistinguishable from \( S^n \) by classical analysis, which is not a bilipschitz
copy of $\mathbb{S}^n$ but may be mapped onto $\mathbb{S}^n$ by a BLD-map. Theorem 1.9 may be used to furnish an example of this type for dimension $n = 4$.

Recall that a discrete and open map $f: (X, d) \to (Y, d')$ between metric spaces is $L$-bounded length distortion ($L$-BLD for short) for $L \geq 1$ if

$$\frac{1}{L}\ell(\gamma) \leq \ell(f \circ \gamma) \leq L\ell(\gamma)$$

for all paths $\gamma$ in $X$, where $\ell(\cdot)$ is the length of a path.

At the core of these examples is a theorem of Martio, Rickman and Väisälä [49, III.5.1]: If $A$ is a closed subset of $\mathbb{S}^n$ of zero $(n - 2)$-dimensional Hausdorff measure, then $\mathbb{S}^n \setminus A$ is simply connected.

The case $n \geq 5$. Siebenmann and Sullivan [58] observed that, for each $n \geq 5$, there exist finite $n$-dimensional polyhedra which are homeomorphic to the standard $\mathbb{S}^n$ but are not bilipschitz to $\mathbb{S}^n$. Their assertion is based on a deep work of Cannon [10] and Edwards [15], see also [16], which asserts that the double suspension $\Sigma^2H^{n-2}$ of any $(n - 2)$-dimensional homology sphere $H^{n-2}$ is homeomorphic to $\mathbb{S}^n$, where the polyhedron $\Sigma^2H^{n-2}$ is equipped with a canonical barycenter metric associated to a fixed triangulation of $H^{n-2}$.

Note that, by the theorem of Alexander, there exists a PL branched covering map $\Sigma^2H^{n-2} \to \mathbb{S}^n$.

The double suspension $\Sigma^2H^{n-2}$ may be considered as the join $\mathbb{S}^1 \ast H^{n-2}$. The complement of the suspension circle $\Gamma$ in $\Sigma^2H^{n-2}$ is not simply connected; therefore every homeomorphism $f: \Sigma^2H^{n-2} \to \mathbb{S}^n$ maps $\Gamma$ onto a curve $f(\Gamma)$ whose complement in $\mathbb{S}^n$ is not simply connected. Thus, by the theorem of Martio, Rickman and Väisälä, $f(\Gamma)$ has positive $(n - 2)$-dimensional Hausdorff measure. Therefore $f$ can not be Hölder continuous of order greater than $1/(n - 2)$, in particular, not bilipschitz. It was asked by Siebenmann and Sullivan in [58] whether $\Sigma^2H^{n-2}$ and $\mathbb{S}^n$ are quasisymmetrically equivalent. This question seems inaccessible at the moment.

Since homology spheres are true spheres in dimensions one and two, the argument above is restricted to dimensions $n \geq 5$.

The cases $n = 3$ or 4. The argument, given here, leading to Corollary 1.10 is not new; it combines the discussions in [24], [25], and [57].

David and Semmes introduced the notion of strong $A_\infty$-weights in [13] and [55]. A strong $A_\infty$-weight $w$ is a nonnegative locally integrable function in $\mathbb{R}^n$, which is doubling and for which the distance function $d_w: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$$d_w(x, y) = \left( \int_{B_{x,y}} w \, dx \right)^{1/n}$$

for $x, y \in \mathbb{R}^n$, where $B_{x,y}$ is the unique $n$-ball containing $x, y \in \mathbb{R}^n$ with diameter $|x - y|$, is comparable to a metric.

If $f: \mathbb{S}^n \to \mathbb{S}^n$ is a quasiregular map, then its Jacobian $J_f$ is a strong $A_\infty$-weight; see [23]. Moreover, if $D_J$ is a metric comparable to $d_J$, then the map $f: (\mathbb{S}^n, D_J) \to \mathbb{S}^n$ is BLD; see [24] Proposition 3.1].
Let $X$ be a self-similar Antoine’s necklace in $S^3$, and $s > 0$. Semmes showed in [57] that the function $w: S^n \to [0, \infty)$,
$$x \mapsto \min\{1, \text{dist}(x,X)^s\},$$
is a strong $A_\infty$-weight for which the distance function $d_w$ is comparable to a metric $D_w$, and that the space $(S^3, D_w)$ is linearly locally contractible, has a Hausdorff measure comparable to the Lebesgue measure on $S^3$, and supports Sobolev and Poincaré inequalities. However, $(S^3, D_w)$ is not bilipschitz to $S^3$ when $s > 3$. The argument follows again from the fact that, when $s > 3$, $X$ has Hausdorff dimension less than one in $(S^3, D_w)$, but $S^3 \setminus X$ is not simply-connected.

There is nothing special about dimension 3. Semmes’ argument works for any quasi-self-similar Cantor set $X$ in $S^4$ as well. When we specialize the Cantor set $X$ and the quasiregular map $f$ to those in Theorem 1.9, the metrics $D_J$ and $D_w$ are comparable. Therefore we obtain the following result as a corollary.

**Corollary 1.10.** There exists a metric $d$ on $S^4$ for which $(S^4, d)$ is not bilipschitz equivalent to $S^4$, but there is a BLD-map $f: (S^4, d) \to S^4$.

If a tame Cantor set is used in Semmes’ argument instead, then the space $(S^n, D_w)$ is bilipschitz equivalent to $S^n$ for any $s > 0$; see [57, Remark 4.24].

1.11. **Jacobians of quasiregular maps.** The theorem on wildly branching quasiregular maps (Theorem 1.9) yields also an example of a quasiregular map $f: S^4 \to S^4$ for which the Jacobian $J_f$ is not comparable to the Jacobian of any quasiconformal map $S^4 \to S^4$. The argument follows almost verbatim from that of Heinonen and Rickman for dimension 3 in [24].

**Corollary 1.11.** There exists a quasiregular map $f: S^4 \to S^4$ for which the Jacobian $J_f$ of the map is not comparable to the Jacobian of any quasiconformal map $S^4 \to S^1$.

1.12. **Radial limit of bounded quasiregular maps.** A classical theorem of Fatou states that bounded analytic functions on the unit disk in $\mathbb{R}^2$ have radial limits almost everywhere on the unit circle. In comparison, the picture for boundary behavior of bounded spatial quasiregular maps in higher dimensions is incomplete.

There are several growth conditions under which a spatial quasiregular map has radial limits; see, for example, Martio and Rickman [30] and Rajala [44]. As for the non-existence, Martio and Srebro [32], and Heinonen and Rickman [25, Section 9.2] have constructed bounded spatial quasiregular mappings on the unit ball for which the radial limit does not exist on a set of Hausdorff dimension arbitrarily close to the dimension of the boundary.

The proof of the 3-dimensional theorem of Heinonen and Rickman [25, Section 9.2] applies verbatim, to all dimensions $n \geq 3$, when the Berstein and Edmonds Theorem is replaced by Theorem 1.5 in the argument. For this reason, we merely state the result.

**Corollary 1.12.** Let $n \geq 3$, and $\Gamma$ be a geometrically finite torsion free Kleinian group without parabolic elements acting on the $(n-1)$-sphere $S^{n-1}$ whose limit set $\Lambda_\Gamma$ is not the whole sphere. Then there exists a bounded
quasiregular map $f : B^n \to B^n$ such that $f$ has no radial limit at points in $A\Gamma$.

1.13. Quasiregular maps with assigned preimages. An immediate consequence of the existence of branched covers between manifolds (Theorem 1.5) is a fact that, for $n \geq 3$, there exist quasiregular maps $\mathbb{S}^n \to \mathbb{S}^n$ with arbitrarily assigned preimages. This result is topological in the sense that there is no control in the distortion. The corresponding statement is false in dimension 2.

**Theorem 1.13.** Let $n \geq 3$, $p \geq 2$, and let $z_1, \ldots, z_p$ be distinct points in $\mathbb{S}^n$ and let $Z_1, \ldots, Z_p$ be mutually disjoint finite non-empty sets in $\mathbb{S}^n$. Then there exists a quasiregular map $f : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $f^{-1}(z_i) = Z_i$ for each $i = 1, \ldots, p$.

Although this result is much simpler, this flexibility of quasiregular maps $\mathbb{S}^n \to \mathbb{S}^n$ in dimensions $n \geq 3$ bears some similarity to the map in Rickman’s Picard construction.

1.14. Julia sets of uniformly quasiregular maps. A composition of two quasiregular maps is again a quasiregular mapping, but the distortion of the composition is, in general, larger than the distortion of the original maps. Therefore, in dynamics of quasiregular mappings, it is natural to consider maps having a uniform bound for the distortions of all the iterates. A quasiregular self-mapping $f : M \to M$ of a Riemannian manifold $M$ is uniformly quasiregular, or UQR, if there exists a constant $K > 1$ for which $f$ and all its iterates are $K$-quasiregular.

Using the method of conformal traps, Iwaniec and Martin [28] constructed UQR maps $\mathbb{S}^n \to \mathbb{S}^n$ whose Julia sets are tame Cantor sets. The existence of UQR maps in $\mathbb{S}^3$ whose Julia sets are wild Cantor sets was established by Fletcher and the first named author in [18]. In this section, we use a quasi-self-similar wild Cantor set in dimension 4, constructed in the appendix, to extend the result in [18] to $\mathbb{S}^4$.

**Theorem 1.14 (Wild Julia set).** For each $k \in \mathbb{N}$, there exists a uniformly quasiregular map $\mathbb{S}^4 \to \mathbb{S}^4$ of degree at least $k$, whose Julia set is a wild Cantor set.

The restriction to dimension $n = 4$ here and in Theorem 1.9 stems from the fact that at present we are only able to construct quasi-self-similar Antoine-Blankinship’s necklaces in $\mathbb{S}^4$. We refer to Blankinship [8] for the topological construction of wild Cantor sets in $\mathbb{R}^n$ for $n \geq 4$; see also [38].

1.15. Open questions. We conclude by listing some open questions on quasiregular maps, related to the topics in this article, which we consider interesting.

Regarding methods, we ask which classes of branched covering maps admit an effective deformation theory, which provides a Hopf-type theorem.

**Question 1.15.** For which class of (PL) branched covering maps $\mathbb{S}^n \to \mathbb{S}^n$ do we have a branched homotopy Hopf theorem? More generally, for which closed $n$-manifolds $M$, is there a Hopf theorem for branched covering maps $M \to \mathbb{S}^n$?
As a particular case of the previous question we ask the following.

**Question 1.16.** Let $f: S^3 \to H^3$ be a covering map onto the Poincaré homology 3-sphere and $F = \Sigma^2 f: S^5 \to \Sigma^2 H^3$ its double suspension. Is $F$ branched cover homotopic to a winding map?

As mentioned previously, Alexander maps $\Sigma^2 H^{n-2} \to S^n$ on the double suspension of homology spheres may be chosen to be BLD, hence quasiregular. In view of the discussion in Section 1.10 we ask for the converse.

**Question 1.17.** Is there a quasiregular map $S^n \to \Sigma^2 H^{n-2}$?

The following extension problem is due to Väisälä [60]. We find the same extension question interesting also in the case of quasiregular mappings between spheres.

**Question 1.18 (Quasiregular extension).** Does every quasiregular mapping $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ extend to a quasiregular map $\mathbb{R}^n \to \mathbb{R}^n$?

By a result of Rajala [44], a quasiregular local homeomorphism has radial limits at least in a countable set. The general case is completely open.

**Question 1.19 (Radial limits).** Does a bounded quasiregular map $B^n \to \mathbb{R}^n$, $n \geq 3$, have a radial limit along even one single radius?

Rickman gave in [46] a version of the Ahlfors defect relation for quasiregular mappings. He improved the result in [50], and showed that this version of the defect relation is sharp in dimension $n = 3$.

**Question 1.20 (Sharpness of Rickman’s defect relation).** Is Rickman’s defect relation in [50] sharp in dimensions $n \geq 4$?

In our construction of Julia sets in $S^4$, the existence of quasi-self-similar wild Cantor sets has a crucial role. Are there quasi-self-similar wild Cantor sets for all dimensions $n \geq 3$?

**Question 1.21 (Wildness construction).** Does there exist wildly branching quasiregular mappings or wild branching uniformly quasiregular mappings $S^n \to S^n$ in all dimensions $n \geq 3$? Further, which Cantor sets in $S^n$ are contained in the branch set of a quasiregular mapping?

In all known constructions, the wild Cantor set is not the entire branch set. Church and Hemmingsen [11] conjectured that a light and open mapping between 3-manifolds has a branch set of dimension at least one. For branched covers $S^3 \to S^3$, this conjecture is equivalent, by the Černavskii–Väisälä theorem, to the conjecture that branch sets are not wild Cantor sets; see [1] for some partial results.

**Question 1.22 (Church and Hemmingsen).** Does there exist a quasiregular mapping $S^3 \to S^3$ whose branch set is a Cantor set?

We refer to Heinonen [22] for related questions on branching of quasiregular mappings.

Finally, we pose a problem in the spirit of Gromov’s ellipticity question [20] for quasiregular maps, recently solved by Prywes [43]; see also Bonk-Heinonen [9] and Rickman [51].
Question 1.23. If a closed n-manifold N admits a non-surjective and non-constant quasiregular mapping from the Euclidean n-space, is N a rational homology sphere?

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2. Preliminaries

In this section we discuss Alexander maps between weakly simplicial complexes. We begin with a short section on terminology related to branched covering maps.

2.1. Branched covering maps. A continuous mapping $f: X \to Y$ between topological spaces is a (generalized) branched covering map if $f$ is discrete and open. Recall that a map $f: X \to Y$ is discrete if $f^{-1}(y)$ is a discrete set for each $y \in Y$, and open if $f(U)$ is open for each open set $U \subset X$. The branch set $B_f$ of $f$ is the set of points $x \in X$ for which $f$ is not a local homeomorphism at $x$.

For us, all mappings between topological spaces are continuous unless otherwise stated, and we call continuous mappings simply as mappings or maps.

For manifolds with boundary, we extend this standard terminology and say that a map $f: M \to N$ between manifolds with boundary is an interior branched covering map if $f|_{\text{int} M}: \text{int} M \to \text{int} N$ is a branched covering map.

A map $F: X \times [0, 1] \to Y \times [0, 1]$ is level preserving if $F(X \times \{t\}) \subset Y \times \{t\}$ for each $t \in [0, 1]$; in this case we denote $F_t: X \to Y$ the map $x \mapsto F(x, t)$. Also, if there is no confusion, we denote $X_s = X \times \{s\}$ for $s \in [0, 1]$.

Lemma 2.1. Let $F': X \times [0, 1] \to Y \times [0, 1]$ be a level preserving map. Then $F'$ is a branched covering map if and only if $F_s: X \to Y$ is a branched covering map for each $s \in [0, 1]$.

Proof. Suppose $F$ is a branched covering map and let $s \in [0, 1]$. Then $F_s$ is clearly discrete. To show that $F_s$ is open, let $U \subset X$ be an open set. Then there exists an open set $V \subset X \times [0, 1]$ satisfying $U = V \cap X_s$. Since $F(V)$ is open and $F_s(U) = F(V) \cap X_s$, we conclude that $F_s(U)$ is open. Thus $F_s$ is open and $F_s: X \to Y$ is a branched covering map.

Suppose now that each map $F_s: X \to Y$ is a branched covering map. We need to show that $F$ is discrete and open. Both are (again) essentially trivial observations. Let $(y, t) \in Y \times [0, 1]$. Since $F^{-1}(y, t) = F_t^{-1}(y)$ and $F_t$ is a branched covering map, $F_t^{-1}(y, t)$ is discrete.

To show that $F$ is open, it suffices to show that sets $F(\Omega \times J)$ are open, where $\Omega \subset X$ is open and $J \subset [0, 1]$ is an interval which is an open set in $[0, 1]$. Let $(y, t) \in F(\Omega \times J)$. Then $t \in J$ and there exists $x \in \Omega$ for which $F_1(x) = y$. Thus there exists a subinterval $J' \subset J$ which is open in $[0, 1]$ so that $y \in F_1(\Omega)$ for each $t \in J'$. By a simple continuity argument, we find a neighborhood $U \subset Y$ of $y$ satisfying $U \subset F_t(\Omega)$ for each $t$. Thus open neighborhood $U \times J' \subset Y \times [0, 1]$ of $(y, t)$ is contained in $F(\Omega \times J)$. Hence $F$ is an open map. \qed
Definition 2.2. A level preserving map \( F: X \times [0, 1] \to Y \times [0, 1] \) is a branched cover homotopy from \( f: X \to Y \) to \( f': X \to Y \) if \( F \) is a branched covering map, \( F_0 = f \), and \( F_1 = f' \).

We say that branched covering maps \( f: X \to Y \) and \( f': X \to Y \) are branched cover homotopic if there exists a branched cover homotopy \( F: X \times [0, 1] \to Y \times [0, 1] \) from \( f \) to \( f' \).

We also say that branched covering maps \( f: X \to Y \) and \( f': X \to Y \) are branched cover homotopic relative to the set \( A \subset X \) (rel \( A \), for short) if there exists a branched cover homotopy \( F: X \times [0, 1] \to Y \times [0, 1] \) from \( f \) to \( f' \) satisfying \( F_s|_A = f_s|_A \) for all \( s \in [0, 1] \); note that, \( f'|_A = F_1|_A = f|_A \).

2.2. Weakly simplicial complexes. Let \( K \) be an \( n \)-dimensional CW-complex consisting of cells \( \{e_\alpha\}_{\alpha \in A} \). We denote by \( [K] \) the space \( \bigcup_{\alpha \in A} e_\alpha \) of the complex \( K \) and by \( K|_A \) the restriction of \( K \) to the closed set \( A \subset [K] \), if \( A \) is a union of cells in \( K \). We also denote by \( K^{[k]} \) the \( k \)-skeleton of \( K \), that is, the subcomplex of \( K \) consisting of all cells of dimensions at most \( k \), and by \( K^{(k)} \) the subcollection of all \( k \)-cells in \( K \). An \( n \)-dimensional CW-complex \( K \) is said to be homogeneous if \( [K] = [K^{(n)}] \).

Two \( k \)-cells \( \sigma \) and \( \sigma' \) in a CW-complex \( K \) are adjacent if \( \sigma \cap \sigma' \) contains an \((k-1)\)-cell in \( K \). More generally, for \( 0 \leq k \leq m \), cells \( \sigma \) and \( \tau \) of dimensions \( k \) and \( m \) in a space \( X \), respectively, are essentially disjoint if \( \sigma \) does not meet the interior of \( \tau \).

Given a subcomplex \( L \) of a CW-complex \( K \), the complement \( L^c \) of \( L \) in \( K \) is the subcomplex consisting of those simplices which do not meet \( |L| \). Note that in general \( L \cup L^c \neq K \). Given a subset \( A \) in a complex \( K \), we denote by \( \text{cl}_K(A) \) the smallest subcomplex of \( K \) containing \( A \); we call \( \text{cl}_K(A) \) the closure of \( A \) in \( K \).

A \( k \)-cell \( \sigma \) in a CW-complex \( K \) is a \( k \)-simplex if the restriction \( K|_\sigma \) of the complex \( K \) to \( \sigma \) is a simplicial complex isomorphic to the standard triangulation of a standard \( k \)-simplex \([e_1, \ldots, e_{k+1}]\) in \( \mathbb{R}^{k+1} \).

In what follows, we consider more general CW\(\Delta\)-complexes as in Figure 5.

\[ \text{Figure 5. A CW}\Delta\text{-complex which is not weakly simplicial, and a weakly simplicial complex which is not simplicial.} \]

Definition 2.3. A collection \( K \) of essentially disjoint cells in a space \( X \) is a CW\(\Delta\)-complex if, for each \( k \)-cell \( \sigma \in K \), the subcollection \( \{\tau \in K : \tau \subset \sigma\} \) is a \( k \)-simplex. A CW\(\Delta\)-complex \( K \) is \( n \)-dimensional if \( n \) is the maximal dimension of cells in \( K \).
Remark 2.5. Suppose that \( f \) is a branched covering map if there exists a map \( \phi \) from \( K \) to \( K' \) such that \( f \) is the composition of \( \phi \) with the restriction of \( \phi \) to \( T \) and \( T' \) is a well-defined simplicial complex, and (3) every \((n-1)\)-simplex in \( K \) is a face of at most two \( n \)-simplices.

Remark 2.7. Suppose that \( K \) is a weakly simplicial complex and that \( T \) and \( T' \) are adjacent simplices whose intersection \( T \cap T' \) is not a simplex. Then \( T \) and \( T' \) have all vertices in common. In this case there are two possibilities.

If \( |T \cap T'| = \bigcup (T \cap T')^{(n-1)} \), then \( T \cap T' \) is an \((n-1)\)-cell and \((T \cap T')^{(n-1)} \) consists of at least two common \((n-1)\)-simplices of \( T \) and \( T' \). Otherwise, \((T \cap T')^{(n-1)} \) consists of exactly one \((n-1)\)-simplex \( \xi \) and the subcomplex \((T \cap T')^{(n-2)} \setminus \{\xi\} \) is non-empty.

Convention 2.6 (\( \Delta \)-complex \( K_{S^n} \)). We fix a canonical \( \Delta \)-structure \( K_{S^n} \) on \( S^n \), for \( n \geq 1 \), by first giving \( S^{n-1} (\subset S^n) \) the simplicial structure of the boundary of an \( n \)-simplex, and then adding two \( n \)-cells (hemispheres \( B^n_+ \) and \( B^n_- \)) identified naturally along the boundary \( S^{n-1} \). This \( \Delta \)-complex \( K_{S^n} \) is a weakly simplicial \( \Delta \)-complex but not simplicial, since \( B^n_+ \cap B^n_- = S^{n-1} \). We label the vertices in \( K_{S^n} \) as \( w_0, \ldots, w_n \); note that there are no vertices in the interiors of the \( n \)-cells \( B^n_+ \) and \( B^n_- \).

Remark 2.8. We emphasize a subtlety in the previous convention. Note that although \( S^{n-1} \subset S^n \), \( K_{S^n-1} \) is not a subcomplex of \( K_{S^n} \). For each \( n \geq 1 \), the complex \( K_{S^n} \) has a subcomplex \( K = K_{S^n}|_{S^n-1} \) having \( S^{n-1} (\subset S^n) \) as its space. However this subcomplex is not isomorphic to \( K_{S^n-1} \), since the numbers of \((n-1)\)-simplices in \( K \) and in \( K_{S^n-1} \) differ; \#\( K^{(n-1)} \) = \#\( K^{(0)} \) = \( n + 1 \) and \#\( K_{S^n-1}^{(n-1)} \) = 2.

2.3. Alexander maps on weakly simplicial complexes. We say that a map \( f : |K| \to |K'| \) between spaces of \( \Delta \)-complexes \( K \) and \( K' \) is \((K,K')\)-simplicial if there exists a map \( \phi : K \to K' \) between \( \Delta \)-complexes satisfying \( f(\sigma) = |\phi(\sigma)| \) for each \( \sigma \in K \).

Definition 2.8. A map \( f : |K| \to S \) onto an \( n \)-sphere \( S \) is a \( K \)-Alexander map if

(1) \( K \) is an \( n \)-dimensional homogeneous weakly simplicial \( \Delta \)-complex,
(2) \( S \) has a \( \Delta \)-structure \( K_S \) isomorphic to \( K_{S^n} \), and
(3) \( f \) is a \((K,K_S)\)-simplicial branched covering map.

We say an Alexander map has degree \( m \) if the interior of each \( n \)-simplex in \( S \) is cover by \( f \) exactly \( m \in \mathbb{N} \) times.

Since a \( K \)-Alexander map \( f \) is a branched covering map, \( f|_\sigma \) is a homeomorphism for each \( \sigma \in K \).

Let \( f : |K| \to S^n \) be an Alexander map. Then, by openness, \( f \) maps adjacent \( n \)-simplices to the opposite hemispheres. Note also that, given an
Figure 6. An Alexander map in dimension 2.

$n$-simplex $\sigma = [v_0, \ldots, v_n]$ in $K$, we may order the vertices so that $f(v_i) = w_i$ for each $i = 0, \ldots, n$. It is easy to see that these properties also characterize Alexander maps.

Remark 2.9. We now call attention to a subtle point: the restriction of an Alexander map to a lower dimensional subcomplex is typically not an Alexander map.

In particular, given a cubical Alexander map $f : |K| \to S^n$ on a manifold $|K|$ with boundary, the restriction of $f$ to the boundary $\partial |K|$ is not a (lower dimensional) Alexander map, and the image $f(\partial |K|)$ is not even an $(n-1)$-sphere. Indeed, in this case, the image $f(\partial |K|)$ is the $(n-1)$-simplex $[w_0, \ldots, w_{n-1}]$ in $K_{S^n}$ since the only vertices in $K_{\Delta}$ which map to $w_n$ are in the interiors of the $n$-cubes in $K$. Thus the mapping $f|_{\partial |K|} : \partial |K| \to S^{n-1}$ is simplicial but not open.

3. Simple covers

In this section, we define simple covers and discuss the expansion of branched covering maps by simple covers. The main result of this section is an isotopy theorem on moving free simple covers within a manifold (Theorem 3.10) in the spirit of Rickman’s description on how to move 2-dimensional simple covers [36, Section 5.1]. This allows free simple covers to be moved at our discretion, and has a crucial role in the reduction of combinatorial complexity of cubical Alexander maps. This reduction, which is a version of Rickman’s 2-dimensional deformation theorem [36, Section 5], is discussed in more detail in Section 6.

3.1. Definitions and basic properties. In this section, we begin by defining simple covers for $S^n$-valued maps, and then discuss the expansion of branched covering maps by free simple covers.

Definition 3.1. Let $E$ be a closed $n$-cell. A map $f : E \to S^n$ is a simple cover if

1. $f(\partial E)$ is a tame $(n-1)$-cell, i.e. $(S^n, f(\partial E)) \approx (S^n, \bar{B}^{n-1})$,
2. $f|_{\text{int}E} : \text{int}E \to S^n \setminus f(\partial E)$ is a homeomorphism, and
3. there exists an essential partition $\{(\partial E)_+, (\partial E)_-\}$ of $\partial E$ into $(n-1)$-cells for which the restrictions $f|(\partial E)_{\pm} : (\partial E)_{\pm} \to f(\partial E)$ are homeomorphisms.

We recall the notion of essential partition of a space as follows.
\textbf{Definition 3.2.} Let $X$ be a space. A finite sequence $(X_1, \ldots, X_m)$ of closed subsets of $X$ is an \textit{essential partition of $X$} if $X_1 \cup \cdots \cup X_m = X$ and the interiors of $\{X_1, \ldots, X_m\}$ are nonempty and pairwise disjoint.

We conclude, directly form the definition, that a simple cover $f : E \to \mathbb{S}^n$ has branch set $B_f = (\partial E)_+ \cap (\partial E)_-$.

In order to define the expansion of a map by a simple cover, we first introduce the notion of a cell-package and fix a Euclidean package $(\tilde{E}_\circ, E_\circ, e_\circ; \rho_\circ)$ in $\mathbb{R}^n$ as a reference, where $e_\circ = B^{n-1} \times \{0\}$, $E_\circ$ is the convex hull of $\{\pm e_n, e_\circ\}$, and $\tilde{E}_\circ$ is the convex hull of $\{\pm 2e_n, e_\circ\}$. Let $\rho_\circ : \tilde{E}_\circ \to E_\circ$ be the unique map which satisfies $\rho_\circ|_{\partial \tilde{E}_\circ} = \text{id}$, and for each $x \in B^{n-1}$,

\begin{enumerate}
  \item $\rho_\circ((\{x\} \times \mathbb{R}) \cap E_\circ) = \{x\}$, and
  \item the map $\rho_\circ|_{(\{x\} \times \mathbb{R}) \cap (\tilde{E}_\circ \setminus \text{int} E_\circ)} : (\{x\} \times \mathbb{R}) \cap (\tilde{E}_\circ \setminus \text{int} E_\circ) \to (\{x\} \times \mathbb{R}) \cap \tilde{E}_\circ$ is linear.
\end{enumerate}

See Figure 7 for an example.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure7.png}
\caption{Euclidean package in dimension $n = 2$.}
\end{figure}

\textbf{Definition 3.3.} A quadruple $(\tilde{E}, E, e; \rho)$, where $\tilde{E}$ and $E$ are $n$-cells, $e$ is an $(n - 1)$-cell, and $\rho : \tilde{E} \to \tilde{E}$ is a map, is an $n$-\textit{dimensional cell-package} if there exists a homeomorphism $\theta : (\tilde{E}, E, e) \to (\tilde{E}_\circ, E_\circ, e_\circ)$ between triples for which $\rho = \theta^{-1} \circ \rho_\circ \circ \theta$. We call $\tilde{E}$ the \textit{support} of the $(\tilde{E}, E, e; \rho)$.

Cell packages $\{(\tilde{E}_j, E_j, e_j; \rho_j) : 1 \leq j \leq m\}$ are said to be \textit{essentially disjoint} (resp. \textit{mutually disjoint}) if their supports are essentially disjoint (resp. mutually disjoint).

\textbf{Remark 3.4.} Note that, if $(\tilde{E}, E, e; \rho)$ is a cell-package then

\begin{enumerate}
  \item $\rho|_{e \cup \partial \tilde{E}} = \text{id}|_{e \cup \partial \tilde{E}}$,
  \item $\rho|_{\tilde{E} \setminus E} : \tilde{E} \setminus E \to \tilde{E} \setminus e$ is a homeomorphism, and
  \item $\rho|_{(\partial E)_{\pm}} : (\partial E)_{\pm} \to e$ are homeomorphisms.
\end{enumerate}

\textbf{Definition 3.5.} Let $M$ be an $n$-manifold (possibly with boundary) and $\hat{f} : M \to \mathbb{S}^n$ a map. A map $f : M \to \mathbb{S}^n$ is
We call \(\tilde{f}\) if there exists an \(n\)-dimensional cell-package \((\tilde{E}, E, e; \rho)\) in \(\text{int} M\) satisfying

1. \(f|_{M\setminus\tilde{E}} = \tilde{f}|_{M\setminus\tilde{E}},\) and
2. \(f|_{E\setminus\text{int} E} = \tilde{f} \circ \rho|_{E\setminus\text{int} E};\)

(b) a \(\rho\)-expansion of \(\hat{f}\) by a simple cover \(f|_E\) if, further,
3. \(f|_E\) is a simple cover; and
4. \(B_f \cap E = e \cap \partial E.\)

We call \(\tilde{E}\) the support of the simple cover \(f|_E.\)

**Remark 3.6.** The existence of a \(\rho\)-expansion of \(\hat{f}\) by a simple cover requires that the map \(\hat{f}|_e\) is an embedding by Definition 3.3(3), and \(f(e)\) is a tame \((n-1)\)-cell by Definition 3.4(3). The existence of a \(\rho\)-expansion of \(\hat{f}\) by a free simple cover implies that \(e \cap B_f \subset e \cap \partial E.\) See Figure 8 for example of these cases.

![Figure 8](image_url)

Figure 8. An Alexander map \(\hat{f}: M \rightarrow \mathbb{S}^n\), an expansion of \(\hat{f}\) by a non-free simple cover, and an expansion of \(\hat{f}\) by a free simple cover.

We observe that branched covers are stable under expansions by simple covers.

**Lemma 3.7.** Let \(M\) be an \(n\)-manifold (possibly with boundary) and \(\hat{f}: M \rightarrow \mathbb{S}^n\) a branched covering map (interior branched covering map if \(\partial M \neq \emptyset\)). Let \((\tilde{E}, E, e; \rho)\) be an \(n\)-dimensional cell-package in \(\text{int} M\) and \(f: M \rightarrow \mathbb{S}^n\) a \(\rho\)-expansion of \(\hat{f}\) by a simple cover. Then \(f\) is a branched covering map (interior branched covering map).

**Proof.** We show first that \(f\) is a discrete map. Let \(y \in \mathbb{S}^n.\) Since \(\hat{f}\) is a branched covering map, the discreteness of \(f^{-1}(y) \setminus E\) follows from the
discreteness of $\hat{f}^{-1}(y) \setminus E$. Further, since $f|_E$ is a simple cover, $f^{-1}(y) \cap E$ consists of at most 2 points. We conclude that the preimage $f^{-1}(y)$ is discrete.

To show that $f$ is open, it suffices to show that every point $x$ in $\partial E$ has a neighborhood $U$ for which $f(U)$ is an open set. Let $\{(\partial E)_{+}, (\partial E)_{-}\}$ be the essential partition of $\partial E$ associated to the simple cover $f|_E$ in Definition 3.5.

When $x \in (\partial E)_{+} \cap (\partial E)_{-} = e \cap \partial E$, choose $U \subset \text{int} \hat{E}$ to be a neighborhood of $x$ in $M$ satisfying $f(U \cap (\partial E)_{+}) = f(U \cap (\partial E)_{-})$ and $f(U \setminus E) \subset f(U \setminus E)$. Since $f|_E$ is a simple cover, $f(U) = (f|_E)(U \cap E)$ is a neighborhood of $f(x)$.

When $x \in (\partial E)_{+}$, choose $U$ to be a neighborhood of $x$ for which $U \cap (\partial E)_{-} = \varnothing$. Then $f(U \setminus E)$ and $f(U \cap \text{int} E)$ are open in $\mathbb{S}^n \setminus f(\partial E)$. Since $f(U \cap \partial E)$ is open in $f(\partial E)$, we conclude that $f(U)$ is open. The case $x \in (\partial E)_{-}$ is similar. Thus $f$ is a branched covering map.

An expansion of a branched covering map by simple covers associated to given cell-packages $(\hat{E}, E, e, \rho)$ is essentially unique. We formulate this observation as follows.

**Lemma 3.8.** Let $M$ be an $n$-manifold (possibly with boundary), $\hat{f}: M \rightarrow \mathbb{S}^n$ an (interior) branched covering map, and let $(\hat{E}, E, e, \rho)$ and $(\hat{E}, E, e, \rho')$ be $n$-dimensional cell-packages in $\text{int} M$. Suppose that $f: \hat{E} \rightarrow \mathbb{S}^n$ is a $p$-expansion of $\hat{f}$ by a simple cover and $f': \hat{E} \rightarrow \mathbb{S}^n$ is a $\rho'$-expansion of $\hat{f}$ by a simple cover. Then there exists a homeomorphism $\vartheta: (\hat{E}, E, e) \rightarrow (\hat{E}, E, e)$ for which $\vartheta|_{\partial \hat{E}} = \text{id}$, $\rho' \circ \vartheta|_{\hat{E} \setminus E} = \partial \circ \rho|_{\hat{E} \setminus E}$, and $f' \circ \vartheta = f$, that is, the diagrams

\[
\begin{array}{ccc}
\hat{E} \setminus E & \xrightarrow{\vartheta} & \hat{E} \setminus E \\
\rho & \downarrow & \rho' \\
\hat{E} \setminus e & \xrightarrow{\vartheta} & \hat{E} \setminus e
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
(\hat{E}, E) & \xrightarrow{\vartheta} & (\hat{E}, E) \\
\partial f|_{\hat{E}} & \downarrow & \partial f'|_{\hat{E}} \\
\mathbb{S}^n, \hat{f}(e) & \rightarrow & \mathbb{S}^n, \hat{f}(e)
\end{array}
\]

commute.

**Proof.** We note first that, since $f|_E$ and $f'|_E$ are simple covers and $E$ is an $n$-cell, the map $\varphi = (f'|_{\text{int} E})^{-1} \circ (f|_{\text{int} E}): \text{int} E \rightarrow \text{int} E$ is a well-defined homeomorphism which admits a homeomorphic extension $\tilde{\varphi}: E \rightarrow E$.

By the definition of cell-package, the map $\psi = (\rho'|_{E \setminus E})^{-1} \circ (\rho|_{E \setminus E}): \hat{E} \setminus E \rightarrow \hat{E} \setminus E$ is a homeomorphism. Since $\hat{f}|_{\text{int} \hat{E}}$ is a branched covering map and maps $f|_E$ and $f'|_E$ are simple covers, the homeomorphism $\psi$ admits a homeomorphic extension $\tilde{\psi}: \hat{E} \setminus \text{int} E \rightarrow \hat{E} \setminus \text{int} E$ satisfying $\tilde{\psi}|_{\partial \hat{E}} = \text{id}$ and $\tilde{\varphi}|_{\partial E} = \tilde{\psi}|_{\partial E}$. Thus, there exists a homeomorphism $\vartheta: E \rightarrow E$ for which $\vartheta|_E = \tilde{\varphi}$ and $\vartheta|_{E \setminus E} = \tilde{\psi}$.

Clearly, $\rho' \circ \vartheta|_{E \setminus E} = \vartheta \circ \rho|_{E \setminus E}$ and $f' \circ \vartheta = f$. Also, in $\hat{E} \setminus E$, we have $f' \circ \vartheta = \hat{f} \circ \rho' \circ \rho'^{-1} \circ \rho = \hat{f} \circ \rho = f$.

This concludes the proof. \qed
We state this theorem formally as follows.

**Definition 3.9.** A branched covering map \( \hat{f} : M \rightarrow \mathbb{S}^n \) is an \( m \)-fold expansion of a branched covering map \( f : M \rightarrow \mathbb{S}^n \) with essentially disjoint free simple covers if there exist essentially disjoint cell-packages \( (\hat{E}_i, E_i, e_i; \rho_i) \), \( i = 1, \ldots, m \), in \( M \) having the properties that

1. \( \hat{f}|_{M \setminus (\cup \hat{E}_i)} = f|_{M \setminus (\cup E_i)} \), and
2. for each \( i = 1, \ldots, m \), \( \hat{f}|_{\hat{E}_i \setminus \text{int} E_i} = f \circ \rho_i|_{\hat{E}_i \setminus \text{int} E_i} \) and \( \hat{f}|_{E_i} \) is a free simple cover.

**3.2. Isotopy theorem for free simple covers.** Heuristically the main result of this section states:

*Free simple covers in a branched cover are movable in the domain.*

We state this theorem formally as follows.

**Theorem 3.10.** Let \( M \) be an \( n \)-manifold (possibly with boundary) and let \( \hat{f} : M \rightarrow \mathbb{S}^n \) be an (interior) branched covering map. For \( i = 0, 1 \), let \( (\hat{E}_i, E_i, e_i; \rho_i) \) be an \( n \)-dimensional cell-package in \( \text{int} M \) and \( f_i : M \rightarrow \mathbb{S}^n \) be a \( \rho_i \)-expansion of \( \hat{f} \) by a free simple cover \( f_i|_{E_i} \). Then there exists a level preserving (interior) branched cover

\[
F : M \times [0, 1] \rightarrow \mathbb{S}^n \times [0, 1]
\]

satisfying \( F_t = f_i \) for \( i = 0, 1 \). Furthermore, if \( \partial M \neq \emptyset \), then \( F_t|_{\partial M} = f_0|_{\partial M} \) for \( t \in [0, 1] \).

**Remark 3.11.** For manifolds without boundary, we have, by the Chernovski-Väisälä theorem [61 59], that the mapping \( \hat{f} \) is either orientation preserving or orientation reversing. Moreover, if \( \hat{f} \) is orientation preserving (reversing) then \( f_0 \) and \( f_1 \) are also orientation preserving (reversing). In particular, if \( M \) is a closed manifold, then

\[
\deg f_0 = \deg f_1 = \deg \hat{f} \pm 1,
\]
where the sign is $+1$ or $-1$ depending on whether $\hat{f}$ is orientation preserving or reversing.

Lemma 2.1 reduces the proof of Theorem 3.10 to the problem of the existence of a homotopy through level preserving branched covering maps. As the first step we prove the uniqueness of the homeomorphism type of a simple cover expansion. The heuristic content of the claim is illustrated in Figure 10 and the formal statement reads as follows.

![Diagram](image)

**Figure 10.** Two expansions $f_0$ and $f_1$ of an interior branched covering map $\hat{f}$ by simple covers. The mapping diagram commutes only for the restrictions $\vartheta|_{\tilde{E}_0}: \tilde{E}_0 \to \tilde{E}_1$ and $f_0|_{\tilde{E}_0}: \tilde{E}_0 \to S^n$; see Lemma 3.12.

**Lemma 3.12.** Let $C$ be an $n$-cell and let $\hat{f}: C \to S^n$ be an interior branched covering map. Suppose that, for each $i = 0, 1$, $(\tilde{E}_i, E_i, e_i; \rho_i)$ is an $n$-dimensional cell-package for which the quadruple $(C, \tilde{E}_i, E_i, e_i)$ of cells is homeomorphic to the quadruple $(\tilde{B}_i^n(4), \tilde{E}_i, E_i, e_i)$, and that $f_i: C \to S^n$ is a $\rho_i$-expansion of $\hat{f}$ by a simple cover $f_i|_{E_i}$. Then there exist homeomorphisms $h: S^n \to S^n$ and $\vartheta: C \to C$ with $\vartheta|_{\partial C} = \text{id}_{\partial C}$ which satisfy the following conditions:

1. $\vartheta|_{\tilde{E}_0} = \tilde{E}_1$, $\vartheta(E_0) = E_1$, $\vartheta(e_0) = e_1$,  
2. $\rho_1 \circ \vartheta|_{\tilde{E}_0} = \vartheta \circ \rho_0|_{\tilde{E}_0}$, and  
3. $f_1 \circ \vartheta|_{\tilde{E}_0} = h \circ f_0|_{\tilde{E}_0}$.

We note that in this lemma above, neither $f_0|_{E_0}$ nor $f_1|_{E_1}$ is assumed to be a free simple cover.

**Proof.** Since $(C, \tilde{E}_i, E_i, e_i) \approx (\tilde{B}_i^n(4), \tilde{E}_i, E_i, e_i)$ for $i = 0, 1$, we may fix a homeomorphism $\varphi: C \to C$ satisfying $\varphi|_{\partial C} = \text{id}$, $\varphi(\tilde{E}_0) = \tilde{E}_1$, $\varphi(E_0) = E_1$, $\varphi(e_0) = e_1$, $\varphi(E_1) = E_0$ and $\varphi(e_1) = e_0$. Then $\varphi|_{\tilde{E}_0}$ is $\rho_0$-oriented if $\varphi|_{\tilde{E}_1}$ is $\rho_1$-oriented. If $\varphi|_{\tilde{E}_0}$ is $\rho_0$-oriented, then for $\rho_1$-oriented $\varphi|_{\tilde{E}_1}$ we have $\varphi|_{\tilde{E}_1} = \varphi|_{\tilde{E}_0}$ by Lemma 2.1. Similarly, $\varphi|_{\tilde{E}_1}$ is $\rho_1$-oriented if $\varphi|_{\tilde{E}_0}$ is $\rho_0$-oriented. Thus $\varphi|_{\tilde{E}_0}$ and $\varphi|_{\tilde{E}_1}$ are both $\rho_0$-oriented or both $\rho_1$-oriented.

We may assume that $\varphi|_{\tilde{E}_0} = \varphi|_{\tilde{E}_1}$ is $\rho_0$-oriented. Fix such a homeomorphism $\varphi$. Then $\varphi|_{\tilde{E}_0}$ is $\rho_0$-oriented and $\varphi|_{\tilde{E}_1}$ is $\rho_1$-oriented. Thus $\varphi|_{\tilde{E}_0}$ and $\varphi|_{\tilde{E}_1}$ are both $\rho_0$-oriented or both $\rho_1$-oriented.

We may assume that $\varphi|_{\tilde{E}_0} = \varphi|_{\tilde{E}_1}$ is $\rho_0$-oriented. Fix such a homeomorphism $\varphi$. Then $\varphi|_{\tilde{E}_0}$ is $\rho_0$-oriented and $\varphi|_{\tilde{E}_1}$ is $\rho_1$-oriented. Thus $\varphi|_{\tilde{E}_0}$ and $\varphi|_{\tilde{E}_1}$ are both $\rho_0$-oriented or both $\rho_1$-oriented.

We may assume that $\varphi|_{\tilde{E}_0} = \varphi|_{\tilde{E}_1}$ is $\rho_0$-oriented. Fix such a homeomorphism $\varphi$. Then $\varphi|_{\tilde{E}_0}$ is $\rho_0$-oriented and $\varphi|_{\tilde{E}_1}$ is $\rho_1$-oriented. Thus $\varphi|_{\tilde{E}_0}$ and $\varphi|_{\tilde{E}_1}$ are both $\rho_0$-oriented or both $\rho_1$-oriented.

We may assume that $\varphi|_{\tilde{E}_0} = \varphi|_{\tilde{E}_1}$ is $\rho_0$-oriented. Fix such a homeomorphism $\varphi$. Then $\varphi|_{\tilde{E}_0}$ is $\rho_0$-oriented and $\varphi|_{\tilde{E}_1}$ is $\rho_1$-oriented. Thus $\varphi|_{\tilde{E}_0}$ and $\varphi|_{\tilde{E}_1}$ are both $\rho_0$-oriented or both $\rho_1$-oriented.
and $\varphi(e_0) = e_1$. Let $\rho' = \varphi \circ \rho_0 \circ \varphi^{-1}|_{E_1} : \tilde{E}_1 \to \tilde{E}_1$. Then $(\tilde{E}_1, E_1, e_1; \rho')$ is a cell-package.

Since $f_1|_{E_1}$ is a simple cover, the restriction $\tilde{f}|_{e_i}$ is an embedding and $\tilde{f}(e_i)$ is a tame $(n-1)$-cell for $i = 0, 1$. Thus there exists a homeomorphism $h : S^n \to S^n$ satisfying $h \circ \tilde{f}|_{e_0} = \tilde{f} \circ \varphi|_{e_0}$. Hence

$$\tilde{f} \circ \varphi \circ \rho_0 \circ \varphi^{-1}|_{\partial E_1} = h \circ \tilde{f} \circ \rho_0 \circ \varphi^{-1}|_{\partial E_1} = h \circ f_0 \circ \varphi^{-1}|_{\partial E_1}.$$ 

Thus there exists a $\rho'$-expansion $f' : C \to S^n$ of $\tilde{f}$ by the simple cover $f'|_{E_1} = h \circ f_0 \circ \varphi^{-1}|_{E_1}$.

By Lemma 3.8 there exists a homeomorphism $\psi : \tilde{E}_1 \to \tilde{E}_1$ for which $\rho' \circ \psi = \psi \circ \rho_1$ and $f' \circ \psi = f_1|_{\tilde{E}_1}$. We extend $\psi$ to be a homeomorphism $C \to C$ satisfying $\psi|_{\partial C} = \text{id}$. Then $\vartheta = \psi \circ \varphi : C \to C$ is a homeomorphism which is identity on the boundary of $C$ and satisfies conditions (1)–(3). \[\square\]

**Proof of Theorem 3.10.** Let $C$ be an $n$-cell in $M$ containing $\tilde{E}_0$ and $\tilde{E}_1$ in its interior and let $\vartheta : C \to C$ be a homeomorphism as in Lemma 3.12. Recall that a homeomorphism of an $n$-cell, which is the identity on the boundary, is isotopic to the identity (modulo boundary).

**Step 1.** We define first an isotopy $\vartheta : C \times [0, 1] \to C \times [0, 1]$ rel $(\partial C) \times [0, 1]$ from $\text{id}_C$ to $\vartheta$ which moves $(\tilde{E}_0, E_0, e_0)$ to $(\tilde{E}_1, E_1, e_1)$, and induces, for every $t \in [0, 1]$, a free simple cover $f_0 \circ \theta_t^{-1}|_{\theta_t(E_0)} : \theta_t(E_0) \to S^n$.

To define $\vartheta$ for $t \in [0, \frac{1}{2}]$, we set $\theta_0 = \text{id}_C$. We may assume that $B_f \cap B_{f_0|_{E_0}} \neq \emptyset$; otherwise, set $\theta_t = \text{id}_C$ for $t \in [0, \frac{1}{2}]$. We choose an initial isotopy $\theta$ which, for $t \in [0, \frac{1}{2}]$, fixes points in $C \setminus \tilde{E}_0$ and contracts $E_0$ (within itself) into an $n$-cell in int$E_0$, and $e_0$ (within itself) into an $(n-1)$-cell. Since $f_0|_{E_0}$ is a free simple cover, the branch sets of $\hat{f}$ and $f_0|_{E_0}$ satisfy

$$e_0 \cap B_f \subseteq e_0 \cap \partial E_0 = B_{f_0|_{E_0}}.$$ 

The contraction of $E_0$ in the isotopy yields that $\theta_t(E_0) \cap B_f = \emptyset$, hence $f_0 \circ \theta_t^{-1}|_{\theta_t(E_0)}$ is a free simple cover for every $t \in (0, \frac{1}{2}]$.

Similarly we choose $\theta$ for $t \in [\frac{1}{2}, 1)$ so that $\theta_1 = \theta$, and that $\theta$ moves $E_1 = \theta_1(E_0)$ to $\theta_{\frac{3}{4}}(E_0)$ within $E_1$, and $e_1 = \theta_1(e_0)$ to $\theta_{\frac{3}{4}}(e_0)$ within $e_1$ while keeping all points in $C \setminus \tilde{E}_1$ fixed. By Lemma 3.12 (2), $\vartheta_1 = \vartheta \circ \rho_0 \circ \vartheta^{-1} = \theta_1 \circ \rho_0 \circ \theta_1^{-1}$ on $\tilde{E}_0$.

To complete the definition of $\vartheta$, we define an isotopy $\theta : C \times \left[\frac{1}{2}, \frac{2}{3}\right] \to C \times \left[\frac{1}{2}, \frac{2}{3}\right]$ rel $(\partial C) \times \left[\frac{1}{2}, \frac{2}{3}\right]$ from $\theta_1$ to $\theta_{\frac{2}{3}}$. Since the dimension of the branch set $B_f$ is at most $n-2$ and $e_0 \cap B_f \subseteq e_0 \cap \partial E_0$, the isotopy may be chosen so that $\theta_t(E_0) \cap B_f = \emptyset$ for all $t$, and hence $f_0 \circ \theta_t^{-1}|_{\theta_t(E_0)}$ is also a free simple cover for $t \in \left[\frac{1}{2}, \frac{2}{3}\right]$.

For each $t \in [0, 1]$, we set $\rho_t = \theta_t \circ \rho_0 \circ \theta_t^{-1}$, and write $\tilde{E}_t, E_t$, and $e_t$ for $\theta_t(E_0)$, $\theta_t(E_1)$, and $\theta_t(e_0)$, respectively. We consider the cell-packages $(\tilde{E}_t, E_t, e_t; \rho_t)$, and note that for $t = 0$ this definition for $\rho_t$ agrees with the given packages.

**Step 2.** We define next an isotopy in the target. Since $\tilde{f}(\theta_t(e_0))$ is a tame $(n-1)$-cell for each $t \in [0, 1]$, each pair $(S^n, \tilde{f}(\theta_t(e_0)))$ is standard.
and there exists an isotopy $H: S^n \times [0, 1] \to S^n \times [0, 1]$ satisfying $H_0 = \text{id}$, $H_1 \circ f_0|_{E_0} = f_1 \circ \partial|_{E_0}$, and $H_t \circ f_1|_{E_0} = f_0 \circ \theta_t|_{E_0}$ for all $t \in [0, 1]$.

Step 3. Having $\theta$ and $H$ at our disposal, the homotopy between branched covering maps $f_0$ and $f_1$ may now be defined as follows. For $t \in [0, 1]$, let $F_t: M \to S^n$ be the mapping given by

1. $F_t|_{M \setminus E_t} = \hat{f}$,
2. $F_t|_{E_t \setminus E_i} = \hat{f} \circ \rho_t$, and
3. $F_t|_{E_i} = H_t \circ f_0 \circ \theta_t^{-1}|_{E_i}.

Then each $F_t$ is a $\rho_t$-expansion of $\hat{f}$ by the simple cover $H_t \circ f_0 \circ \theta_t^{-1}|_{E_t}$. Thus the mapping $F: M \times [0, 1] \to S^n \times [0, 1]$, $(x, t) \mapsto (F_t(x), t)$, is a branched covering map by Lemma 2.1. If $\partial M \neq \emptyset$, we have, by construction, that $F_t|_{\partial M} = f_0|_{\partial M}$ for $t \in [0, 1]$.

It remains to check that $F_i = f_i$ for $i = 0, 1$. Observe first that

\[ F_i|_{M \setminus E_i} = \hat{f}|_{M \setminus E_i} = f_i|_{M \setminus E_i} \quad \text{and} \quad F_i|_{E_i \setminus E_i} = \hat{f} \circ \rho_i = f_i \]

for $i = 0, 1$. Since

\[ F_0|_{E_0} = H_0 \circ f_0 \circ \theta_0^{-1}|_{E_0} = f_0|_{E_0} \]

and

\[ F_1|_{E_1} = H_1 \circ f_0 \circ \theta_1^{-1}|_{E_1} = f_1 \circ \theta_0 \circ \theta_1^{-1}|_{E_1} = f_1|_{E_1}, \]

we conclude that $F_i = f_i$ for $i = 0, 1$. The proof is complete. \hfill \Box

3.3. Extension of codimension one simple covers. In this section we prove the following extension result for expansion of a branched cover with a codimension one simple cover on the boundary. This notion of expansion is use for weaving; see Theorem 10.5.

**Proposition 3.13.** Let $M$ be an $n$-manifold with boundary and $\hat{f}: M \to \bar{B}^n$ a branched covering map. Let $(\bar{E}, E, e; \rho)$ be an $(n-1)$-dimensional cell-package on $\partial M$ and let $\varphi: \partial M \to S^{n-1}$ be a $\rho$-expansion of $\hat{f}|_{\partial M}: \partial M \to \bar{E}$ by a free simple cover $\varphi|_E$. Then there exists a branched covering map $f: M \to \bar{B}^n$ extending $\varphi$.

Note that under the assumptions of the proposition, $\hat{f}|_{\partial M}: \partial M \to S^{n-1}$ is a branched covering map.

**Proof.** Since $\varphi|_E$ is a free simple cover, $\hat{f}|_e$ is an embedding. Hence there exists an $(n-1)$-cell $d$ in $M$ for which $d \cap \partial M = e$ and $\hat{f}|_d$ is an embedding. Fix now an $n$-cell $D$ in $M$ for which $D \cap \partial M = E$, $d \subset D$, and $\partial d \subset \partial D$. Let $D_\pm$ be the closures of the components of $D \setminus d$ and $E_\pm$ be the closures of the components of $E \setminus e$ labeled so that $E_+ \subset D_+$; note that $E_- = D_- \cap \partial M$. Let also $(\partial D)_\pm = (\partial D \cap D_\pm) \setminus \text{int} E$. Finally, fix an $n$-cell $\bar{D}$ in $M$ for which $D \subset \bar{D}$, $\bar{D} \cap \partial M = \bar{E}$, and $(\partial \bar{D} \cap \partial D) \setminus \partial M = \partial d \setminus \partial M$.

Let $R: \bar{D} \to \bar{D}$ be a map satisfying the following conditions:

1. $R|_{d \setminus (\partial D \setminus E)} = \text{id}|_{d \setminus (\partial D \setminus E)}$,
2. $R|_{\bar{D} \setminus D}: \bar{D} \setminus D \to \bar{D} \setminus d$ is a homeomorphism,
3. $R|_{(\partial D)_\pm}: (\partial D)_\pm \to d$ are homeomorphisms, and
4. $R|_{E} = \rho$.
Let also $\psi : D \to \bar{B}^n$ be a map, homeomorphic in $\text{int} D$, for which

$$\psi|_E = \varphi|_E, \quad \text{and} \quad \psi|_{(\partial D)_\pm} = \hat{f} \circ R|_{(\partial D)_\pm}.$$  

We define now a branched covering map $f : M \to \bar{B}^n$ by setting

$$f|_{M \setminus \tilde{D}} = \hat{f}|_{M \setminus \tilde{D}}, \quad f|_{D \setminus D} = \hat{f} \circ R|_{D \setminus D}, \quad \text{and} \quad f|_D = \psi.$$  

We may conclude by the construction that $f$ extends $\varphi$. Indeed, we have

$$f|_{\partial M \setminus \tilde{E}} = \hat{f}|_{\partial M \setminus \tilde{E}}, \quad f|_{\tilde{E} \setminus E} = \hat{f} \circ R|_{\tilde{E} \setminus E} = \hat{\varphi} \circ \rho|_{\tilde{E} \setminus E} = \varphi|_{\tilde{E} \setminus E}, \quad \text{and} \quad f|_E = \psi|_E = \varphi|_E. \quad \Box$$

\section*{Part 1. Deformation}

In this part, we establish Rickman’s planar deformation theory ([48, Section 5]) for cubical Alexander maps on shellable complexes in all dimensions, as discussed in the introduction.

As a consequence, we obtain two versions of Hopf degree theorem for cubical Alexander maps.

4. Local deformation of Alexander maps

The deformation considered in this section is local, and takes place in a neighborhood of a simplicial star $\text{St}(v_0)$ of a vertex $v_0$. We discuss the deformation of Alexander maps in a single star in Section 4.1, and examine the transformation of branched covering maps in the ambient complex in Section 4.2.

4.1. Deformation in a single star. In this section, we show that an Alexander map $f : |\text{St}(v_0)| \to S^n$ on a single star $\text{St}(v_0)$ deforms to an Alexander map $f^{\text{cl}} : |\text{Clover}(v_0; w_n)| \to S^n$ on a clover complex $\text{Clover}(v_0; w_n)$. The deformation of the map and the transformation of the domain occur simultaneously; the end map $f^{\text{cl}}$ consists purely of simple covers. The main result of this section is Theorem 4.7.

Let $\text{St}(v_0)$ be a simplicial complex which is a star of a vertex $v_0$ on which there exists an Alexander map $f : |\text{St}(v_0)| \to S^n$.

Recall that the sphere $S^n$ has the CW$_\Delta$-structure $K_{S^n}$ consisting of two $n$-simplices $B^n_+$ and $B^n_-$ and having vertices $w_0, \ldots, w_n \in S^{n-1}$ as given in Convention 2.6.

Here and in what follows, we assume that $f(v_0) = w_0$, and adopt the convention that the labeling, $x_0, x_{j(1)}, \ldots, x_{j(k)}$, of the vertices of a $k$-simplex $\sigma$ in $\text{St}(v_0)$ respects the mapping $f$, that is,

$$f(x_0) = w_0, \quad f(x_{j(i)}) = w_{j(i)} \quad \text{for} \quad i = 1, \ldots, k.$$  

A vertex $v \in \text{St}(v_0)$ is called a $w_j$-vertex if $f(v) = w_j$. 

4.1.1. Simple pairs in St($v_0$). Recall that the link Lk($v_0$) of $v_0$ is the subcomplex of St($v_0$) consisting of all simplices not having $v_0$ as a vertex. We call the subcomplex Lk($v_0; w_n$) $\subset$ Lk($v_0$), consisting of all simplices not meeting $f^{-1}(w_n)$, the link of $v_0$ modulo $w_n$. Similarly, we define St($v_0; w_n$), which is the subcomplex consisting of all simplices in St($v_0$) not meeting $f^{-1}(w_n)$, to be the reduced star of $v_0$ modulo $w_n$. Naturally, Lk($v_0; w_n$) and St($v_0; w_n$) depend on the mapping $f$ although we do not emphasize this in the notation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{A reduced star St($v_0; w_n$) in green in a star St($v_0$); $w_n$-vertices in red.}
\end{figure}

We consider a pairing of $n$-simplices of St($v_0$) as follows. First, we say that an $(n - 1)$-simplex $\tau$ is $w_n$-avoiding if $\tau \cap f^{-1}(w_n) = \emptyset$. Note that an $(n - 1)$-simplex is $w_n$-avoiding if and only if it is contained in St($v_0; w_n$).

Given a $w_n$-avoiding $(n - 1)$-simplex $\tau$ in St($v_0$), there exist exactly two $n$-simplices $T$ and $T'$ having $\tau$ as a face. Since St($v_0$) is simplicial, $T \cap T' = \tau$. The only vertices $x_T$ and $x_{T'}$ of $T$ and $T'$, respectively, which are not contained in $\tau$, are distinct and satisfy $f(x_T) = f(x_{T'}) = w_n$, $T = \tau \ast \{x_T\}$ and $T' = \tau \ast \{x_{T'}\}$; here the join is taken in the simplicial structure of St($v_0$). Since $f|_{\text{int}[T \cup T']} : \text{int}[T \cup T'] \to S^n$ is an embedding, we call $\{T, T'\}$ a simple pair sharing $\tau$. In fact, $f|_{\text{int}[T \cup T']}$ is a simple cover. Since each $n$-simplex has a $w_n$-avoiding face, all $n$-simplices in St($v_0$) can be paired, and the pairing is unique. We denote $\mathcal{S}(v_0; w_n)$ the family of all such simple pairs in St($v_0$).

4.1.2. Leaves and clovers – a special class of simple covers. Our first goal is to deform all simple cover pairs in $\mathcal{S}(v_0; w_n)$ into leaves of a clover – a special class of simple covers for which the moving theorem (Theorem 3.10) is applicable.

**Definition 4.1.** An $n$-leaf $L$ is an $n$-dimensional weakly simplicial complex which consists of two adjacent $n$-simplices $T$ and $T'$ whose intersection $T \cap T'$ is a subcomplex containing exactly $n$ common $(n - 1)$-simplices. The space $r_L = |T \cap T'|$ of the intersection $T \cap T'$ is called the midrib of $L$. The unique vertex $v_L$ of $L$ in the interior of $|L|$ is called the center of the leaf $L$; the set $q_L = r_L \cap \partial |L|$ is called the rim of the leaf $L$.

Let $L$ be an $n$-leaf $L$ consisting of $n$-simplices $T$ and $T'$. Then $|L|$ is an $n$-cell, the rim $q_L$ of $L$ is an $(n - 2)$-sphere, the midrib $r_L$ of $L$ is an $(n - 1)$-cell, and the union of the boundaries $\partial |T| \cup \partial |T'|$, consisting of three $(n - 1)$-cells, is homeomorphic to $S^n \cup B^n$. 

Definition 4.2. A clover \( C \) with \( m \) leaves is an \( n \)-dimensional weakly simplicial complex which consists of \( m \)-leaves \( L_1, \ldots, L_m \) with a common vertex \( v_C \) satisfying \( L_i \cap L_j = \{v_C\} \) for all \( i \neq j \); the vertex \( v_C \) is called the node of the clover \( C \). The union \( r_C = \bigcup_{j=1}^{m} r_{L_j} \) of the midribs of the leaves is called the midrib of the clover \( C \).

The midrib \( r_C \) of a clover \( C \) carries also a natural complex which is simpler than \( L_{\mid r_C} \). We call this complex the midrib complex.

Definition 4.3. The midrib complex \( \text{Midrib}(L) \) of a leaf \( L \) is the simplicial complex which consists of a single \((n-1)\)-cell \( r_L \) and the subcomplex \( L_{\mid \partial L} \).

The midrib complex \( \text{Midrib}(C) \) of a clover \( C \) is the union \( \bigcup L_{\mid \partial L} \text{Midrib}(L) \) with space \( |\text{Midrib}(C)| = r_C \).

Observe that the space of the \((n-2)\)-skeleton of \( L_{\mid \partial L} \) is exactly the rim \( \partial L \) and that \( L_{\mid \partial L} \) has \( n \) vertices. Hence \( \text{Midrib}(L) \) gives the \((n-1)\)-cell \( r_L \) the structure of an \((n-1)\)-complex, which is combinatorially simpler than that of \( T \cap T' \).

Figure 12. A 2-dimensional clover complex, and its midrib complex (in green).

Definition 4.4. Let \( \text{St}(v_0) \) be a star, \( f: |\text{St}(v_0)| \to S^n \) be an Alexander map, and \( \text{St}(v_0; w_n) \) be a reduced star. Let \( C \) be a clover. We say \( C \) is a clover corresponding to the reduced star \( \text{St}(v_0; w_n) \) if \( \text{Midrib}(C) = \text{St}(v_0; w_n) \), and we write, in this case, \( C = \text{Clover}(v_0; w_n) \).

Also, in this case, we call the (topologically) unique Alexander map \( f^{cl}: \text{Clover}(v_0; w_n) \to S^n \) for which \( (f^{cl})^{-1}(w_0) \) is the set of centers of the leaves in \( \text{Clover}(v_0; w_n) \), \( (f^{cl})^{-1}(w_n) = \{v_0\} \), and \( (f^{cl})^{-1}(w_i) = f^{-1}(w_i) \) for \( i = 1, \ldots, n-1 \), the clover map corresponding to \( f \).

Remark 4.5. Note that the restrictions of \( f^{cl} \) to the leaves of \( \text{Clover}(v_0; w_n) \) are essentially disjoint free simple covers as in Definition 3.9.

4.1.3. Flow from a star to a clover – deformation of the space. We begin the deformation of an Alexander map \( f: |\text{St}(v_0)| \to S^n \) by modifying the simplicial structure for all pairs in \( T(v_0; w_n) \) simultaneously. We first define, for each \( n \)-simplex \( T \) in \( \text{St}(v_0) \) and each \( s \in [0, 1] \), an \( n \)-cell \( T(s) \subset T \) satisfying \( T(s) \supset T(s') \) for \( 0 \leq s < s' \leq 1 \).
We introduce now some auxiliary notations. For a given \( w_n \)-avoiding simplex \( \tau = [v_0, x_1, \ldots, x_{n-1}] \) in \( \text{St}(v_0) \), we denote
\[
y_{\tau} = (v_0 + x_1 + \cdots + x_{n-1})/n
\]
the barycenter of \( \tau \). For \( s \in [0, 1] \), let \( y_{\tau}(s) = (1-s)v_0 + sy_{\tau} \in \tau \). Also, let \( h: f^{-1}(w_n) \times [0, 1] \to \text{St}(v_0) \) be the map \((x, s) \mapsto (1-s)x + sv_0\).

Let \( T \) be an \( n \)-simplex in \( \text{St}(v_0) \), \( x_T \) be the vertex of \( T \) for which \( f(x_T) = w_n \), and \( \tau = [v_0, x_1, \ldots, x_{n-1}] \) be the \( w_n \)-avoiding face in \( T \). We set, for each \( s \in [0, 1] \),
\[
\tau_T(s) = [y_{\tau}(s), x_1, \ldots, x_{n-1}] \subset \tau
\]
and, for each \( j \in \{1, \ldots, n-1\} \),
\[
\sigma_{T,j}(s) = [v_0, x_1, \ldots, x_{j-1}, y_{\tau}(s), x_{j+1}, \ldots, x_{n-1}]
\]
\[
\cup [v_0, x_1, \ldots, x_{j-1}, h(x_T, s), x_{j+1}, \ldots, x_{n-1}].
\]
Note that, for each \( j \in \{1, \ldots, n-1\} \), \( \sigma_{T,j}(s) \) is an \((n-1)\)-cell, and the set
\[
\beta_T(s) = \tau_T(s) \cup \bigcup_{j=1}^{n-1} \sigma_{T,j}(s)
\]
is also an \((n-1)\)-cell.

Finally, we choose, for each \( s \in [0, 1] \), an \((n-1)\)-cell \( \sigma_T(s) \) contained in \( T \) so that the boundary of \( \sigma_T(s) \) coincides with the boundary of \( \beta_T(s) \), and the union
\[
\eta(s) = \tau_T(s) \cup \bigcup_{j=1}^{n-1} \sigma_{T,j}(s) \cup \sigma_T(s)
\]
is the boundary of an \( n \)-cell contained in \( T \). Denote by \( T(s) \) the \( n \)-cell having boundary \( \eta(s) \). We require, in addition, the choices be made so that the \( n \)-cells \( \{T(s): 0 \leq s \leq 1\} \) are monotone in the following sense:

(M1) \( T(s) \supset T(s') \) for \( 0 \leq s < s' \leq 1 \),

(M2) \( T(s) = \text{cl}(\bigcup_{s<s' \leq 1} T(s')) \) for each \( s \in [0, 1) \), and
We may take each one of the simplices $s_j$ to vertices $x$ and $y$ such that $s_j - 1$, and with $z = (1 - s) x + s z$.

We may take $\sigma_T(s)$ to be the graph of the PL map over $\tau$, which is affine on each one of the simplices $T_j(s)$, and $[v_0, x_1, \ldots, x_{j-1}, y_T(s), x_{j+1}, \ldots, x_n]$, $j = 1, \ldots, n - 1$, and which maps points $x_1, x_2, \ldots, x_{n-1}, v_0, y_T(s)$ to points $x_1, x_2, \ldots, x_{n-1}, h(x_T, s)$ and $(y_T + z(s))/2$, respectively.

For each $s \in [0, 1]$, the $n$-cell $T(s)$ has the structure of an $n$-simplex with vertices $y_T(s)$, $x_1, \ldots, x_{n-2}$, and $h(x_T, s)$; see Figure 13.

For $s \in (0, 1)$, the simplicial structures of the individual $n$-simplices in $\{T(s) : T \in St(v_0)^{(n)}\}$ induce a CW structure on $\bigcup T(s)$, which is not weakly simplicial for any $s \in (0, 1)$.

On the other hand, for all $s \in [0, 1]$, the individual structures on neighboring $n$-cells in $\bigcup_{T \in St(v_0)} T(s)$ are compatible in the following sense:

(C1) If $T = [v_0, x_1, \ldots, x_n]$ and $T' \in St(v_0)$ is an $n$-simplex which shares an $w_n$-avoiding $(n - 1)$-simplex $\tau$ with $T$, then

(a) $\tau_T(s) = \tau_{T'}(s)$, and

(b) $\sigma_{T,j}(s) \cap \tau = \sigma_{T',j}(s) \cap \tau$ for $j = 1, \ldots, n - 1$.

(C2) If $T = [v_0, x_1, \ldots, x_n]$ and $T'' = [v_0, x_1, \ldots, x_{\ell-1}, x'_\ell, x_{\ell+1}, \ldots, x_n]$ are $n$-simplices in $St(v_0)$ sharing the face $[v_0, x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n]$, then

$\sigma_{T,\ell}(s) \cap \partial T'' = \sigma_{T'',\ell} \cap \partial T$.

For each pair $(T, T') \in S(v_0; w_n)$ and $s = 1$, the simplicial structures of $T(1)$ and of $T'(1)$, together with the compatibility relations, induce a weakly simplicial structure of their union $L(T, T') = T(1) \cup T'(1)$, making it an $n$-leaf having the barycenter $y_T$ of $\tau = T \cap T'$ as its center. The structures on the leaves ascend to a weakly simplicial structure on their union

$$ \bigcup_{(T, T') \in S(v_0; w_n)} L(T, T') $$

making it a clover complex $C$ with leaves $L(T, T') = S(v_0; w_n)$. Moreover, the midrib $Midrib(C)$ of this clover complex is the reduced star $St(v_0; w_n)$. Hence $C$ is the clover complex $Clover(v_0; w_n)$ corresponding to the reduced star $St(v_0; w_n)$.

The compatibility relations also induce for each $n$-simplex $T$ in $St(v_0)$ and each $s \in [0, 1]$, a simplicial homeomorphism $\varphi_{T,s} : T(s) \to T$

having the following properties:

(a) $\varphi_{T,s}(y_T(s)) = v_0$, $\varphi_{T,s}(h(x_n, s)) = x_n$, and $\varphi_{T,s}(x_j) = x_j$ for each $j = 1, \ldots, n - 1$,

(b) $f \circ \varphi_{T,s}|_\tau = f \circ \varphi_{T',s}|_\tau$ if $T$ and $T'$ is a simple pair sharing $\tau$,

(c) $f \circ \varphi_{T,s}|_T = f \circ \varphi_{T'',s}|_T$ if $T'' \neq T'$ shares a face with $T$.

Thus, there is a well-defined Alexander map

$$ g : |Clover(v_0; w_n)| \to S^n $$

which satisfies

$$ g|_{T(1)} = f \circ \varphi_{T,1} \quad \text{and} \quad g|_{T'(1)} = f \circ \varphi_{T',1} $$
on the leaf $L_{T,T'} = T(1) \cup T'(1)$ for each $\{T,T'\} \in \mathcal{S}(v_0;w_n)$, and for which $g^{-1}(w_0)$ is the set of centers of the leaves in Clover($v_0;w_n$), $g^{-1}(w_n) = \{v_0\}$, and $g^{-1}(w_i) = f^{-1}(w_i)$ for $i = 1,\ldots,n-1$. In view of Definition 4.4 this map $g$ is the clover map $f^\text{cl}$ corresponding to the Alexander map $f : St(v_0) \to S^n$.

Heuristically, the flow $s \mapsto T(s)$ of $n$-simplices, with $T(0) = T$ in $St(v_0)$ transforms the star $St(v_0)$ to a weakly simplicial Clover($v_0;w_n$) through CW$\Delta$-complexes, and it deforms the Alexander map $f : |St(v_0)| \to S^n$ to the clover map $f^\text{cl} : |Clover(v_0;w_n)| \to S^n$. We record first the transformation of the complexes in Lemma 4.6 and next the deformation of maps in Theorem 4.7.

**Lemma 4.6.** Let Clover($v_0;w_n$) be the clover corresponding to the reduced star $St(v_0;w_n)$ of a star $St(v_0)$. Then there exists a collection

$$\mathcal{F} = \bigcup_{T \in St(v_0)} \{s \mapsto T(s) : 0 \leq s \leq 1\}$$

of maps from simplices in $St(v_0)$ to simplices in Clover($v_0;w_n$) which satisfies the following properties:

1. for each $s \in [0,1]$, $T(s)$ is an $n$-simplex for which $T(0)$ belongs to $St(v_0)$, and $T(1)$ belongs to Clover($v_0;w_n$);
2. for each $T \in St(v_0)$, the simplices $\{T(s) : 0 \leq s \leq 1\}$ satisfy the monotonicity conditions (M1), (M2), and (M3);
3. for all neighboring pairs in $\mathcal{F}(v_0;w_n)$, the compatibility relations (C1) and (C2) are satisfied.

We call the collection $\mathcal{F}$ of maps in Lemma 4.6 a flow from the star $St(v_0)$ to the clover Clover($v_0;w_n$).

**4.1.4. Deformation from $f|_{St(v_0)}$ to $f^\text{cl}|_{Clover(v_0;w_n)}$.** We are now ready to summarize the deformation from an Alexander map $f : |St(v_0)| \to S^n$ to the clover map $f^\text{cl} : |Clover(v_0;w_n)| \to S^n$.

The heuristic content of the following result is that there exists an $(n+1)$-dimensional space $\Omega(v_0) \subset St(v_0) \times [0,1]$, which has copies of $|St(v_0)|$ and $|Clover(v_0)|$ on its boundary, and which admits a branched covering map $F : \Omega(v_0) \to S^n \times [0,1]$ extending $f$ and $f^\text{cl}$.

**Theorem 4.7.** Consider a simplicial complex consisting of a single star $St(v_0)$. Let $f : |St(v_0)| \to S^n$ be an interior simplicial branched covering map, and let $f^\text{cl} : |Clover(v_0;w_n)| \to S^n$ be a clover map corresponding to $f$. Then there exist an open $(n + 1)$-cell in $St(v_0) \times [0,1]$ whose closure $\Omega(v_0)$ has the following properties:

1. $\Omega(v_0) \cap (\{St(v_0)\} \times \{0\}) = \{St(v_0)\} \times \{0\}$,
2. $\Omega(v_0) \cap (\{St(v_0)\} \times \{1\}) = |Clover(v_0;w_n)| \times \{1\}$, and
3. $\Omega(v_0) \cap (\{St(v_0)\} \times \{s\})$ is an $n$-cell for each $s \in (0,1)$,

and also an interior branched covering map $F : \Omega(v_0) \to S^n \times [0,1]$ which satisfies

4. $F$ is level preserving in the sense that $F^{-1}(S^n \times \{s\}) = \Omega(v_0) \cap (\{St(v_0)\} \times \{s\})$ for each $s \in [0,1]$,
5. $F|_{St(v_0) \times \{0\}} = f$, and
Figure 14. Deformation of a 3-dimensional Alexander map on a star, to one on a clover, then to simple covers. On the domain side of the first three maps, 10 of the 12 outer faces (in red) are removed for viewing.
Proof. Let \( U(s) = \bigcup_T T(s) \), where \( T(s) \) is the set defined in Section 4.1.3 and the union is taken over all \( n \)-simplices in \( \text{St}(v_0) \). We define
\[
\Omega(v_0) = \bigcup_{s \in [0,1]} U(s) \times \{s\}
\]
and set
\[
F : \Omega(v_0) \to S^n \times [0,1], \quad (x, s) \mapsto (f \circ \varphi_{T,s}(x), s).
\]
Since \( U(s) \) is a closed \( n \)-cell for \( s \in [0,1) \), we conclude that \( \Omega(v_0) \) is a closure of an open \((n + 1)\)-cell. An argument similar to that of Lemma 2.1 shows that \( F \) is an interior branched covering map. Since \( F|_{\text{St}(v_0) \times \{0\}} = f \) and \( F|_{\text{Clover}(v_0,w_n) \times \{1\}} = f^{\text{cl}} \), the claim follows. \( \square \)

4.1.5. Comments on the local deformation in weakly simplicial stars. Let \( \text{St}(v_0) \) be a weakly simplicial star, but not simplicial, which supports an Alexander map \( f : |\text{St}(v_0)| \to S^2 \). The \( n \)-simplices in \( \text{St}(v_0) \) may still be paired into simple pairs sharing a common \( w_n \)-avoiding \((n - 1)\)-simplex in \( \text{St}(v_0) \). We also have that, for any simple pair \( \{T, T'\} \) in \( \text{St}(v_0) \), the restriction \( f|_{\text{int}(T \cup T')} \) is an embedding.

In two dimensions, Rickman’s deformation of map complexes \([48, \text{Section 5}]\) gives a local deformation for Alexander maps \( |\text{St}(v_0)| \to S^2 \) in such weakly simplicial stars. In our terms, the special property of these stars is the following.

Let \( \{T_1, T_2\} \) be a pair of adjacent \( 2 \)-simplices in a weakly simplicial star \( \text{St}(v_0) \) for which \( |\text{St}(v_0)| \neq |T_1 \cup T_2| \). Then \( |T_1 \cap T_2| \) is either a common face, a pair of common faces, or a union of a common face and all vertices in \( \text{St}(v_0) \). In the second case, the pair \( \{T_1, T_2\} \) is a leaf. In the third case, Rickman shows, that there exists another pair \( \{T'_1, T'_2\} \) in \( \text{St}(v_0) \) which is a leaf. Clearly, this property does not hold in higher dimensions; consider for example a pair of \( 3 \)-simplices having two faces in common.

4.2. Local deformation in a simplicial complex. The goal of the deformation is to simplify the combinatorics of the Alexander map and the underlying complex.

In this section we study how an Alexander map \( f \) evolves in an ambient complex when the deformation of the map is performed locally in a star. We examine the global evolution following the local transformation in two steps. In the first step, map \( f \) on a complex is deformed in a star to a map \( \tilde{f} \) on a new complex with the star replaced by a clover (Theorem 4.7 and Proposition 4.8). Heuristically, the second step, the deformed \( \tilde{f} \) is moved by an isotopy to a map \( \hat{f} \) (Theorem 3.10): the moving collapses map \( \hat{f} \) in the clover to its midrib and creates new free simple covers at assigned locations. The combined process is summarized in Theorem 4.17.

Since there is a freedom in choosing the new locations for the simple covers, the deformation simplifies the combinatorics of the Alexander map and the underlying complex.
4.2.1. Local deformation of simplicial maps. We begin by giving a version of Theorem 4.7 for an embedded star in the ambient complex.

**Proposition 4.8.** Let $M$ be an $n$-manifold (possibly with non-empty boundary), $P$ a weakly simplicial complex having $M$ as its space, and $f : M \to \mathbb{S}^n$ be a $P$-Alexander map. Suppose that $\text{St}(v_0)$ is a simplicial star in $P$ satisfying $|\text{St}(v_0)| \cap \partial M \subset |\text{St}(v_0; w_n)|$, and that $f(v_0) = w_0$. Let $	ext{Clover}(v_0; w_n)$ be the clover complex corresponding to the reduced star $\text{St}(v_0; w_n)$, and let $f^\text{cl} : |\text{Clover}(v_0; w_n)| \to \mathbb{S}^n$ be the clover map corresponding to the map $f|_{\text{St}(v_0)}$.

Then there exist a weakly simplicial complex $R$ on $M$ which contains $\text{Clover}(v_0; w_n) \cup (\text{Clover}(v_0; w_n))^\circ$ as a subcomplex, and an $R$-Alexander map $f_1 : M \to \mathbb{S}^n$ satisfying

$$f_1|_{\text{Clover}(v_0; w_n)} = f^\text{cl} \quad \text{and} \quad f_1|_{\text{St}(v_0; w_n)} = f|_{\text{St}(v_0; w_n)}.$$

Furthermore, there is a level preserving branched covering map

$$F : M \times [0, 1] \to \mathbb{S}^n \times [0, 1]$$

for which $F_0 = f$, $F_1 = f_1$, and $F_s|_{\partial M} = f|_{\partial M}$ for each $s \in [0, 1]$.

**Remark 4.9.** In the Proposition, the branched covering maps $F_s : M \to \mathbb{S}^n$ are not Alexander maps for $0 < s < 1$. See Figure 15 for an example of complex $R$.

![Figure 15](https://example.com/figure15.png)

**Figure 15.** The underlying complex $P$ corresponding to an Alexander maps $f$ and the underlying complex $R$ corresponding to the deformed Alexander map $f^\text{cl}$, as in Proposition 4.8.

**Proof of Proposition 4.8.** Recall first that the midrib complex $\text{Midrib}(v_0; w_n)$ of the clover $\text{Clover}(v_0; w_n)$ is the reduced star $\text{St}(v_0; w_n)$. Recall also, from Section 4.1.3 that for each $n$-simplex $T$ in $\text{St}(v_0)$, the deformed $T(1)$ is an $n$-cell in $\text{Clover}(v_0; w_n)$. Let $\psi_T : T \to T(1)$ be the inverse of the homeomorphism $\varphi_{T,1} : T(1) \to T$. By the compatibility of homeomorphisms $\psi_T$, there exists a well-defined map $\psi : \partial|\text{St}(v_0)| \to \partial|\text{Clover}(v_0; w_n)|$ which, for each $T$, $\psi$ coincides with $\psi_T$ on $|T| \cap \partial|\text{St}(v_0)|$.

Since $\text{St}(v_0)$ is simplicial, we may fix a collar $W_S$ of $\partial|\text{St}(v_0)|$ in $M \setminus \text{int}|\text{St}(v_0)|$, and take $W_C$ to be $(W_S \cup |\text{St}(v_0)|) \setminus \text{int}|\text{Clover}(v_0; w_n)|$. We extend $\psi|_{\partial|\text{St}(v_0)|}$ to a map $\Psi : W_S \to W_C$ for which

$$\Psi|_{W_S \setminus \partial|\text{St}(v_0)|} : W_S \setminus \partial|\text{St}(v_0)| \to W_C$$
is a homeomorphism, and \( \Psi|_{\partial W} = \int_{\partial W} \partial (\text{St}(v_0)) \to M \times \text{St}(v_0) \times \{0, 1\} \) is a homeomorphism, and \( \Psi = \int_{\partial W} \partial (\text{St}(v_0)) \). Let now \( \Omega(v_0) \) be the closure of the open \((n + 1)\)-cell in \( |\text{St}(v_0)| \times [0, 1] \) and \( F:\Omega(v_0) \to S^n \times [0, 1] \) be the branched covering map fixed in Theorem 4.7.

To simplify notations, we write \( M_s = M \times \{s\} \) and \( \Omega_s(v_0) = \Omega(v_0) \cap (M \times \{s\}) \) for each \( s \in [0, 1] \). Write also \( D = W_s \cup |\text{St}(v_0)| \) and \( \hat{D} = (D \times [0, 1]) \setminus \int \Omega(v_0) \), where the interior \( \int \Omega(v_0) \) of \( \Omega(v_0) \) is taken with respect to \( M \times [0, 1] \). Then \( \hat{D} \cap M_0 = (W_s \cup |\text{St}(v_0)|) \times \{0\} \), and \( \hat{D} \cap M_1 = (W_s \cup |\text{St}(v_0)|) \times \{1\} \). Moreover, for each \( s \in (0, 1) \), \( \hat{D} \cap M_s = D \times \{s\} \) is a collar of \( \partial \Omega(v_0) \cap M_s \) in \( M_s \cap \Omega(v_0) \).

Thanks to the product structure of collars, \( F \) may be extended over \( \hat{D} \) to a level preserving branched covering map \( F:\hat{D} \to S^n \times [0, 1] \) for which \( F_1|_{\partial \hat{D}} = f|_{\partial D} \) for each \( s \in [0, 1] \) and \( F_1 \circ \Psi = f|_{\partial W} \circ |\text{St}(v_0)| \). Finally we extend \( F \) to a level preserving branched covering map \( F:M \times [0, 1] \to S^n \times [0, 1] \) by setting \( F_1|_{\partial \Omega(v_0)} = f \).

Denote now by \( R \) the unique weakly simplicial complex for which \( F_1 \) is an \( R \)-Alexander map. Since \( F_1|_{\text{Clover}(v_0,w_n) \times \{1\}} = f_{\text{cl}} \), the claim holds by taking \( f_1 = F_1 \).

4.2.2. Simplicial collapsing. The goal of deformation is to simplify the complexity of the branched covering maps and the underlying complexes. To keep track of the progress, we introduce the notions of collapse of maps and reduction of complexes.

**Definition 4.10.** Let \( P \) and \( Q \) be weakly simplicial complexes, and let \( P' \) be a subcomplex of \( P \) and \( Q' \) a subcomplex of both \( P' \) and \( Q \). A simplicial map \( \tilde{\varphi}:(P,P') \to (Q,Q') \) is a simplicial collapsing map, if

1. \( \tilde{\varphi}(P') = Q' \),
2. \( \tilde{\varphi}|_Q = \text{id} \), and
3. \( \tilde{\varphi}|_{\text{cl}_P(P \setminus P')} : \text{cl}_P(P \setminus P') \to \text{cl}_Q(Q \setminus Q') \) is an isomorphism.

Write, in this case, \( \varphi:(P,P') \sim (Q,Q') \) for the collapse map, and say that \( Q \) is a simplicial collapse of \( P \) through the map \( \varphi:(P,P') \sim (Q,Q') \).

**Remark 4.11.** This terminology stems from the our applications, where \( |P| = |Q| \) and part of the complex \( P' \) collapses to a lower dimensional subcomplex; see Figure 4.17 for an illustrative example and forthcoming Theorem 4.17 for a concrete statement.

**Definition 4.12.** A map \( \varphi:|P| \to |Q| \) is a realization of a simplicial collapsing map \( \tilde{\varphi}:(P,P') \sim (Q,Q') \) if \( \varphi(|P'|) = |Q'| \) and, for each cell \( \sigma \) in \( P \setminus P' \), \( \varphi(\sigma) = \tilde{\varphi}(\sigma) \) and \( \varphi|_{\sigma} \) is an embedding.

**Remark 4.13.** By Definition 4.10(3), the restriction \( \varphi|_{P \setminus |P'|} : |P| \setminus |P'| \to |Q| \setminus |Q'| \) of a realization \( \varphi:|P| \to |Q| \) of a simplicial collapsing map \( \varphi:(P,P') \sim (Q,Q') \), is a homeomorphism.

We rephrase now Lemma 4.6 in terms of collapse maps.

**Corollary 4.14.** Let \( M \) be an \( n \)-manifold (possibly with non-empty boundary \( \partial M \)), and let \( P \) be a weakly simplicial complex having space \( M \) and
A simplicial collapse map $\bar{\varphi}: (P, P') \to (Q, Q')$. Complex $Q'$ in red and complex $P' \supset Q'$ in blue (and red).

containing a clover $C$ with node $v_0$ so that $|C| \cap \partial M \subset |\text{Midrib}(C)|$ when $\partial M \neq \emptyset$. Then there exists a weakly simplicial complex $Q$ which has $M$ as a space and contains $\text{Midrib}(C) \cup C^c$ as a subcomplex, and for which there exists a simplicial collapse map

$$\bar{\varphi}: (P, C \cup C^c) \sim (Q, \text{Midrib}(C) \cup C^c).$$

In addition, there exists a realization $\varphi: M \to M$ of $\bar{\varphi}$ with the following property: Let $f: M \to S^n$ be a $P$-Alexander map. Then there exists a $Q$-Alexander map $\hat{f}: M \to S^n$ satisfying $f = \hat{f} \circ \varphi$ and $f|\partial M = \hat{f}|\partial M$ if $\partial M \neq \emptyset$.

**Remark 4.15.** The $Q$-Alexander map $\hat{f}$ in Corollary 4.14 may be viewed as the collapse of $f$ associated to $\bar{\varphi}$. Conversely, the $P$-Alexander map $f$ may be regarded as expansion of $\hat{f}$ by essentially disjoint free simple covers.

**4.2.3. Deforming-collapsing-expanding.** We sum up the deforming-collapsing-expanding process as follows. For the statement we need the following definition.

**Definition 4.16.** A subcomplex $S \subset P$ is a reduced star at a vertex $v$ of a simplicial complex $P$ if there exists an Alexander map $g: |\text{St}(v)| \to S^n$ satisfying $g(v) \neq w_n$ and $S = \text{St}(v; w_n)$.

**Theorem 4.17.** Let $M$ be an $n$-manifold (possibly with non-empty boundary $\partial M$), $P$ a weakly simplicial complex having space $M$, and let $v_0 \in P$ be a vertex in the interior of $M$ and $S \subset \text{St}(v_0)$ a reduced star at $v_0$ for which $|\text{St}(v_0)| \cap \partial M \subset S$ if $\partial M \neq \emptyset$. 

![Figure 16](image1.png)

**Figure 16.** Deformation from a star to a clover, and then from a clover to a reduced star expanded by simple covers in an ambient complex.
Then there exist a weakly simplicial complex \( Q \) with space \( M \) which contains \( S \cup \text{St}(v_0)^c \) as a subcomplex, a simplicial collapse map

\[
\bar{\varphi} : (P, \text{St}(v_0) \cup \text{St}(v_0)^c) \sim (Q, S \cup \text{St}(v_0)^c),
\]

and a realization \( \varphi : M \to M \) of \( \bar{\varphi} \) for the following.

Suppose \( f : M \to S^n \) is a \( P \)-Alexander map. Then there exists a \( Q \)-Alexander map \( \hat{f} : M \to S^n \) satisfying \( f = \hat{f} \circ \bar{\varphi} \), and \( \hat{f}|_{\partial M} = f|_{\partial M} \) when \( \partial M \neq \emptyset \). Moreover, if \( 2m \) is the number of \( n \)-simplices in \( \text{St}(v_0) \) and \( \hat{f} : M \to S^n \) is a branched covering map obtained from \( f \) by an \( m \)-fold expansion with essentially disjoint free simple covers in \( \text{int} M \), then there exists a branched covering map

\[
F : M \times [0, 1] \to S^n \times [0, 1]
\]
satisfying \( F_0 = f, \ F_1 = \hat{f}, \ \text{and} \ F_s|_{\text{St}(v_0)^c} = f|_{\text{St}(v_0)^c} \) for each \( s \in [0, 1] \).

Proof. Since the reduced star \( S = \text{St}(v_0; w_n) \) coincides with the midrib complex \( \text{Midrib}(	ext{Clover}(v_0; w_n)) \), the simplicial collapse map in the first claim may be obtained from Proposition 4.8 and Corollaries 4.14. The second claim follows from Remark 4.15 and the isotopy theorem for free simple covers i.e. Theorem 3.10. \( \square \)

Remark 4.18. Theorem 4.17 implicitly yields that \( \bar{\varphi}|_{S \cup \text{St}(v_0)^c} = \text{id} \); in other words, the deformation occurs only in a regular neighborhood of \( S \).

Remark 4.19. If \( M \) is a closed manifold in Theorem 4.17, branched covering maps \( M \to S^n \) have non-zero degree. More precisely, we have in this case

\[
\deg f = \deg \hat{f} = \deg \hat{f} \pm m
\]
in Theorem 4.17, c.f. Remark 3.11. Thus \( \hat{f} \) and \( f \) are not homotopic even as maps \( M \to S^n \), whereas \( f \) and \( \hat{f} \) are homotopic in the strong sense of Theorem 4.17.

4.2.4. Reduction of complexes. In this short section, we shift our attention momentarily from the deformation of Alexander maps to the transformation of the underlying complexes. At the same time, the terminology changes from the collapse of maps to the reduction of complexes.

Definition 4.20. Let \( P \) be a weakly simplicial complex. A weakly simplicial complex \( Q \) on \( |P| \) is a reduction of the complex \( P \) at a vertex \( v \in P \), denoted \( P \setminus_v Q \), if \( \text{St}_P(v) \) is a simplicial complex that supports an Alexander map, a reduced star \( S = \text{St}(v; w_n) \) of \( \text{St}_P(v) \), and a simplicial collapse map \( (P, \text{St}_P(v)) \sim (Q, S) \).

In Figure 16 the complex on the right is a reduction of the complex on the left.

Convention 4.21. If there is no specific need, we usually do not mention the Alexander map associated to the reduction.

If there is no confusion, we often leave out the vertex in the notation \( P \setminus v Q \) and merely denote \( P \setminus Q \). We also denote by

\[
P_0 \setminus P_1 \setminus \ldots \setminus P_m
\]
a sequence of consecutive reductions \( P_k \setminus_{v_k} P_{k+1} \) for \( k = 0, \ldots, m - 1 \).
Having this terminology at our disposal, we may formulate the following corollary of Theorem 4.17.

**Corollary 4.22.** Let \( P = P_0 \searrow P_1 \searrow \cdots \searrow P_m \) be a sequence of consecutive reductions and \( f: |P| \to S^n \) a \( P \)-Alexander map. Then for each \( k = 1, \ldots, m \), there exist a \( P_k \)-Alexander map \( f_k: |P| \to S^n \) and an expansion \( f_k': |P| \to S^n \) of \( f_k \) with essentially disjoint free simple covers for which \( f \) and \( f_k' \) are branched cover homotopic rel \( \partial|P| \).

5. **Deformation of Alexander star pairs**

The reduction of a simplicial complex, obtained by the collapsing a simplicial star, may yield a weakly simplicial complex. To repeat the reduction procedure, we need to find a simplicial star in the reduced complex.

In this section, we study a condition – formulated using adjacent stars in a complex – which allows the merge of an adjacent star pair to a single star.

In the next section, we discuss the canonical cubical triangulation (see also [13] Section 6) and a topological property of cubical complexes which, when combined with the merging method of this section, allow the reduction of a simplicial \( n \)-cell to a single star.

5.1. **Alexander star pairs.**

**Definition 5.1.** An \( n \)-dimensional simplicial complex \( K \) is an **Alexander star** if

1. the space \( |K| \) is an \( n \)-cell,
2. there exists a vertex \( v_K \in \text{int}|K| \) for which \( St_K(v_K) = K \), and
3. there exists a \( K \)-Alexander map \( |K| \to S^n \).

Note that, since the Alexander star \( K \) is simplicial, the vertex \( v_K \) is unique.

**Definition 5.2.** A pair \( \{K_1, K_2\} \) of \( n \)-dimensional Alexander stars is an **Alexander star pair** if

1. the intersection \( |K_1| \cap |K_2| \) is an \((n-1)\)-cell,
2. \( K_1 \cap K_2 \) is a simplicial complex with space \( |K_1| \cap |K_2| \), and
3. \( K_1 \cap K_2 \) has a vertex in \( \text{int}|K_1 \cap K_2| \).

On the union of any given Alexander pair \( \{K_1, K_2\} \), there always exists a \( K_1 \cup K_2 \)-Alexander map \( f: |K_1 \cup K_2| \to S^n \).

**Lemma 5.3.** Let \( \{K_1, K_2\} \) be an Alexander star pair. Then there exists a \((K_1 \cup K_2)\)-Alexander map \( f: |K_1 \cup K_2| \to S^n \) such that \( f|_{|K_1|}: |K_1| \to S^n \) is a \( K_1 \)-Alexander map and \( f|_{|K_2|}: |K_2| \to S^n \) is a \( K_2 \)-Alexander map.

Moreover, if \( K = St_K(v_K) \) is a star contained in \( |K_1 \cup K_2| \), which satisfies \( K|_{\partial|K|} = (K_1 \cup K_2)|_{\partial|K|} \), then \( K \) is an Alexander star.

**Proof.** Let \( f_1: |K_1| \to S^n \) be a \( K_1 \)-Alexander map and \( f_2: |K_2| \to S^n \) a \( K_2 \)-Alexander map. We assume as we may that \( f_1(v_{K_1}) = f_2(v_{K_2}) = w_n \).

Let \( D \) be the \((n-1)\)-cell \( |K_1 \cap K_2| \). Then \( f_1|_D: D \to S^{n-1} \) and \( f_2|_D: D \to S^{n-1} \) are \((K_1 \cap K_2, K_{S^{n-1}})\)-simplicial maps which map \( D \) onto the unique \( w_n \)-avoiding \((n-1)\)-simplex in \( K_{S^n} \). Since \( f_1|_D \) and \( f_2|_D \) are simplicial and
nondegenerate, we obtain, by rearranging the vertices in $K_{S^n}$, a $(K_{S^n}, K_{S^n})$-simplicial homeomorphism $\varphi: S^n \to S^n$ satisfying $\varphi(w_n) = w_n$ and $f_1(v) = \varphi(f_2(v))$ for each vertex $v \in D$.

Let now $f'_2: |K_2| \to S^n$ be the (combinatorially unique) Alexander map which extends $\varphi \circ f_2|_D$. Then the map $f: |K_1 \cup K_2| \to S^n$, defined by $f|_{|K_1|} = f_1|_{|K_1|}$ and $f|_{|K_2|} = f'_2|_{|K_2|}$, is a well-defined $(K_1 \cup K_2)$-Alexander map. This proves the first claim.

As for the second claim, let $f: |K| \to S^n$ to be the $(K_1 \cup K_2)$-Alexander map above. Then $f(v_{K_1}) = f(v_{K_2})$. Since $K$ is a star, we have that either $v_K \in \{v_{K_1}, v_{K_2}\}$ or $v_K \in K_1 \cap K_2$. In both cases, $g = f|_{|K|} : |K| \to S^n$ is an Alexander map.

Having this lemma at our disposal, we may introduce the notion of merging; see Figure 18 for an illustration.

**Definition 5.4.** An Alexander star $K$ is called a **merge of the Alexander star pair** $\{K_1, K_2\}$ if $|K| = |K_1 \cup K_2|$ and $K|_{\partial|K|} = (K_1 \cup K_2)|_{\partial|K|}$.

![Figure 18. Merge of an Alexander star pair.](image)

**5.2. Merge of a simple Alexander star pair.** We now discuss a condition on an Alexander pair which allows simple merging; see Figure 19.

**Definition 5.5.** An Alexander star pair $\{K_1, K_2\}$ is a **simple Alexander star pair** if $K_1 \cap K_2$ is the star $\text{St}_K(v_{K_1 \cap K_2})$ for a vertex $v_{K_1 \cap K_2} \in \text{int}|K_1 \cap K_2|$

![Figure 19. A simple Alexander star pair and its merge.](image)

**Lemma 5.6.** Let $\{K_1, K_2\}$ be a simple Alexander star pair and $K$ a merge of $\{K_1, K_2\}$. Then the reduction $K_1 \cup K_2 \setminus v_{K_1 \cap K_2}$ $K$ may be realized through the simplicial collapsing $(K_1 \cup K_2, \text{St}_{K_1 \cup K_2}(v_{K_1 \cap K_2})) \sim (K, \text{St}_K(v_{K_1 \cap K_2}))$

Furthermore, let $f: |K| \to S^n$ be a $(K_1 \cup K_2)$-Alexander map, $g: |K| \to S^n$ a $K$-Alexander map with $g|_{\partial|K|} = f|_{\partial|K|}$, and $\hat{g}: |K| \to S^n$ be an $m$-fold expansion of $g$ with essentially disjoint simple covers, where

$$m = \left(\#(K_1 \cup K_2)^{(n)} - \#K^{(n)}\right)/2.$$
Then $f$ and $\tilde{g}$ are branched cover homotopic rel $\partial |K|$.

**Proof.** Let $\{K_1, K_2\}$ be a simple Alexander star, and $f : |K_1 \cup K_2| \to \mathbb{S}^n$ be an Alexander map. Let $v_{K_1}$ and $v_{K_2}$ be the only vertices of $K_1 \cup K_2$ in $\text{int}|K_1|$ and $\text{int}|K_2|$, respectively. We may assume $f(v_{K_1}) = f(v_{K_2}) = w_n$. Then the complex $\text{St}_{K_1 \cup K_2}(v_{K_1} \cap K_2)$ contains both $v_{K_1}$ and $v_{K_2}$ as vertices, and has $\text{St}_{K_1 \cup K_2}(v_{K_1} \cap K_2, w_n) = K_1 \cap K_2$ as a reduced star. In view of Theorem 4.17, the pair $\{K_1, K_2\}$ may be merged to an Alexander star $K$ through a reduction $(K_1 \cup K_2) \diagdown_{w_{K_1} \cap K_2} K$.

The second claim in the lemma follows from the deformation of the maps (Theorem 4.7, Theorem 4.17) that leads to the reduction of the complexes. Note also that $m = \#(K_1 \cap K_2)^{(n-1)}$. \hfill $\square$

### 5.3. Reduction towards a simple Alexander pair.

In this section, we give the first step towards reducing a general Alexander star pair to a simple Alexander star pair; see Figure 20 for an illustration.

**Lemma 5.7.** Let $\{K_1, K_2\}$ be an Alexander star pair, and $v$ be a vertex in $K_1 \cap K_2$ contained in $\text{int}|K_1 \cap K_2|$. Suppose that a simplicial complex $Q$ is a reduction of $K_1 \cap K_2$ at $v$. Then there exists a reduction complex $P$ of $K_1 \cup K_2$ at $v$ satisfying $P|_{K_1 \cap K_2} = Q$. Moreover, $\{P|_{K_1}, P|_{K_2}\}$ is an Alexander star pair.

![Figure 20](image.png)

**Figure 20.** An Alexander star pair $\{K_1, K_2\}$, and a simple Alexander star pair after reduction at $v$, as in Lemma 5.7.

**Proof.** Since $Q$ is a reduction of $K_1 \cap K_2$ at a vertex $v \in K_1 \cap K_2$, there exist an Alexander map $g : |K_1 \cap K_2| \to \mathbb{S}^{n-1}$ with $g(v) = w_0$ and a simplicial collapsing $(K_1 \cap K_2, \text{St}_{K_1 \cap K_2}(v)) \sim (Q, \text{St}_{K_1 \cap K_2}(v; w_{n-1}))$.

Let $v_{K_1}$ and $v_{K_2}$ be the only vertices of $K_1 \cup K_2$ in $\text{int}|K_1|$ and $\text{int}|K_2|$, respectively. Since an $n$-simplex $\sigma$ in $\text{St}_{K_1 \cup K_2}(v)$ is a cone of an $(n-1)$-simplex $\sigma$ in $\text{St}_{K_1 \cap K_2}(v)$ with either $v_{K_1}$ or $v_{K_2}$, in view of Lemma 5.3, there exists an Alexander map $f : |K_1 \cup K_2| \to \mathbb{S}^n$ with $f^{-1}(w_i) = g^{-1}(w_i)$ for $i = 0, \ldots, n-1$ and $f^{-1}(w_n) = \{v_{K_1}, v_{K_2}\}$.

It suffices now to observe that the reduced star $\text{St}_{K_1 \cup K_2}(v; w_{n-1})$ is the join $\{v_{K_1}, v_{K_2}\} \ast \text{St}_{K_1 \cap K_2}(v; w_{n-1})$. Thus, the collapse map

$$(K_1 \cap K_2, \text{St}_{K_1 \cap K_2}(v)) \sim (Q, \text{St}_{K_1 \cap K_2}(v; w_{n-1}))$$

associated to the reduction $K_1 \cap K_2 \diagdown_{v} Q$ yields a collapse map

$$(K_1 \cup K_2, \text{St}_{K_1 \cup K_2}(v)) \sim (P, \text{St}_{K_1 \cup K_2}(v; w_{n-1})).$$
Thus \( K_1 \cup K_2 \setminus \_{v_0} P \).

By repeating the process in Lemma 5.7, we obtain a simple Alexander star pair, assuming that the intersection of the given pair may be reduced to a star.

**Corollary 5.8.** Let \( K \) be a weakly simplicial complex on an \( n \)-cell. Let \( K_1 \) and \( K_2 \) be subcomplexes of \( K \) such that \( |K_1 \cup K_2| = |K| \) and \( \{K_1,K_2\} \) form an Alexander star pair. Suppose \( K_1 \cap K_2 \) may be reduced, through a reduction sequence

\[
K_1 \cap K_2 = P_0 \setminus v_1 P_1 \setminus v_2 \cdots \setminus v_m P_m
\]
to an Alexander star \( P_m \). Then \( K_1 \cup K_2 \) may be reduced, through a reduction sequence

\[
K_1 \cup K_2 = \tilde{P}_0 \setminus v_1 \tilde{P}_1 \setminus v_2 \cdots \setminus v_m \tilde{P}_m = Q
\]
to a simplicial complex \( Q \) having space \( |Q| = |K| \) and the property that \( \{Q|_{K_1},Q|_{K_2}\} \) is a simple Alexander star pair with \( Q|_{K_1 \cap K_2} = \tilde{P}_m \).

### 6. Deformation of Alexander maps in cubical complexes

In this section, we prove a Cubical reduction Theorem and a Cubical Deformation Theorem for shellable cubical complexes discussed in the introduction. The main results are formulated as Theorem 6.12 and Theorem 6.13.

To start, we show that the canonical triangulation of a shellable cubical complex may be reduced to its star-replacement. This is obtained by repeated merges of Alexander star pairs using Corollary 5.8.

#### 6.1. Cubical complexes

We begin by reviewing the notions of cubical complexes and cubical Alexander maps studied in [12].

We say that a \( k \)-cell \( \sigma \) in a CW-complex \( K \) is a \( k \)-cube if \( K|_{\sigma} \) is isomorphic to the standard CW-structure on the unit cube \( [0,1]^k \) having vertices \( \{0,1\}^k \). We say that \( K \) is a CW-\( \square \)-complex if each cell is a cube. We call cells in \( \text{CW}_\square \) simply as cubes.

An \( n \)-dimensional \( \text{CW}_\square \)-complex \( K \) is a cubical complex if for all adjacent \( n \)-cubes, \( \sigma' \) and \( \sigma '' \) the intersection \( \sigma \cap \sigma '' \) is an \((n-1)\)-cube in \( K \). This is analogous to simplicial complexes, and similarly, as for simplicial complexes, cubes are uniquely determined by their vertices. We tacitly extend the terminology introduced for \( \text{CW}_\Delta \)-complexes, also for \( \text{CW}_\square \)-complexes.

### 6.1.1. Canonical triangulation of a \( \text{CW}_\square \)-complex

A cubical complex \( K \) admits a subdivision to a simplicial complex \( K^\Delta \) as follows. Let \( \mathcal{K}_{\text{ord}} \) be the collection of sequences \( (q_0,\ldots,q_k) \) for \( k \geq 0 \), where each \( q_i \) is a cube of dimension \( i \) in \( K \) and the sequence satisfies the inclusion relation \( q_0 \subset q_1 \subset \cdots \subset q_k \). Let \( \mathcal{K} \) be the collection of all subsets \( \{q_0,\ldots,q_k\} \) to the sequences \((q_0,\ldots,q_k) \in \mathcal{K}_{\text{ord}} \) together with \( \{\emptyset\} \). Then \( \mathcal{K} \) is an abstract simplicial complex. For each \( Q \in K \), let \( \mathcal{K}|_Q \) be the subcollection of all the subsets \( \{q_0,\ldots,q_k\} \) in \( \mathcal{K} \) for which \( q_k = Q \).

There exists now a simplicial complex \( K^\Delta \) so that, for each \( Q \in K \), the subcomplex \( K^\Delta|_Q \) is a triangulation of \( Q \) isomorphic to \( \mathcal{K}|_Q \). Indeed, we may associate to each \( \{q_0,\ldots,q_k\} \) in \( \mathcal{K} \) the \( k \)-simplex \( [v_{q_0},\ldots,v_{q_k}] \), where \( v_{q_i} \) is an \( i \)-cube \( q_i \).
$v_q$ is a (fixed) vertex in the interior of cube $q_i$. We call the complex $K^\Delta$ the canonical triangulation of the cubical complex $K$; see Figure 21 for a simple example.

The complex $K^\Delta$ is canonical in the following sense: Let $K'$ be a cubical subcomplex of $K$. Then $(K')^\Delta$ is isomorphic to $K^\Delta|_{K'}$.

![Figure 21. A cubical complex $K$ and its triangulation $K^\Delta$.](image)

We say an $n$-dimensional cubical complex $K'$ refines an $n$-dimensional cubical complex $K$ if $|K| = |K'|$ and every $k$-cube, for $1 \leq k \leq n$, in $K$ is the union of a collection of essentially disjoint $k$-cubes in $K'$.


Definition 6.1. Let $K$ be an $n$-dimensional cubical complex. A mapping $f: |K| \to S^n$ is called a cubical $K$-Alexander map if $f$ is a $K^\Delta$-Alexander map.

More specifically, a cubical $K$-Alexander map $f: |K| \to S^n$ is a $(K^\Delta, K_{S^n})$-simplicial branched cover, where $K_{S^n}$ is the CW $\Delta$-complex fixed in Convention 2.6.

Remark 6.2. Let $f: |K| \to S^n$ be a cubical $K$-Alexander map. By relabeling the vertices of $K_{S^n}$ if necessary, we may always assume (as we do) that $f(v) = w_k$ for each $v \in (K^\Delta)^{(0)}$ which is in the interior of a $k$-cube in $K$.

There are no obstructions for the existence of cubical Alexander maps. We formulate this observation as follows.

Proposition 6.3. Let $K$ be a cubical structure on an orientable connected manifold $M$ with boundary. Then there exists a cubical $K$-Alexander map $M \to S^n$.

Readers familiar with Alexander maps notice immediately that this theorem is essentially a result of Alexander [2] stated in our terminology. The main difference is that we consider here maps with respect to a fixed triangulation $K^\Delta$. For other results on the existence of Alexander maps with prescribed local behavior; see e.g. Rickman [48, Section 2] and Peltonen [41].

Proof of Proposition 6.3. Since we may map the vertices of $K^\Delta$ by the formula $v \mapsto w_k$ if $v$ is in the interior of an $k$-cube in $K$, it suffices to construct a parity function $\nu: (K^\Delta)^{(n)} \to \{-1, 1\}$.
Let $\omega = \sum_{\sigma \in (K^\Delta)^{(n)}} \omega_\sigma$ be a representative of a fundamental class in $H_n([K], \partial [K])$, where $\omega_\sigma$ is an oriented $n$-simplex, i.e. an $n$-simplex $\sigma$ with an choice of a permutation class of vertices.

Let now $\sigma \in (K^\Delta)^{(n)}$. Then $\sigma$ has a (unique) preferred choice for the order of its vertices induced by the canonical triangulation, namely $\sigma = [v_0, \ldots, v_n]$, where each $v_k$ is in the interior of a $k$-cube for $k > 0$. Let $\nu: (K^\Delta)^{(n)} \to \{-1,1\}$ be the function satisfying $[v_0, \ldots, v_n] = \nu(\sigma)\omega_\sigma$, as oriented simplices, for each $\sigma = [v_0, \ldots, v_n]$. 

To verify that $\nu$ is a parity function, let $\sigma$ and $\sigma'$ be adjacent $n$-simplices in $K^\Delta$. Then $\sigma = [v_0, \ldots, v_n]$ and $\sigma' = [v'_0, \ldots, v'_n]$, where $v_k$ and $v'_k$ are vertices in $K^\Delta$ which are in the interior of a $k$-cube in $k$ for $k > 0$. Since $\sigma \cap \sigma'$ is an $(n-1)$-simplex, we conclude that there exists unique $j \in \{0, \ldots, n\}$ for which $v'_j \neq v_j$ and $v_k = v'_k$ for $k \neq j$. Since $\sigma \cap \sigma' = [v_0, \ldots, \hat{v}_j, \ldots, v_n]$, exactly one of $[v_0, \ldots, v_n]$ and $[v'_0, \ldots, v'_n]$ is positively oriented with respect to $\omega_\sigma$ and $\omega_{\sigma'}$ in the fixed chain $\omega$. Thus $\nu$ is a parity function. 

6.3. Shellable cubical complexes. In this section we consider a particular class of cubical complexes of $n$-cells, which we call shellable cubical complexes, and show that all cubical complexes on a 2-cell are shellable.

**Definition 6.4.** A cubical complex $K$, having an $n$-cell as its space $|K|$, is **shellable** if there exists an order $q_1, q_2, \ldots, q_m$ of the $n$-cubes of $K$ for which $$(q_1 \cup \cdots \cup q_i) \cap q_{i+1}$$ is an $(n-1)$-cell for each $i = 1, \ldots, m - 1$.

**Remark 6.5.** Note that, in Definition 6.4, the union $q_1 \cup \cdots \cup q_i$ is an $n$-cell for each $i = 1, \ldots, m$.

We also say that $K'$ is a **totally shellable refinement of a complex $K$** if $K'$ is a refinement and, for $k \in \{0, \ldots, n\}$ and each cube $Q \in K^{(k)}$, the restriction $K'|_Q$ is shellable. Note that this definition is relative in the sense that we do not assume complex $K$ to be shellable.

As our first observation on shellable complexes, we note that all cubical complexes on a 2-cell are shellable.

**Proposition 6.6.** Let $K$ be a cubical complex for which $|K|$ is a 2-cell. Then $K$ is shellable.

**Proof.** Let $C$ be the collection of all 2-cubes $q \in K^{(2)}$ whose intersection $q \cap \partial |K|$ contains at least one 1-cube. We first show that there is a cube $q_1 \in C$ for which $q_1 \cap \partial |K|$ is connected. It follows then that $|K| \setminus q_1$ is a 2-cell.

Suppose that for all $q \in C$ the intersection $q \cap \partial |K|$ is not connected, and let $q' \in C$. Then $|K| \setminus q'$ has more than one components each of which meets $\partial |K|$ in a 1-dimensional set. Let $D_1$ be any one of these components, and $q'' \in C$ be a 2-cube contained in $D_1$.

Since $q'' \cap \partial |K|$ is disconnected, $|D_1| \setminus q''$ has a component $D_2$ which does not meet $q'$ and whose intersection with $\partial |K|$ is 1-dimensional. Let $q''' \in C$ be a 2-cube contained in $D_2$. Since this process may be continued indefinitely and $C$ is finite, we conclude that there exists a cube $q_1 \in C$ for which $q_1 \cap \partial |K|$ is connected.
Now let $m = \# K^{(2)}$, $K_m = K$, and $K_{m-1}$ the subcomplex of $K$ having space $|K| \setminus q_1$. The claim now follows by induction.

For the forthcoming induction argument in the proof of Cubical Reduction Theorem, we record a simple observation: $(n-1)$-cells on the boundary of $[0,1]^n$ are shellable, in the standard cubical structure.

**Lemma 6.7.** Let $Q$ be an $n$-cube, and $K_Q$ be the cubical complex on $Q$ that is isomorphic to the standard CW-$\mathbb{Z}$-structure of the Euclidean cube $[0,1]^n$. Let $P$ be a subcomplex of $K_Q|_{\partial Q}$ whose space $|P|$ is an $(n-1)$-cell. Then $P$ is a shellable cubical $(n-1)$-complex.

**Proof.** For each $(n-1)$-cube $q$ in $K_Q|_{\partial Q}$, denote by $\hat{q}$ the unique cube in $K_Q|_{\partial Q}$ opposite to $q$, that is, $q \cap \hat{q} = \emptyset$. Then the intersection $q \cap q'$ is an $(n-2)$-cube, for any $q', q \in K_Q|_{\partial Q} \setminus \{q, \hat{q}\}$.

We claim that $P$ contains a face $q_0$ of $Q$ for which the opposite face $\hat{q}_0$ is not in $P$. Suppose this is not the case. Then the cubes in $P^{(n-1)}$ may be grouped into $i$ pairs of opposite faces for some $i \in [1,n-1]$. Thus, $|P|$ is homeomorphic to $S_i^{n-1} \times [0,1]^{n-i}$, which is not an $(n-1)$-cell.

With this information, we now fix an ordering, $q_1, q_2, \ldots, q_m$, of the $(n-1)$-cubes in $P$ by setting $q_1 = q_0$. Since all other cubes connect to $q_1$ in an $(n-1)$-face, the sequence $q_1, \ldots, q_m$ satisfies the defining property for the shellability.  

**Remark 6.8.** Shellability for simplicial complexes has a long history. It is well-known that not all triangulations of a tetrahedron are shellable; see M. Rudin [53]. To the other direction, it is a result of Frankl [19] that every triangulation of a 2-cell is shellable and a result of Sanderson [54] that every triangulation of a 3-cell has a shellable subdivision.

Not all cubical $n$-complexes of cells are shellable for any $n \geq 3$. See Bing [7] Example 2] for a 3-dimensional example; higher dimensional examples may be obtained by taking the product of Bing’s example with an Euclidean cubes.

6.4. Cubical reduction and Cubical Deformation Theorems. We begin with the following definition; see Figure 22 for an illustration.

**Definition 6.9.** Let $K$ be a cubical complex $K$ on an $n$-cell. A simplicial complex $K^*$ is a star-replacement of $K_{\Delta}$ if $K^*|_{\partial |K|} = K_{\Delta|\partial |K|}$, $|K^*| = |K|$, and $K^*$ has a unique vertex in the interior of $|K|$.

The star-replacement $K^*$ of a canonical triangulation $K_{\Delta}$ admits Alexander maps. We record this fact in the following stronger form.

**Lemma 6.10.** Let $K$ be a cubical complex on an $n$-cell, $K_{\Delta}$ a canonical triangulation, and $K^*$ its star-replacement. Let $f: |K| \to S^n$ be a $K_{\Delta}$-Alexander map. Then there exists a $K^*$-Alexander map $f^*: |K| \to S^n$ for which $f^*|_{\partial |K|} = f|_{\partial |K|}$. Moreover, $f^*$ is unique up to isotopy.

**Definition 6.11.** Let $K$ be a cubical complex on an $n$-cell. A $K^*$-Alexander map $f^*: |K| \to S^n$ is a star-replacement of a $K_{\Delta}$-Alexander map $f: |K| \to S^n$ if $f^*|_{\partial |K|} = f|_{\partial |K|}$. 


The canonical triangulation $K^\Delta$ of a cubical complex $K$ in Figure 21 and its star-replacement $K^\ast$. 

The canonical triangulation $K^\Delta$ reduces to its star-replacement $K^\ast$ through a reduction sequence.

**Theorem 6.12** (Cubical Reduction). Let $K$ be a shellable cubical complex on an $n$-cell. Then the canonical triangulation $K^\Delta$ of $K$ may be reduced to its star-replacement $K^\ast$ through a reduction sequence $K^\Delta \searrow \cdots \searrow K^\ast$.

In terms of mappings, there exist a branched cover homotopy between a $K^\Delta$-Alexander map and an expansion of a $K^\ast$-Alexander map of the same degree, we call this result the Cubical Deformation Theorem. The claim follows immediately from the Cubical Reduction Theorem and Theorem 4.17.

**Theorem 1.1** (Cubical deformation). Let $n \geq 2$, and let $K$ be a shellable cubical $n$-complex, $K^\Delta$ a canonical triangulation of $K$, and $K^\ast$ a star-replacement of $K^\Delta$. Let $f : |K| \to S^n$ be a $K^\Delta$-Alexander map and $f^\ast : |K| \to S^n$ a $K^\ast$-Alexander map. Let 

$$m = \left( \#(K^\Delta)^{(n)} - \#(K^\ast)^{(n)} \right) / 2,$$

and let $\tilde{f} : |K| \to S^n$ be a branched covering map obtained from $f^\ast$ by an expansion with $m$ mutually essentially disjoint free simple covers. Then $f$ and $\tilde{f}$ are branched cover homotopic rel $\partial|K|$.

The remaining part of this section is devoted to the proofs.

6.4.1. **Proof of the Cubical reduction Theorem.** The proof of Theorem 6.12 is by a double induction – first on the dimension and then on the complexity of the cell. We first prove the theorem in dimension two, which then serves as the initial step in the induction by dimension.

**Lemma 6.13.** Let $K$ be a 2-dimensional cubical complex whose space is a 2-cell, $K^\Delta$ its canonical triangulation and $K^\ast$ a star-replacement. Then there exists a reduction sequence $K^\Delta = K_0 \searrow \cdots \searrow K_j = K^\ast$.

**Proof.** We prove the claim by induction on the number of 2-cubes in $K$. The claim clearly holds when $\#K^{(2)} = 1$.

Assume now that $\ell \geq 1$ and that the claim holds for all complexes $K'$ satisfying $\#(K')^{(2)} \leq \ell$, and let $K$ be a cubical complex with $\#K^{(2)} = \ell + 1$.

By Proposition 6.6, we may assume that there exist a subcomplex $\tilde{K}$ of $K$ and a 2-cube $q \in \tilde{K}$ for which $K = \tilde{K} \cup q$ and $|\tilde{K} \cap q|$ is an 1-cell. Then $\{\tilde{K}^\ast, q^\ast\}$ is an Alexander pair. Since $\#\tilde{K}^{(2)} = \ell$, there exists a reduction
sequence $\tilde{K}^\Delta \setminus \cdots \setminus \tilde{K}^*$. This sequence immediately yields a reduction sequence $\tilde{K}^\Delta \cup q^* \setminus \cdots \setminus K^* \cup q^*$.

Let $\tilde{v}$ and $v_q$ be the only vertices of $\tilde{K}^*$ and $q^*$ contained in $\operatorname{int}|\tilde{K}|$ and $\operatorname{int}|q|$, respectively. We assume, as we may, that $f(\tilde{v}) = f(v_q) = w_2$ and that the corners of the 2-cube $q$ are $w_0$-vertices. Then $L = \tilde{K}^* \cap q^*$ is a 1-complex having only $w_0$ and $w_1$ vertices, and the $w_1$ vertices are contained in the interior of $|L|$.

![Diagrams](image)

**Figure 23.** Joins of $L$ with vertices $\tilde{v}$ and $v_q$ in all possible cases

There are two possibilities. If $|L|$ consists of exactly one face of $q$ then $\{\tilde{K}^*, q^*\}$ is a simple Alexander pair; the claim follows from Lemma 5.6. Otherwise, $|L|$ is a union of either two or three faces of $q$. See Figure 23 for an illustration. In either case there exists a reduction sequence $L = L_0 \setminus v_{w_1} L_1 \setminus v_{w_2} \cdots \setminus v_{w_k} L_k$, where $L_k$ is a 1-dimensional Alexander star consisting of two 1-simplices and three vertices; see Figure 24 for an illustration. Taking the join with the interior vertices of $\tilde{K}^*$ and $q^*$, we get by Lemma 5.7 and Corollary 5.8 a reduction sequence

$$\tilde{K}^* \cup q^* = P_1 \setminus v_{w_1} P_2 \setminus v_{w_2} \cdots \setminus v_{w_k} P_k$$

to a complex $P_k$ for which $\{P_k|\tilde{K}^*, P_k|q^*\}$ is a simple Alexander star pair. The claim now follows from Lemma 5.6 and concatenation of the reduction sequences.

**Proof of Theorem 6.12.** We start with the induction on the dimension and then on the number of top-dimensional cubes.

The induction assumption with respect to dimension states: *Suppose that $n \geq 2$ and that, for each shellable cubical complex $K$ whose space $|K|$ is an $n$-cell, there exists a reduction sequence $K^\Delta \setminus \cdots \setminus K^*$*. This statement holds for $n = 2$ by Lemma 6.13.

Assume next that in dimension $n+1$ the following holds: *Suppose that $\ell \geq 1$ and that, for each shellable cubical complex $K$ for which $|K|$ is an $(n+1)$-cell and $\#(K^{(n+1)}) = \ell$, there exists a reduction sequence $K^\Delta \setminus \cdots \setminus K^*$*. This statement holds trivially for $\ell = 1$.

For the induction step, let $K$ be a shellable cubical complex for which $|K|$ is an $(n+1)$-cell and $\#(K^{(n+1)}) = \ell + 1$. We proceed as in Lemma 6.13.

By the cubical shellability of $K$, there exist a shellable subcomplex $\tilde{K}$ of $K$, whose space is an $(n+1)$-cell, and an $(n+1)$-cube $q \in K$ for which $K = \tilde{K} \cup q$ and $|\tilde{K} \cap q|$ is an $n$-cell; $q^\Delta = q^*$. The cubical complex $\tilde{K} \cap q$ is shellable by Lemma 6.7, and complex $\tilde{K}$ is shellable since $K$ is shellable.
By the (second) induction assumption, there exist a reduction sequence
\( \tilde{K}^\Delta \setminus \cdots \setminus K^* \), and hence an induced reduction sequence \( \tilde{K}^\Delta \cup q^* \setminus \cdots \setminus K^* \cup q^* \).

Since \( \tilde{K} \cap q \) is shellable and \( |\tilde{K}^* \cap q^*| = |\tilde{K} \cap q| \) is an \( n \)-cell, there exists, by the induction hypothesis on the dimension, a reduction sequence \( \tilde{K}^* \cap q^* = L_0 \setminus L_1 \setminus \cdots \setminus L_k = (K^* \cap q^*)^* \). In view of Corollary 5.8, the reduction on \( K^* \cap q^* \) induces the reduction,

\[
\tilde{K}^* \cup q^* = P_1 \setminus P_2 \setminus \cdots \setminus P_k,
\]

of \( K^* \cup q^* \). Furthermore, \( P_k|_{\tilde{K} \cap q} = (\tilde{K}^* \cap q^*)^* \) and \( \{P_k|_{\tilde{K}}, P_k|_{q}\} \) is a simple Alexander star pair. Lemma 5.6 and concatenation of the reduction sequences complete the induction step for the second induction. Thus the claim holds for dimension \( n + 1 \). This completes the induction step on the dimension, and the proof of the theorem. \( \square \)

6.5. **Homotopical uniqueness theorems.** The Cubical Deformation Theorem has an important consequence in terms of homotopy.
Theorem 6.14 (Homotopy theorem of Alexander maps). Let $K_1$ and $K_2$ be two shellable cubical complexes on an $n$-cell $E$ for which $K_1|_{\partial E} = K_2|_{\partial E}$. Let $f_1 : E \to \mathbb{S}^n$ be a $K_1^\Delta$-Alexander map and $f_2 : E \to \mathbb{S}^n$ a $K_2^\Delta$-Alexander map. For $i = 1, 2$, let $f_i : E \to \mathbb{S}^n$ be an expansion of $f_i$ by free simple covers for which $\deg f_1 = \deg f_2$ have the same degree. Then $f_1$ and $f_2$ are branched cover homotopic.

Proof. Since $E$ is a cell and $K_1|_{\partial E} = K_2|_{\partial E}$, the star-replacements $K_1^s$ and $K_2^s$ of $K_1^\Delta$ and $K_2^\Delta$ are isomorphic. We may therefore assume that $K_1^s = K_2^s$. Then, by the Cubical Deformation Theorem, each map $f_i$ is a branched cover homotopic to a $K_1^s$-Alexander map $h_i : E \to \mathbb{S}^n$ expanded by free simple covers. Since $f_1$ and $f_2$ are either both orientation preserving or both orientation reversing and $K_1^s$ is a star, we may assume by Theorem 3.10 that $h_1 = h_2$. Thus both $f_1$ and $f_2$ are branched cover homotopic to the same $K_1^s$-Alexander map $E \to \mathbb{S}^n$ expanded by a same number of free simple covers. The claim follows. \hfill $\Box$

We record a useful corollary of this uniqueness theorem. Note that we do not assume the complex $K$ in the statement to be shellable.

Corollary 6.15. Let $K$ be a cubical $n$-complex and $K'$ be a totally shellable refinement of $K$. Then each $(K')^\Delta$-Alexander map $|K'| \to \mathbb{S}^n$ is branched cover homotopic to a $K^\Delta$-Alexander map $|K| \to \mathbb{S}^n$ expanded by free simple covers.

Proof. Let $f : |K'| \to \mathbb{S}^n$ be an $(K')^\Delta$-Alexander map. We may assume that $f(v) = w_k$ if $v \in ((K')^\Delta)^{(0)}$ is in the interior of a $k$-cube in $K$. We construct a sequence of simplicial complexes

$$(K')^\Delta = L_n, L_{n-1}, \ldots, L_1 = K^\Delta$$

inductively by star-replacements of decreasing dimension. Let $L_n = (K')^\Delta$ and $f_n = f : |L_n| \to \mathbb{S}^n$.

Let $L_{n-1}$ be the complex having the property that, for each $Q \in K^{(n)}$, the restriction $L_{n-1}|Q$ is the star-replacement of $L_n|Q$. Since $L_{n-1}|\partial Q = L_n|\partial Q$ for each $Q \in K^{(n)}$, the complex $L_{n-1}$ is well-defined. Since $K'|Q$ is shellable for each $Q \in K^{(n)}$, it follows from Theorem 6.14 that $f_n$ is branched cover homotopic to an $L_{n-1}$-Alexander map $f_{n-1} : |K'| \to \mathbb{S}^n$ expanded by free simple covers. We note that, for $Q \in K^{(n)}$, $L_{n-1}|Q = St_{L_{n-1}}(v_Q)$ where $v_Q$ is the unique vertex of $K^\Delta$ in the interior of $Q$.

For each $k = n-1, \ldots, 1$, the complex $L_{k-1}$ is now obtained from the complex $L_k$ by a similar star-replacement in $k$-cubes in $K^{(k)}$. As a consequence, for each $\ell > k$ and for all $Q \in K^{(\ell)}$, $L_{k}|Q$ is a (simplicial) refinement of $L_{\ell}|Q$. We omit the details. The simplicial complex $L_1$ has the same vertices (and the same space) as $K^\Delta$. Hence $L_1 = K^\Delta$. \hfill $\Box$

7. Hopf theorem for Alexander maps

As a direct consequence of Theorem 6.14, we obtain two versions of the Hopf degree theorem for Alexander maps. For the statements, we say that
a cubical complex $K$, for which $|K| \approx S^n$, is shellable if there exists an $n$-cube $Q \in K^{(n)}$ for which $K' = K \setminus \{Q\}$ is shellable cubical complex. These statements prove the following theorems in the introduction.

**Theorem 1.2** (First Hopf theorem for cubical Alexander maps). Let $K_1$ and $K_2$ be two shellable cubical complexes on $S^n$ having the same number of $n$-cubes. Then a $K_1^\Delta$-Alexander map and a $K_2^\Delta$-Alexander map, with the same orientation, are branched cover homotopic.

**Theorem 1.3** (Second Hopf theorem for cubical Alexander maps). Let $K$ be a shellable cubical complex on $S^n$. Then an orientation preserving $K^\Delta$-Alexander map is branched cover homotopic to

1. the identity map $S^n \to S^n$ expanded by free simple covers, and to
2. a winding map $S^n \to S^n$.

In the above two theorems, only part (2) of Theorem 1.3 requires a proof.

We first recall the notion of a winding map. For each $k \in \mathbb{N}$, let $re^{i\theta} \mapsto re^{ik\theta}$ be the standard winding map of order $k$ in $\mathbb{R}^2$. We also call its restriction to $S^1 \to S^1$ a winding map on $S^1$. Inductively, for each $n \geq 2$, we call a map $F = f \times 1: \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}$ the winding map of order $k$ on $\mathbb{R}^n$ if $f$ is a winding of order $k$ in $\mathbb{R}^{n-1}$. Similarly, we call the restriction $F|_{S^{n-1}}: S^{n-1} \to S^{n-1}$ a winding map of order $k$ on $S^{n-1}$.

**Proof of part (2) of Theorem 1.3.** The proof is by induction on dimension. In any case, $n = 2$, it suffices to observe that a winding map $S^2 \to S^2$ is branched cover homotopic to the identity id: $S^2 \to S^2$ expanded by free simple covers. Thus, for $n = 2$, the claim follows from Part (1) of Theorem 1.3.

Suppose now that the claim holds on $S^n$ for some $n \geq 2$. Let $K$ be a shellable cubical complex on $S^{n+1}$ and let $f$ be a $K^\Delta$-Alexander map. Let $Q \in K$ be an $(n+1)$-cube for which $K' = K \setminus \{Q\}$ is shellable. By Theorems 6.12 and 1.1, we may reduce $(K')^\Delta$ to its star-replacement, which is the canonical triangulation $(Q')^\Delta$ of a single $(n+1)$-cube $Q'$, and deform the $K^\Delta$-Alexander map $f$ to a $(Q \cup Q')^\Delta$-Alexander map expanded by simple covers. Since $|Q \cap Q'|$ is an $n$-sphere, a $(Q \cup Q')^\Delta$-Alexander map is a suspension of a $(Q \cap Q')^\Delta$-Alexander map. Since a winding map $S^{n+1} \to S^{n+1}$ is a suspension of a winding map $S^n \to S^n$, the claim now follows from the induction assumption. □

**Part 2. Weaving**

In this part, we describe the method of weaving, which is used to combine individual branched covering maps $M_i \to E_{c(i)}$, associated to a partition $\{M_i\}$ of a manifold $M$ and a partition $\{E_j\}$ of the sphere $S^n$, into a single branched covering map $M \to S^n$. Weaving is possible when the boundary maps of the given branched covers are Alexander that agree on the $(n-2)$-skeletons.

The process of weaving is topological. However we may obtain a mild geometric control on the resulting map for a special class of branched covers.

The idea of weaving has its origin in Rickman’s Picard construction [48] and has been further developed by Heinonen and Rickman in [24] and [25].
In these articles, the maps and the methods are referred indirectly as sheets or the sheet construction; see also [14] Section 7.

In what follows, we remove some of the restrictions implicitly presented in the previous studies. In particular, we allow the partition $\{E_j\}$ of $\mathbb{S}^n$ to have fewer elements than $\{M_i\}$, and allow adjacent submanifolds in $\{M_i\}$ to be mapped to non-adjacent components in $\{E_j\}$.

The usefulness of this version of weaving extends outside the scope of geometric function theory. For example, we establish in Theorem 1.5 the existence of branched covers from an $n$-manifold with $m$ boundary components to a Euclidean $n$-sphere with $p$ balls removed, with respect to an arbitrarily pre-assigned boundary match, for $n \geq 3$ and $2 \leq p \leq m$. This topological result on branched covers, in the spirit of a 3-dimensional existence theorem of Hirsch [27] and an extension theorem of Berstein and Edmonds [5], is new. The method of weaving allows us to bypass the handlebody decomposition in dimension 3 used in [5], and to ignore the global topological structure of $M$ in constructing branched covers.

To aid in understanding of the technical discussions in the subsequent sections, the reader may wish to glance at Theorem 1.5 and its proof for motivation.

8. Definition of weaving

8.1. Manifold partition. Let $M$ be an $n$-manifold (possibly with boundary). Let $M = (M_1, \ldots, M_m)$ an essential partition of $M$. We call the set

$$\partial_{\cup} M = \bigcup_{i \neq j} (M_i \cap M_j)$$

pairwise common boundary of the partition $M$. In the other direction, we say that a closed subset $X$ of $M$ induces an essential partition $M = (M_1, \ldots, M_m)$ of $M$ if $X = \partial_{\cup} M$.

Definition 8.1. An essential partition $M = (M_1, \ldots, M_m)$ of an $n$-manifold $M$ is a manifold partition if each $M_i$ is a connected $n$-manifold with boundary. A manifold partition $M$, together with a CW $\Delta$-complex $K_{\partial_{\cup} M}$ on $\partial_{\cup} M$, is an Alexander manifold partition of $M$ if $K_{\partial_{\cup} M}|_{\partial M \cap \partial_{\cup} M}$ supports an Alexander map for each $i = 1, \ldots, m$.

8.2. Cyclic partition of $\mathbb{S}^n$ and branched spheres. Given $n \geq 2$ and $p \geq 3$, we fix a cyclic cell partition $E = (E_{p,1}^n, \ldots, E_{p,p}^n)$ of $\mathbb{S}^n$ with the following properties:

1. each $E_{p,j}^n$ is an $n$-cell, with $E_{p,1}^n = \mathbb{S}^n \setminus \text{int} B^n$ and $E_{p,2}^n = B^n \cap \mathbb{R}^n$,

2. the intersections $E_{p,1}^n \cap E_{p,2}^n, \ldots, E_{p,p-1}^n \cap E_{p,p}^n, E_{p,p}^n \cap E_{p,1}^n, E_{p,1}^n \cap E_{p,2}^n$ are $(n-1)$-cells having mutually disjoint interiors and having a pairwise common boundary $\mathbb{S}^{n-2}$, and the intersection $E_{p,j}^n \cap E_{p,1}^n = \mathbb{S}^{n-2}$ if $|i-j| \neq 1$ or $\{i, j\} \neq \{1, p\}$; see Figure 25 for illustration.

The pairwise common boundary $\partial_{\cup} E$ is a branched $(n-1)$-sphere. We denote by $K_{\partial_{\cup} E}$ the CW $\Delta$-complex on $\partial_{\cup} E$ whose $(n-1)$-simplices are $\{E_{p,1}^n \cap$
E_p^1, \ldots, E_p^{n-1} \cap E_p^n, E_p^n \cap E_p^{n-1}\) and whose \((n-2)\)-skeleton \(K_{n-2}^{\partial_\sigma, \mathcal{E}}\) is \(K_{\partial_\sigma, \mathcal{E}}^{\partial_\sigma, \mathcal{E}}\).

For each \(i = 1, \ldots, p\), the subcomplex \(K_{\partial_\sigma, \mathcal{E}}^{\partial_\sigma, \mathcal{E}}\) is isomorphic to \(K_{\partial_\sigma, \mathcal{E}}^{\partial_\sigma, \mathcal{E}}\). It consists of two \((n-1)\)-simplices of the form \(E_p^{n-1} \cap E_p^n\): one with \(i' - i = 1\) mod \(p\), and the other with \(i' - i = -1\) mod \(p\).

For \(n \geq 2\) and \(p = 2\), we set \(E_1^n = S^n \setminus \text{int}B^n\) and \(E_2^n = B^n\). In this case, \(K_{\partial_\sigma, \mathcal{E}}^{\partial_\sigma, \mathcal{E}} = K_{\partial_\sigma, \mathcal{E}}^{\partial_\sigma, \mathcal{E}}\) and \(E_2^n \cap E_2^{n-1}\) is isomorphic to \(K_{\partial_\sigma, \mathcal{E}}^{\partial_\sigma, \mathcal{E}}\).

When the dimension \(n\) is understood from the context, we simply write \((E_1^n, \ldots, E_p^n)\) for \((E_{p,1}^n, \ldots, E_{p,p}^n)\).

**Definition 8.2.** Let \(n \geq 2\), \(p \geq 2\), and let \(\mathcal{E} = (E_1, \ldots, E_p)\) be a cyclic cell partition of \(S^n\). A space \(S\) homeomorphic to \(\partial_\sigma, \mathcal{E}\) is a branched \((n-1)\)-sphere of rank \(p\) if it is equipped with a \(\text{CW}_{\Delta}\)-structure \(K_S\) that is isomorphic to the \(\text{CW}_{\Delta}\)-complex \(K_{\partial_\sigma, \mathcal{E}}^{\partial_\sigma, \mathcal{E}}\). The complex \(K_S\) is called a standard complex of the branched sphere \(S\).

**Figure 25.** Cyclic cell partitions \(\mathcal{E}_p = (E_{p,1}^2, \ldots, E_{p,p}^2)\) of \(S^2\) and branched 1-spheres, for \(p = 2, 3, 4\).

### 8.3. Alexander maps of higher rank
Before starting the discussion on weaving, we introduce first the notion of higher rank Alexander maps, and branched sphericalization of complexes.

**Definition 8.3.** An \(n\)-dimensional \(\text{CW}_{\Delta}\)-complex \(\bar{K}\) is a branched sphericalization of a simplicial \(n\)-complex \(K\) if

\[
\bar{K} = K^{[n-1]} \cup \bigcup_{\sigma \in K^{(n)}} K_{S_\sigma},
\]

where either \(S_\sigma\) is a branched \(n\)-sphere and \(K_{S_\sigma}\) is a standard branched sphere on \(S_\sigma\) satisfying \(K_{S_\sigma}^{[n-1]} = K^{[n-1]}|\partial_\sigma\), or \(S_\sigma = \sigma\) and \(K_{S_\sigma} = K|\sigma\).

For an example of sphericalization, see Figure 27. Note that the 1-complex on the left-hand side of Figure 27 is a branched sphericalization of the 1-complex on the left-hand side of Figure 26.

We take \(r(\sigma)\) to be the rank of \(S_\sigma\) when \(S_\sigma\) is a branched sphere, and \(r(\sigma) = 1\) when \(S_\sigma = \sigma\). We call \(r: K^{(n)} \to \mathbb{Z}_+\) the rank function associated to the sphericalization \(\bar{K}\).

With a slight abuse of notation, for any given function \(r: K^{(n)} \to \mathbb{Z}_+\), we write \(S(K; r)\) for either the sphericalization complex \(\bar{K}\) of \(K\) having rank function \(r\), or the space \(|\bar{K}|\).
Definition 8.4. Let $n \geq 1$, $p \geq 2$, $P$ an $n$-dimensional CW $\Delta$-complex, and $K_S$ a standard complex on a branched $n$-sphere $S$ of rank $p$. A branched covering map $f: |P| \to S$ is an Alexander map of rank $p$ if it is surjective and $(P, K_S)$-simplicial.

8.4. Alexander Sketches. Let $M$ be an oriented $n$-manifold and let $\mathcal{M} = (M_1, \ldots, M_m)$ be an Alexander manifold partition associated with a CW $\Delta$-structure $K_{\partial_\mathcal{M}}$. Let $\mathcal{E} = (E_1, \ldots, E_p)$ be a cyclic cell partition of $\mathbb{S}^n$ and whose pairwise common boundary $\partial_i \mathcal{E}$ is a branched $(n-1)$-sphere equipped with a CW $\Delta$-structure $K_{\partial_i \mathcal{E}}$.

Definition 8.5. A sequence $\mathcal{F} = (f_1, \ldots, f_m)$ of orientation preserving interior branched covering maps $f_i: M_i \to \mathbb{S}^n$ is an $(\mathcal{M}, \mathcal{E})$-Alexander sketch, if there exists a function

$$c: \{1, \ldots, m\} \to \{1, \ldots, p\}$$

such that, for $i = 1, \ldots, m$,

1. $f_i(M_i) = E_{c(i)}$,
2. $f_i|_{\partial M_i \cap \partial \mathcal{M}}: \partial M_i \cap \partial \mathcal{M} \to \partial E_{c(i)}$ is $K_{\partial_\mathcal{M}}|_{\partial M_i}$-Alexander map, and
3. $f_i|_{M_i \cap M_j}$ and $f_j|_{M_i \cap M_j}$ agree on the $(n-2)$-skeleton of $K_{\partial_\mathcal{M}}|_{M_i \cap M_j}$ for all indices $i \neq j$.

We say that $\mathcal{F}$ is an Alexander sketch if it is an $(\mathcal{M}, \mathcal{E})$-Alexander sketch for a manifold partition $\mathcal{M}$ of $M$ and a cyclic partition $\mathcal{E}$ of $\mathbb{S}^n$.

An $(\mathcal{M}, \mathcal{E})$-Alexander sketch is surjective if the function $c$ is surjective.

Note that, in this definition, we do not impose the condition $p \leq m$, and that typically maps $f_i$ and $f_j$ in an Alexander sketch $\mathcal{F}$ do not agree on $M_i \cap M_j$, as seen in Figure 26.

Heuristically, it is useful to assume that open cells $\{\text{int} E_1, \ldots, \text{int} E_p\}$ are coded with distinct colors and that open manifolds $\{\text{int} M_1, \ldots, \text{int} M_m\}$ may also be colored via the coloring function $c: \{1, \ldots, m\} \to \{1, \ldots, p\}$; see Figure 26.

![Figure 26](image)

**Figure 26.** An essential partition $\mathcal{M} = \{M_1, M_2\}$ of $\mathbb{R}^2$, a cyclic cell partition $\mathcal{E} = (E_1, E_2, E_3, E_4)$ of $\mathbb{S}^2$, and two branched covering maps $f_1: M_1 \to E_4$ and $f_2: M_2 \to E_1$ forming an Alexander sketch $\mathcal{F} = (f_1, f_2)$.

Remark 8.6. Suppose that $\mathcal{F} = (f_1, \ldots, f_m)$ is an $(\mathcal{M}, \mathcal{E})$-Alexander sketch. Then, by Definition 8.5, each map $f_i: M_i \to E_{c(i)}$ is orientation preserving.

Let $\sigma$ be an $(n-1)$-simplex in $K_{\partial_\mathcal{M}}$ and $i \neq j$ be indices in $\{1, \ldots, m\}$ for which $\sigma \subset M_i \cap M_j$. Since $\sigma$ has opposite orientations with respect to $M_i$ and $M_j$, its image $f_i(\sigma)$ as a face of $E_{c(i)}$ and its image $f_j(\sigma)$ as a face of $E_{c(j)}$ have opposite signs.
8.5. **Weaving of sketches.** Let $\mathcal{F}$ be a $(\mathcal{M}, \mathcal{E})$-Alexander sketch. We define the weaving $f: M \to S^n$ of the maps $f_i: M_i \to E_{c(i)}$ for $i = 1, \ldots, m$. Recalling that maps $f_i$ and $f_j$ in $\mathcal{F}$ do not a priori match on the entire $M_i \cap M_j$.

**Definition 8.7.** Let $\mathcal{F} = (f_1, \ldots, f_m)$ be an $(\mathcal{M}, \mathcal{E})$-Alexander sketch. An (interior) branched covering map $f: M \to S^n$ is a weaving of $\mathcal{F}$, if there exist an essential partition $\mathcal{M}' = (M'_1, \ldots, M'_m)$ of $M$ and, for each $i = 1, \ldots, m$, an orientation preserving homeomorphism $\lambda_i: \text{int} M_i \to \text{int} M'_i$ which extends to a map $\bar{\lambda}_i: M_i \to M'_i$ and satisfies

$$ f_i = f \circ \bar{\lambda}_i. $$

We write $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)$ and define the *support* of $\bar{\lambda}$ to be the set

$$ \text{spt}(\bar{\lambda}) = \bigcup_{i=1}^m \{ x \in M_i : \bar{\lambda}_i(x) \neq x \}. $$

In the definition above, as well as in future applications, the essential partition $\mathcal{M}'$ for the weaving typically is not a manifold partition and the extended maps $\bar{\lambda}_i: M_i \to M'_i$ are not open maps. However, maps $\bar{\lambda}_i$ are discrete.

9. **Partial weaving of Alexander sketches**

In this section, we show that an Alexander sketch induces a branched covering map which we call partial weaving.

**Definition 9.1.** Let $\mathcal{F} = (f_1, \ldots, f_m)$ be an $(\mathcal{M}, \mathcal{E})$-Alexander sketch on $M$. A branched covering map $f: M \to S^n$ is a partial weaving of $\mathcal{F}$ if there exists a sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of embeddings, $\lambda_i: M_i \to M$, which satisfy $f_i = f \circ \lambda_i$ for all $i$, and whose images $\{\lambda_1(M_1), \ldots, \lambda_m(M_m)\}$ have mutually disjoint interiors.

Note that, in a partial weaving, the images $\{\lambda_1(M_1), \ldots, \lambda_m(M_m)\}$ need not cover the manifold $M$; see Figure 27.

![Figure 27. A partial weaving $f: \mathbb{R}^2 \to S^2$ of the $(\mathcal{M}, \mathcal{E})$-Alexander sketch $\mathcal{F} = (f_1, f_2)$ in Figure 26.](image)

In the construction of a partial weaving of an $(\mathcal{M}, \mathcal{E})$-Alexander sketch $\mathcal{F}$, we replace the complex $K_{\partial \mathcal{M}}$ by a suitable sphericalization, and the maps $f_i|_{\partial M_i}: \partial M_i \to \partial E_{c(i)}$, $i = 1, \ldots, m$, by Alexander maps of higher rank; see Figure 27.

Definition 9.2. An $n$-dimensional CW$_\Delta$ complex $K$ is said to be simplicially connected if the adjacency graph of the $n$-simplices in $K$ is connected.

Proposition 9.3. Let $M$, $\mathcal{M}$, $K_{\partial_\mathcal{M}}$, and $\mathcal{E}$, respectively, be the $n$-manifold, the Alexander manifold partition, the CW$_\Delta$-complex, and the cyclic cell partition of $\mathbb{S}^n$, respectively, as in Definition 8.8, and let $m, n \geq 2$ and $p \geq 2$. Let also $F = (f_1, \ldots, f_m)$ be an $(\mathcal{M}, \mathcal{E})$-Alexander sketch, and let $U$ be a neighborhood of $\partial\mathcal{M}$. Then, there exist

1. an Alexander manifold partition $\mathcal{M}' = (M_1', \ldots, M_m', M_{m+1}', \ldots, M_{m'})$ of $M$ and a CW$_\Delta$-complex $K_{\partial_\mathcal{M}'}$ for which
   
   (a) $(K_{\partial_\mathcal{M}'})^{[n-2]} = (K_{\partial_\mathcal{M}})^{[n-2]}$
   
   (b) $(K_{\partial_\mathcal{M}'})^{[n-2]}|_{\partial_\mathcal{M}'} = (K_{\partial_\mathcal{M}}|_{\partial_\mathcal{M}'}))^{[n-2]}$ for each $1 \leq i \leq m$, and
   
   (c) $K_{\partial_\mathcal{M}'}|_{\partial_\mathcal{M}'}$ is isomorphic to $K_{\mathbb{S}^{n-1}}$ for each $i = m + 1, \ldots, m'$,

2. a branched covering map $f': M \to \mathbb{S}^n$ for which the restriction $f'|_{\partial_\mathcal{M}'}: \partial_\mathcal{M}' \to \partial_\mathcal{E}$ is an Alexander map of rank $p$, and

3. for $1 \leq i \leq m$, a homeomorphism $\lambda_i': M_i \to M_i'$, supported in $U$, which satisfies $f' \circ \lambda_i' = f_i$ and is the identity map on $(K_{\partial_\mathcal{M}})^{[n-2]}$.

In particular, the branched covering map $f'$ is a partial weaving of $F$.

Proof. Let $c: \{1, \ldots, m\} \to \{1, \ldots, p\}$ be the coloring function associated to the Alexander sketch $F$. We fix a rank function $r: K_{\mathcal{M}}^{(n-1)} \to \mathbb{Z}$ as follows.

Let $\sigma$ be an $(n - 1)$-simplex in $K_{\partial_\mathcal{M}}$, and $i \neq j$ be the indices for which $\sigma \subset M_i \cap M_j$. It follows from Definition 8.5 and Remark 8.6 that the sign (plus or minus) of $f_i(\sigma)$ with respect to the cell $E_{c(i)}$ and the sign of $f_j(\sigma)$ with respect to $E_{c(j)}$ are opposite and are determined by the orientation of $\sigma$ with respect to $M_i$.

Before fixing the rank $r(\sigma)$, we define an $n$-cell $D_\sigma$, possibly degenerate, which is the union of a number of components in $\mathcal{E}$.

Assume first that $p \geq 3$. Suppose that $c(i) = c(j)$. Then $E_{c(i)} = E_{c(j)}$, $f_i(\sigma) \neq f_j(\sigma)$, and $\partial E_{c(i)} = f_i(\sigma) \cup f_j(\sigma)$. We set $E_\sigma$ to be the unique $n$-cell whose interior is $\mathbb{S}^n \setminus E_{c(i)}$. Note that $\partial D_\sigma = f_i(\sigma) \cup f_j(\sigma)$.

Suppose next that $|c(i) - c(j)| = 1 \mod p$. Then $E_{c(i)} \cap E_{c(j)}$ is an $(n - 1)$-simplex. We have two cases:

(i) If $f_i(\sigma) \neq f_j(\sigma)$, we let $D_\sigma$ be the $n$-cell having $\mathbb{S}^n \setminus (E_{c(i)} \cup E_{c(j)})$ as its interior. In this case, $\partial D_\sigma = f_i(\sigma) \cup f_j(\sigma)$.

(ii) If $f_i(\sigma) = f_j(\sigma)$, we let $D_\sigma = \emptyset$.

Suppose now that $|c(i) - c(j)| \geq 2 \mod p$. We take $D_\sigma$ to be the unique $n$-cell in $\mathbb{S}^n$ which is enclosed by $f_i(\sigma) \cup f_j(\sigma)$ and whose interior is a component of $\mathbb{S}^n \setminus \{E_{c(i)}, E_{c(j)}\}$.

Next, in the case $p = 2$, there are only two possibilities. If $c(i) = c(j)$, then $f_i(M_i) = f_j(M_j) = E_{c(i)}$, and $f_i(\sigma)$ and $f_j(\sigma)$ are mapped to the opposites faces of $E_{c(i)}$. Let $D_\sigma = E_k$ with index $k \neq c(i)$. If $c(i) \neq c(j)$, then, by orientation, $f_i(\sigma)$ and $f_j(\sigma)$ are mapped to the same $(n-1)$-cell in the branched sphere $\partial_\mathcal{E}$. Let $D_\sigma = \emptyset$ in this case.

For any $p \geq 2$, take $r_\sigma$ to be the number of the elements of $\mathcal{E}$ contained in $D_\sigma$. 

Before continuing, we make an observation for adjacent \((n - 1)\)-simplices \(\sigma\) and \(\sigma'\) in \(M_i \cap M_j\) with \(i \neq j\). Note first that \(r_\sigma + r_{\sigma'} = 2p - 2\) when \(c(i) = c(j)\), and then that \(r_\sigma + r_{\sigma'} = p - 2\) in all other cases. Therefore,
\[
r(\sigma) + r(\sigma') + 2 \text{ is a multiple of } p.
\]

We define a rank function \(r: K^{(n-1)}_{(n-2)\partial M} \to \mathbb{Z}_+\) by setting
\[
r(\sigma) = 2p + r_\sigma,
\]
for each \((n - 1)\)-simplex \(\sigma\) in \(K^{(n-1)}_{\partial M}\).

Let now \(S(K^{(n-1)}_{\partial M}; r)\) be the sphericalization of \(K^{(n-1)}_{\partial M}\), associated to the rank function \(r\), for which \(|S(K^{(n-1)}_{\partial M}; r)|\) is contained in the open set \(U\). Note that \(S(K^{(n-1)}_{\partial M}; r)^{(n-2)} = K^{(n-2)}_{\partial M}\) by the definition of the sphericalization. Since \(r(\sigma) \geq 2\) for all \(\sigma \in K^{(n-1)}_{\partial M}\), each branched sphere \(S_\sigma\) in \(S(K^{(n-1)}_{\partial M}; r)\) bounds an \(n\)-cell \(G_\sigma\) in \(M\) that contains \(S_\sigma\) and does not meet \(K^{(n-2)}_{\partial M}\) | \(S_\sigma\). So the closures of the components of \(G_\sigma \setminus S_\sigma\) form an essential partition \((G_{\sigma,1}, \ldots, G_{\sigma,r(\sigma)\setminus 1})\) of \(G_\sigma\). Let, for \(1 \leq i \leq m\),
\[
M'_i = M_i \setminus \bigcup_{\sigma \in K_{\partial M}^{(n-1)}} \text{int } G_\sigma
\]
and let \(\{M'_{m+1}, \ldots, M'_{m'}\}\) be the collection \(\bigcup_{\sigma \in K_{\partial M}^{(n-1)}} \{G_{\sigma,1}, \ldots, G_{\sigma,r(\sigma)\setminus 1}\}\) of the \(n\)-cells. Then
\[
\mathcal{M}' = (M'_1, \ldots, M'_m, M'_{m+1}, \ldots, M'_{m'});
\]
is a manifold partition of \(M\) with the \(\CW_\Delta\)-structure \(S(K^{(n-1)}_{\partial M}; r)\) on \(\partial \mathcal{M}'\); see Figure 28 for an example. By requiring in the sphericalization that \(\sigma \subset G_\sigma \subset U\) for each \(\sigma \in K^{(n-1)}_{\partial M}\), we may assume that \(M'_i \subset M_i\) for \(1 \leq i \leq m\).

For each \(1 \leq i \leq m\), let \(\lambda'_i: M_i \to M'_i\) be a homeomorphism which is identity on \((K^{(n-1)}_{\partial M} | \partial M_i)^{(n-2)} \cup (M_i \setminus U)\), and satisfies
\[
K^{(n-1)}_{\partial M} | \partial M_i = \{\lambda'_i(\sigma): \sigma \in K_{\partial M}^{(n-1)}\}.
\]
For \(i = 1, \ldots, m\), the complex \(K^{(n-1)}_{\partial M'} | \partial M'_i\) is isomorphic to \(K_{\partial M}^{(n-1)} | \partial M_i\) and the map \(f'_i = f_i \circ (\lambda'_i)^{-1}: M'_i \to E_{c(i)}\) is a branched covering map with the property that the restriction \((f'_i|_{\partial M'_i})^{-1}: \partial M'_i \to \partial E_{c(i)}\) is a \((K^{(n-1)}_{\partial M'} | \partial M'_i)^{2}\)-Alexander map.

Since \(r(\sigma) + r(\sigma') + 2\) is a multiple of \(p\) for adjacent \((n - 1)\)-simplices \(\sigma\) and \(\sigma'\) in \(M_i \cap M_j\), there exists, for each \(\sigma \in K^{(n-1)}_{\partial M}\), a well-defined interior branched covering map \(f'_\sigma: G_\sigma \to \mathbb{S}^n\) for which
\[
\begin{align*}
&f'_\sigma|_{G_\sigma \cap \partial M'_i} = f'_i|_{G_\sigma \cap \partial M'_i} \text{ and } f'_\sigma|_{G_\sigma \cap \partial M'_j} = f'_j|_{G_\sigma \cap \partial M'_j}, \quad \text{where } i \neq j \text{ are the indices for which } \sigma \subset M_i \cap M_j, \text{ and} \\
&f'_\sigma(G_{\sigma,k}) \text{ is an element in } \mathcal{E} \text{ for each } k = 1, \ldots, r(\sigma) - 1.
\end{align*}
\]
A partial weaving \(f': M \to \mathbb{S}^n\) of \(\mathcal{F}\) may be defined on the components of the essential partition \(\mathcal{M}'\) by \(f'|_{M'_i} = f'_i: M'_i \to E_{c(i)}\) for \(i = 1, \ldots, m\), and \(f'|_{G_\sigma} = f'_\sigma|_{G_\sigma}\) for each \(\sigma \in K^{(n-1)}_{\partial M}\). In particular, \(f'|_{\partial \mathcal{M}'}: \partial \mathcal{M}' \to \partial \mathcal{E}\) is an Alexander map of rank \(p\). This completes the proof.
Figure 28. An example of partial weaving $f': M \to S^3$ of an Alexander sketch $(f_1, f_2, f_3, f_4)$ – a local detail.

9.2. **Neighborly trees.** We now introduce the notions of neighborly pairs and neighborly connected graphs. The neighborly connected graphs are used in the weaving process to combine adjacent domains in a manifold partition provided by a partial weaving.

Again we assume that $M$ is an oriented $n$-manifold, $\mathcal{M} = (M_1, \ldots, M_m)$ is an Alexander manifold partition of $M$ with a CW$_\Delta$-structure $K_{\partial,\mathcal{M}}$. Let $\mathcal{E} = (E_1, \ldots, E_p)$ be a cyclic cell partition of $S^n$ for which $\partial_c \mathcal{E}$ is a branched $(n - 1)$-sphere with a standard CW$_\Delta$-structure.

**Definition 9.4.** Let $\mathcal{F} = (f_1, \ldots, f_m)$ be an $(\mathcal{M}, \mathcal{E})$-Alexander sketch. Elements $M_i$ and $M_j$ in $\mathcal{M}$ are $\mathcal{F}$-neighborly if $f_i(M_i) = f_j(M_j)$, and $K_{\partial,\mathcal{M}|M_i\cap M_j}$ contains at least one $(n - 2)$-simplex in $K_{\partial,\mathcal{M}}$.

The graph $G(\mathcal{M}; \mathcal{F})$ on $\mathcal{M}$ whose edges consists of all $\mathcal{F}$-neighborly pairs is called the **neighborly graph on** $\mathcal{M}$ **associated to** $\mathcal{F}$.

**Definition 9.5.** Let $R$ be a subcollection of $\{M_1, \ldots, M_m\}$. The neighborly graph $G(\mathcal{M}; \mathcal{F})$ is **$R$-neighborly connected** if there exists a collection $\Sigma$ of subtrees of $G(\mathcal{M}; \mathcal{F})$ such that each element in $\mathcal{M}$ is the vertex of precisely one tree in $\Sigma$, and each tree in $\Sigma$ contains exactly one element of $R$. The forest $\Sigma$ is called a **$R$-forest** of $G(\mathcal{M}; \mathcal{F})$.

In an $R$-forest of $G(\mathcal{M}; \mathcal{F})$, each element in $R$ is considered as the root of a tree to which it belongs; see Figure 29. Note that if $M_i$ and $M_j$ belong to the same tree in $\Sigma$ then $f_i(M_i) = f_j(M_j)$.

We record as a corollary the existence of a forest associated to a partial weaving.

**Corollary 9.6.** Suppose that the CW$_\Delta$-complex $K_{\partial,\mathcal{M}}$ in Proposition 9.3 is simplicially connected. Then the Alexander manifold partition $\mathcal{M}'$, the partial weaving $f'$, and an $(\mathcal{M}', \mathcal{E})$-Alexander sketch

$$\mathcal{F}' = (f'|_{M'_1}, \ldots, f'|_{M'_{m'}}, f'|_{M'_{m'+1}}, \ldots, f'|_{M'_{m''}})$$
Proof. As in Proposition 9.3, let $F' = (f'|_{M'_1}, \ldots, f'|_{M'_m}, f'|_{M'_{m+1}}, \ldots, f'|_{M'_{m'}})$ be the $(M', E)$-Alexander sketch, where the map $f'|_{M'_j}$, for each $j \geq m+1$, is a restriction $f'_i|_{G_{\sigma,k}}$ for some $\sigma \in K^{(n-1)}$ and some $1 \leq k \leq r(\sigma) - 1$.

The neighborly graph $G(M'; F')$ consists of $p$ subgraphs $G_1, \ldots, G_p$, one for each color. Under the assumption that $K_{\partial_sM}$ is simplicially connected and the fact that rank $r(\sigma) \geq 2p$ for each $\sigma \in K^{(n-1)}$, each $G_i$ is connected and contains a subgraph that is isomorphic to the adjacency graph on the $(n-1)$-simplices in $K_{\partial_sM}$. We choose, for each $i = 1, \ldots, p$, a maximal tree $T_i$ from the graph $G_i$. By removing some edges if necessary, we trim each tree $T_i$ into a family $\Sigma_i$ of mutually disjoint subtrees, each of which has precisely one vertex in $\{M'_j; 1 \leq j \leq m, c(j) = i \}$ and $c(j) = i$. Then $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_p$ is an $\{M'_1, \ldots, M'_m\}$-forest in $G(M'; F')$. \hfill \Box

10. WEAVING OF ALEXANDER SKETCHES

A branched covering map $M \to S^n$ obtained as a partial weaving of an Alexander sketch $F$, associated to a manifold partition $\mathcal{M}$ of $M$, is typically based on a finer manifold partition $\mathcal{M}'$ than $\mathcal{M}$; cf. Proposition 9.3. In many applications, such as in the Picard problem [48, 14] or in the generalization of Heinonen–Rickman branching theorem (Theorem 1.9), this change of essential partition is undesirable.

The aim in this section is to obtain a branched covering map $M \to S^n$ associated to an essential partition $\mathcal{M}''$ having the same number of elements as $\mathcal{M}$. This is obtained by constructing a new branched covering map $M \to S^n$ from the partial weaving of $F$ by adjoining the components of $\mathcal{M}'$.

Before constructing the branched covering map $M \to S^n$, we introduce Alexander–Rickman maps which replace the Alexander maps of higher rank associated to the sphericalization. These maps are present in Rickman’s original sheet construction in [48, Section 7] and have been implicitly used in the constructions of Heinonen and Rickman [24] and [25], and in [14]. They can be viewed as a non-simplicial variant of Alexander maps of higher rank.
10.1. **Alexander–Rickman maps.** The definition of Alexander-Rickman maps is based on the notion $K$-projection illustrated in Figure 30.

![Figure 30](image_url)

**Figure 30.** An example of a $K$-projection $\pi: X \to |K|$ when $\dim |K| = 1$.

**Definition 10.1.** Let $K$ be an $n$-dimensional CW$_\Delta$-complex with $n \geq 1$. A map $\pi: X \to |K|$, from a space $X$, is a $K$-projection if

1. $\pi|_{\pi^{-1}(\sigma)}: \text{cl}(\pi^{-1}(\text{int} \sigma)) \to \sigma$ is a homeomorphism for each $\sigma \in K^{(n)}$, and
2. when $n \geq 2$, $\pi|_{\pi^{-1}(|K|^{n-2})}: \pi^{-1}(|K|^{n-2}) \to |K|^{n-2}$ is a homeomorphism.

**Remark 10.2.** A $K$-projection $\pi: X \to |K|$ induces a natural collection $\pi^{-1}(K) = \{\text{cl}(\pi^{-1}(\text{int} \sigma)): \sigma \in K\}$ of simplices on $X$. This collection need not be a CW$_\Delta$-complex. The $n$-simplices in $\pi^{-1}(K)$ are called sheets in [48].

**Definition 10.3.** Let $X$ be an $(n-1)$-dimensional space and $E$ be a cyclic cell partition of $S^n$ of rank $p$. A branched covering map $f: X \to \partial \sqcup E$ onto the branched sphere $\partial \sqcup E$ is an Alexander–Rickman map (of rank $p$) if there exist an $(n-1)$-dimensional CW$_\Delta$-complex $K$, a $K$-projection $\pi: X \to |K|$, and a $K$-Alexander map $\bar{f}: |K| \to \partial \sqcup E$ of rank $p$ for which the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \partial \sqcup E \\
\pi \downarrow & & \downarrow \bar{f} \\
|K| & & \\
\end{array}
\]

commutes. The mapping $\bar{f}$ is called a $K$-model for $f$;

We now give an example of $K$-projection in $\mathbb{R}^3$, modeled on the idea of creating an opening of Rickman in [48, Section 7], which serves as a prototype for the construction in Theorem 10.5. We denote by $\text{conv} S$ the convex hull of a set $S$ in $\mathbb{R}^3$.

**Example 10.4.** Let $\mathcal{M}$ be an essential partition of $\mathbb{R}^3$ which consists of tetrahedra

$\xi = \text{conv} \{(0,0,0), (0,1,0), (1,0,0), (1,0,-1)\}$

and

$\xi' = \text{conv} \{(0,0,0), (0,1,0), (-1,0,0), (-1,0,-1)\}$,
and the set $\text{cl}(\mathbb{R}^3 \setminus (\xi \cup \xi'))$. Let $K = K_{\partial,\mathcal{M}}$ be a CW$_\Delta$-complex on $\partial_\lambda \mathcal{M}$ whose 2-simplices consists of precisely 2-faces of $\xi$ and $\xi'$. Let $\ell = \xi \cap \xi' = \text{conv}\{(0,0,0), (0,1,0)\}$. Given $\frac{1}{10} < a < \frac{9}{10}$ and $0 < \rho < \frac{1}{100}$, let $p_a$ be the point $(0, a, 0)$, $D$ be the half-ball $B^3(p_a, \rho) \cap (\mathbb{R}^2 \times [0, \infty))$, $\delta$ be the semi-disk $D \cap \{(0) \times \mathbb{R}^2\}$, and let $l$ be the segment $D \cap \ell$. Let $\alpha = D \cap (\xi \cup \xi') = \partial D \cap (\mathbb{R}^2 \times \{0\})$, and $\beta = \text{cl}(\partial D \setminus \alpha)$.

**Figure 31.** An example of a $K$-projection $\pi: X \to |K|$, when $\dim |K| = 2$ – local picture.

Let $\Pi: \mathbb{R}^3 \to \mathbb{R}^3$ be a map which is

1. the identity on $\partial(\xi \cup \xi') \setminus \text{int}(\alpha \setminus l)$,
2. the natural projection, $(x_1, x_2, x_3) \mapsto (x_1, x_2, 0)$, from $\beta$ onto $\alpha$ and from $\delta$ onto $l$, and
3. a homeomorphism from $\mathbb{R}^3 \setminus \delta$ onto $\mathbb{R}^3 \setminus l$.

Let $X = \Pi^{-1}\left(\partial_\lambda \mathcal{M}\right) \setminus \text{int} \delta$. Then $\pi = \Pi|_X: X \to \partial_\lambda \mathcal{M}$ is a $K$-projection.

In this example, the segment $\ell$ may be viewed as a barrier between two chambers $\xi$ and $\xi'$. A larger chamber $\text{cl}(\xi \cup D \cup \xi')$ combining $\xi$ and $\xi'$ is built by lifting $\alpha$ to $\beta$ to create a passage $\delta$. In this process the space $|K| \setminus l$ is topologically unchanged. The $K$-projection $\pi$ reverses this procedure.

### 10.2. A weaving theorem

We now formulate the main theorem on the existence of weaving.

**Theorem 10.5.** Let $n \geq 3$, $m \geq 2$, $\mathcal{M}$ be an oriented $n$-manifold, and let $\mathcal{M} = (M_1, \ldots, M_m)$ be an Alexander manifold partition associated with a CW$_\Delta$-structure $K_{\partial,\mathcal{M}}$ on the pairwise common boundary $\partial_\lambda \mathcal{M}$. Let $p \geq 2$, and $\mathcal{E} = (E_1, \ldots, E_p)$ be a cyclic cell partition of $\mathbb{S}^n$. Let also $\mathcal{F} = (f_1, \ldots, f_m)$ be an $(\mathcal{M}, \mathcal{E})$-Alexander sketch, $c: \{1, \ldots, m\} \to \{1, \ldots, p\}$ the corresponding coloring function, and let $U$ be a neighborhood of $\partial_\lambda \mathcal{M}$. Suppose that $K_{\partial,\mathcal{M}}$ is simplicially connected.

Then there exist

1. an essential partition $\mathcal{M}' = (M_1', \ldots, M_m')$ of $\mathcal{M}$ and a CW$_\Delta$-complex $K_{\partial,\mathcal{M}'}$ on the pairwise common boundary $\partial_\lambda \mathcal{M}'$, 
2. homeomorphisms $\lambda_i: \text{int} M_i \to \text{int} M_i'$ for $i = 1, \ldots, m$, which extend to maps $\tilde{\lambda}_i: M_i \to M_i'$, and
3. a branched covering map $f'': M \to \mathbb{S}^n$ supported in $U$, for which

   i. the restriction $f''|_{\partial_\lambda \mathcal{M}'}: \partial_\lambda \mathcal{M}' \to \partial_\lambda \mathcal{E}$ is a $K_{\partial,\mathcal{M}'}$-Alexander–Rickman map of rank $p$. 


(ii) on the pairwise common boundary \(\partial_i M\), for each \(i = 1, \ldots, m\), the composition
\[
f'' \circ \overline{\lambda}_i|_{\partial_i M \cap \partial M} : \partial_i M \cap \partial M \to \partial E_c(i)
\]
is the \(K_{\partial_i M|\partial M, \partial_i M}\)-Alexander map \(f_i|_{\partial_i M \cap \partial M} : \partial_i M \cap \partial M \to \partial E_c(i)\) expanded by simple covers, and

(iii) on the boundary \(\partial M\) of the manifold, for each \(i = 1, \ldots, m\), the restriction \(f''|_{\partial M \cap \partial M}\) is the branched cover \(f_i|_{\partial M \cap \partial M}\) expanded by simple covers.

Proof. Let \(M' = (M'_1, \ldots, M'_m, M'_m+1, \ldots, M'_n)\) be the Alexander manifold partition of \(M\), \(f' : M \to S^n\) the partial weaving of \(F\), and let \(F' = (f'_1|_{M'_1}, \ldots, f'|_{M'_m}, f'|_{M'_m+1}, \ldots, f'|_{M'_n})\) be the \((M', E')\)-Alexander sketch constructed in Proposition 9.3. Let also \(\lambda'_i : M_i \to M'_i\), for \(i = 1, \ldots, m\), be homeomorphisms satisfying \(f' \circ \lambda'_i|_{M_i} = f_i|_{M_i}\).

Since \(K_{\partial_i M}\) is simplicially connected, we may choose and fix, as in the proof of Corollary 9.6, an \(\{M'_1, \ldots, M'_m\}\)-forest \(\Sigma\) of the neighborly graph \(G(M'; F')\). Assign, to each \(F'-\)neighborly pair \(\{\xi, \xi'\}\) in a tree in the forest \(\Sigma\), a designated common \((n-2)\)-simplex \(\ell_{\xi, \xi'}\) in \(K_{\partial_i M}\). Note that, in general, \(\xi \cap \xi'\) may contain several common \((n-2)\)-simplices; see Figure 32. Recall also that \(K^{[n-2]}_{\partial_i M} = K^{[n-2]}_{\partial_j M}\).

Step 1. Alexander-Rickman map – a local construction. The procedure in this step resembles that of Example 10.4.

For each tree \(T\) in \(\Sigma\) and each edge \(\{\xi, \xi'\}\) (an \(F'\)-neighborly pair) of the tree \(T\), we do the following.

Let \(\ell = \ell_{\xi, \xi'}\). Note that, in general, there may be other \(F'\)-neighborly pairs \(\{\eta, \eta'\}\), not necessarily on the same tree \(T\), for which the designated common \((n-2)\)-simplex \(\ell_{\eta, \eta'}\) equals \(\ell\).

![Figure 32. Neighborly pairs \(\{\xi, \xi'\}, \{\eta, \eta'\}, \{\eta', \eta''\}\) and \(\{\zeta, \zeta'\}\) having a common designated face \(\ell\) – a local picture.](image)

Let \(M'(\ell)\) be the collection of elements in \(M'\) which contain \(\ell\), and let \(U(\ell)\) be their union. Typically \(U(\ell)\) is not an \(n\)-cell. We fix a closed neighborhood
We set $\Pi_\xi G$ by as its $K$ $X$ is denoted by $(\Omega(\ell))$

The adjacency graph of $M'$ is cyclic, and, heuristically, the $(n-2)$-simplex $\ell$ is a barrier between any two elements in $M'(\ell)$. The construction below opens up the barrier between $\xi$ and $\xi'$ by creating a clearance $D_{\xi,\xi'}$ and a passage $\delta_{\xi,\xi'}$ inside $\Omega(\ell)$, while keeping the space $\partial_\ell M' \cup \Omega(\ell) \cup \xi,\xi$ unaltered and the space $\partial_\ell M' \setminus \ell$ topologically intact. The new space replacing $\partial_\ell M'$ is denoted by $X_{\xi,\xi'}$.

We now proceed to define the space $X_{\xi,\xi'}$, a $K_{\partial_\ell M'}$-projection $\pi_{\xi,\xi'} : X_{\xi,\xi'} \to \partial_\ell M'$, and an Alexander–Rickman map $f_{\xi,\xi'} : X_{\xi,\xi'} \to \partial_\ell E$ having $f'|\partial_\ell M'$ as its $K_{\partial_\ell M'}$-model.

The open set $\Omega(\ell) \setminus (\xi \cup \xi')$ has two components. We denote their closures by $G^\pm_{\xi,\xi'}$ respectively, and observe that $G^+_{\xi,\xi'}, G^-_{\xi,\xi'}, \xi,\xi \cup \xi,\xi'$, and $G^{-\xi,\xi'} \cup \xi \cup \xi'$ are all $n$-cells.

We fix a closed PL $n$-cell $D_{\xi,\xi'}$ in $G^+_{\xi,\xi'} \cap \Omega(\ell)$ and a closed PL $(n-1)$-cell $\delta_{\xi,\xi'}$ in $D_{\xi,\xi'}$, subject to the following conditions:

1. $\alpha_{\xi,\xi'} = D_{\xi,\xi'} \cap (\xi \cup \xi')$ is an $(n-1)$-cell,
2. $l_{\xi,\xi'} = D_{\xi,\xi'} \cap \ell$ is an $(n-2)$-cell,
3. $\text{int} \delta_{\xi,\xi'} \subset \text{int} D_{\xi,\xi'}$, $\partial \delta_{\xi,\xi'} = \delta_{\xi,\xi'} \cap \partial D_{\xi,\xi'}$, and $\ell \cap \delta_{\xi,\xi'} = l_{\xi,\xi'}$, and
4. there is a homeomorphism

$$h_{\xi,\xi'} : (D_{\xi,\xi'}, \delta_{\xi,\xi'}, \alpha_{\xi,\xi'}, l_{\xi,\xi'}) \to (B^+, B^+_1, B^+_n, B^+_n)$$

of quadruples, where $B^+ = B^n(0, 1) \cap (\mathbb{R}^{n-1} \times [0, \infty))$, $B^+_1 = B^+ \cap \{0\} \times \mathbb{R}^{n-1}$, $B^+_n = B^+ \cap (\mathbb{R}^{n-1} \times \{0\})$, and $B^+_1, B^+_n = B^+_1 \cap B^+_n$.

Denote by $\beta_{\xi,\xi'} = \text{cl} (\partial D_{\xi,\xi'} \setminus \alpha_{\xi,\xi'})$, and fix an $n$-cell $V_{\xi,\xi'}$ satisfying

$$D_{\xi,\xi'} \subset \text{int} V_{\xi,\xi'} \subset V_{\xi,\xi'} \subset \Omega(\ell).$$

Let $p : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, 0)$ be the projection in $\mathbb{R}^n$, and let $\Pi_{\xi,\xi'} : M \to M$ be a map which satisfies the following conditions:

1. $\Pi$ is the identity map on $\partial (\xi \cup \xi') \setminus \text{int} (\alpha_{\xi,\xi'} \setminus l_{\xi,\xi'})$;
2. the restriction $\Pi_{\xi,\xi'}|_{\beta_{\xi,\xi'}} : \beta_{\xi,\xi'} \cup \delta_{\xi,\xi'} \to \alpha_{\xi,\xi'} \cup \xi,\xi'$ is the projection

$$x \mapsto h_{\xi,\xi'}^{-1} \circ p \circ h_{\xi,\xi'}(x);$$
3. $\Pi_{\xi,\xi'}(l_{\xi,\xi'}) = \delta_{\xi,\xi'}$, $\Pi_{\xi,\xi'}^{-1}(M \setminus l_{\xi,\xi'}) = M \setminus \delta_{\xi,\xi'}$, and $\Pi_{\xi,\xi'}^{-1}(G^+_{\xi,\xi'}) = \text{cl} (G^+_{\xi,\xi'} \setminus D_{\xi,\xi'})$;
4. the restrictions $\Pi_{\xi,\xi'}|_{\xi,\xi'}(M \setminus l_{\xi,\xi'})$, $\Pi_{\xi,\xi'}|_{\xi,\xi'}|_{G^+_{\xi,\xi'}}$, and $\Pi_{\xi,\xi'}|_{G^-_{\xi,\xi'}}$ are embeddings; and
5. $\Pi_{\xi,\xi'}|_{M \setminus V_{\xi,\xi'}}$ is the identity map.

We set

$$X_{\xi,\xi'} = \Pi_{\xi,\xi'}^{-1}(\partial_\ell M') \setminus \text{int} \delta_{\xi,\xi'}$$

and

$$\pi_{\xi,\xi'} = \Pi_{\xi,\xi'}|_{X_{\xi,\xi'}} : X_{\xi,\xi'} \to \partial_\ell M'.$$

Then $\pi_{\xi,\xi'}$ is a $K_{\partial_\ell M'}$-projection.

We define the Alexander-Rickman map $f_{\xi,\xi'} : X_{\xi,\xi'} \to \partial_\ell E$ by

$$f_{\xi,\xi'} = f' \circ \pi_{\xi,\xi'}.$$
Step 2. Alexander-Rickman map – a global construction. We repeat Step 1 for every tree \( T \) in \( \Sigma \) and for each edge \( \{ \xi, \xi' \} \) in \( T \). We require, as we may, that all \( n \)-cells \( V_{\xi, \xi'} \) are pairwise disjoint and that each \( V_{\xi, \xi'} \) meets exactly one \((n - 2)\)-simplex, namely \( \ell_{\xi, \xi'} \), in \( K_{\partial M'} \). Since for each edge \( \{ \xi, \xi' \} \), the restriction \( \Pi|_{M \setminus V_{\xi, \xi'}} \) is an identity map, we may construct passages associated to all edges of all trees in the forest \( \Sigma \) independently of each other inside mutually disjoint \( n \)-cells \( V_{\xi, \xi'} \).

Therefore the composition of the maps \( \pi_{\xi, \xi'} \) over all edges in the forest \( \Sigma \), in any order, yields a well-defined map \( \Pi: \tilde{M} \to M \). The closed set defined by

\[
X = \Pi^{-1}(\partial_{\xi} M') \setminus \bigcup_{\{\xi, \xi'\}} \text{int} \delta_{\xi, \xi'}
\]

admits a \( K_{\partial_{\xi} M'} \)-projection \( \pi = \Pi|_X: X \to \partial_{\xi} M' \), and an Alexander-Rickman map

\[
\varphi = f' \circ \pi: X \to \partial_{\xi} \mathcal{E}
\]

of rank \( p \).

Since \( \Sigma \) is an \( \{M'_1, \ldots, M'_m, \ldots\} \)-forest, the set \( M \setminus X \) has \( m \) components. The closures \( (M''_1, \ldots, M''_m) \) of these components form an essential partition \( \mathcal{M}'' \) of \( M \) satisfying \( \partial_{\xi} \mathcal{M}'' = X \). Since \( X \subset U \), we may label the elements in the partition so that \( M_i \setminus U = M''_i \setminus U \) for each \( i = 1, \ldots, m \). Furthermore, the construction of \( \Pi \) induces homeomorphisms \( \lambda_i: \text{int} M_i \to \text{int} M''_i \) which extend to maps \( \tilde{\lambda}_i: M_i \to M''_i \) for which each \( \varphi \circ \tilde{\lambda}_i|_{\partial M_i} \) is the Alexander map \( f_i \) expanded by simple covers.

Step 3. Extension from \( \partial_{\xi} M'' \) to \( M \). We now summarize the previous steps. Beginning with an Alexander manifold partition \( M \) of \( M \) associated with a \( \text{CW}_\Delta \)-structure \( K_{\partial_{\xi} M} \) on \( \partial_{\xi} M \) and an Alexander sketch \( \mathcal{F} \), we, by sphericalizing \( K_{\partial_{\xi} M} \), have constructed a refined Alexander partition \( \mathcal{M}' = (M'_1, \ldots, M'_m, M''_{m+1}, \ldots, M''_n) \) and a partial weaving \( f': M \to S^n \) of \( \mathcal{F} \). The restriction \( f'|_{\partial_{\xi} \mathcal{M}'}: \partial_{\xi} \mathcal{M}' \to \partial_{\xi} \mathcal{E} \) of the partial weaving is an Alexander map...
of a higher rank. Guided by the trees in a \{\mathcal{M}_1', \ldots, \mathcal{M}_m'\}-forest \Sigma, we have reconnected, after Steps 1 and 2, some of the elements in \mathcal{M}'. This process transforms the manifold partition \mathcal{M}' to another essential partition \mathcal{M}'' of \mathcal{M} that has the same number of elements as that in \mathcal{M}, and the map \tilde{f}'|_{\partial_n\mathcal{M}'} to an Alexander–Rickman map \varphi: \partial_n\mathcal{M}'' \to \partial_n\mathcal{E}'. Furthermore, there are homeomorphisms \lambda_i: \text{int} M_i \to \text{int} M'_i, which extend to maps \lambda_i: M_i \to M''_i, so that \varphi \circ \lambda_i|_{\partial M_i} = f_i|_{\partial M_i}, expanded by simple covers. 

To obtain a weaving \tilde{f} : \mathcal{M} \to \mathbb{S}^n from the Alexander-Rickman map \varphi, it remains to extend \varphi to a branched covering map \mathcal{M} \to \mathbb{S}^n. Since the components in \mathcal{M}'' in general are not cells, there is no obvious way to extend the map from the boundaries to the interiors. In fact, the extension of \varphi needs to be applied concurrently with the construction of the passages. We now retrieve the steps.

For \(i = 1, \ldots, m\), let \(T_i\) be the tree in \(\Sigma\) containing \(M'_i\) as a vertex. We fix an increasing sequence
\[
\{M'_i\} = T_{i,0} \subset T_{i,1} \subset \cdots \subset T_{i,s_i} = T_i
\]
of subtrees of \(T_i\) having the property that, for each \(k = 0, \ldots, s_i-1\), trees \(T_{i,k}\) and \(T_{i,k+1}\) differ only by one vertex that is a leaf of \(T_{i,k+1}\). Let \(s = \max s_i\) and set \(T_{i,k} = T_i\) for \(s_i < k \leq s\).

For each \(k = 0, \ldots, s\), let \(\Sigma_k\) be the forest that is composed of trees \(T_{i,k}\), for \(i = 1, \ldots, m\), and trees consisting of isolated vertices which are the remaining elements of \(\mathcal{M}'\). Thus, \(\Sigma_0\) consists only of isolated vertices \(\{M'_i: i = 1, \ldots, m, m+1, \ldots, m'\}\) and \(\Sigma_s = \Sigma\).

Following Steps 1 and 2 above, we find, for each \(k = 0, \ldots, s\),
\begin{enumerate}
\item[(a)] an essential partition \(\mathcal{M}'_k = (M'_1,k, \ldots, M'_{m,k}, \ldots, M'_{m(k),k})\) of \(\text{int} M\) associated to the vertices of the trees in the forest \(\Sigma_k\), where \(m(k)\) is the number of trees in \(\Sigma_k\), and
\item[(b)] an Alexander-Rickman map \(\varphi_k: \partial_n\mathcal{M}'_k \to \partial_n\mathcal{E}\).
\end{enumerate}

Since each tree \(T_{i,k+1}\) is either the tree \(T_{i,k}\), or is obtained by adding one leaf to \(T_{i,k}\) the expansion from \(M''_{i,k}\) to \(M''_{i,k+1}\) adds two \((n-1)\)-simplices to the boundary. It may further be arranged so that, for each \(k\) and each \(i \in [1, m(k)]\), there is a homeomorphism \(\lambda'_{i,k}: \text{int} M_i \to \text{int} M''_{i,k}\) which extends to a map \(\lambda'_{i,k}: M_i \to M''_{i,k}\), having the property that \(\varphi_k \circ \lambda_{i,k+1} = \text{int} M_i\) is the map \(\varphi_k \circ \lambda_{i,k}|_{\partial M_i}\) expanded by one (or none) simple cover.

Since the cells \(V_{\xi,\xi'}\), over all edges \(\{\xi, \xi'\}\) in \(\Sigma\), are mutually disjoint, the expansions may be carried out by adding free simple covers whose supports are mutually disjoint.

We proceed to the extension by induction in \(k = 0, \ldots, s\). Assume now the existence of a branched cover extension \(f''_k: \mathcal{M} \to \mathbb{S}^n\) of \(\varphi_k|_{\partial_n\mathcal{M}'_k}\) for a certain \(k\). In view of the relation between maps \(\varphi_k \circ \lambda_{i,k+1}\) and \(\varphi_k \circ \lambda_{i,k}|_{\partial M_i}\), we may, by Proposition 3.13, find, an interior branched covering map \(f''_{i,k+1}: M''_{i,k+1} \to \mathcal{E}_{\xi(i)}(i)\) which extends \(\varphi_k \circ \lambda_{i,k+1}|_{\partial_n\mathcal{M}'_{i,k+1}}\) for each \(i = 1, \ldots, m(k+1)\). Since these mappings extend the same map \(\varphi_{k+1}: \partial_n\mathcal{M}'_{k+1} \to \partial_n\mathcal{E}\), we conclude that there exists a branched covering map \(f''_{k+1}: \mathcal{M} \to \mathbb{S}^n\) which extends \(\varphi_{k+1}\) .
The mapping $f''_s : M \to S^n$ defined at the end of the induction process is the weaving which satisfies the conditions of the theorem.

**Remark 10.6.** In dimension 2, $\ell = \xi \cap \xi'$ is a single point. Thus it is not possible to build more than one passage through $\ell$. For this reason, we have assumed $n \geq 3$ in the theorem. In other words, this weaving does not exist in dimension two.

We finish this section with a remark on the metric properties of the weaving constructions. By suitable choices the map produced by the weaving can be taken to be a BLD-map. The BLD-constant, however, depends on the choices made in the construction.

**Remark 10.7.** Let the pairwise common boundary $\partial_\cup M$ be equipped with a path metric $d$ for which each $(n-1)$-simplex in $K_{\partial_\cup M}$ is isometric to a regular $(n-1)$-simplex in $\mathbb{R}^n$ of side length 1. Suppose that the metric $d$ is bilipschitz equivalent to the Riemannian metric on $M$ restricted to $\partial_\cup M$, and suppose that the branched covering maps in the Alexander sketch $\mathcal{F}$ are BLD. Then the weaving $f''_s : M \to S^n$ of $\mathcal{F}$ may also be chosen to be a BLD branched covering map.

Indeed, each domain $M'_i$ may be chosen to be bilipschitz to $M_i$ with a constant depending only on the numbers $m$ and $p$, the complex $K_{\partial_\cup M}$ (more precisely, the lengths of the trees in the forest $\Sigma$), and the choice of cells $V_{\xi,\xi'}$, $D_{\xi,\xi'}$ and $\delta_{\xi,\xi'}$ for the edges $\{\xi,\xi'\}$ in the forest $\Sigma$. Thus in order to obtain an $L$-BLD weaving, we need to choose, for each $k$ and $i$, a BLD-homotopy from $\varphi_{k+1} \circ \lambda_{i,k+1}|_{\partial M_i}$ to an expansion of $\varphi_k \circ \lambda_{i,k}|_{\partial M_i}$ by a simple cover.

11. Berstein–Edmonds type extension theorems

In this section we show that the weaving theorem (Theorem 10.5) yields a dimension independent theorem on the existence of branched covers between manifolds, in the spirit of a 3-dimensional result of Berstein and Edmonds [5]. See also Heinonen and Rickman [25, Theorem 0.3] and [37, Theorem 3.1] for other generalizations of the result of Berstein and Edmonds.

In this existence result, on the domain side of the map we have a $n$-manifold with $m$ boundary components, and on the target side we have a $n$-sphere with $p$ Euclidean balls removed; here $n \geq 3$ and $2 \leq p \leq m$. The novelty here is that the numbers $m$ and $p$ need not be the same and that the boundary components may be arbitrarily pre-matched.

In this dimension free construction, the topology does not have a role. We replace a handlebody decomposition in the proof of Berstein and Edmonds [5] by working directly with a simplicial structure. As discussed in the introduction, the method of Berstein and Edmonds in dimension two is more robust and yields extension for all branched covers. We are not aware of its higher dimensional counterpart.

**Theorem 1.5** (Branched cover realization of boundary assignments). Let $n \geq 3$, $m \geq p \geq 2$, and let $c: \{1, \ldots, m\} \to \{1, \ldots, p\}$ be a surjection. Suppose that $M$ is an oriented compact PL $n$-manifold having boundary components $\Sigma_1, \ldots, \Sigma_m$, and $N = S \setminus \text{int} (B_1 \cup \cdots \cup B_p)$ is a PL $n$-sphere $S$ with $p$ pairwise disjoint closed $n$-cells $B_1, \ldots, B_p$ removed.
Then there exist a simplicial complex $K$ on $\partial M$ and a branched covering map $f: M \to N$ for which the image $f(B_f)$ of the branch set $B_f$ is an $(n-2)$-sphere, and each restriction $f|_{\Sigma_i}: \Sigma_i \to \partial B_{c(i)}$ is an Alexander map expanded by free simple covers.

**Proof.** For the use of the weaving theorem (Theorem 10.5), we fix pairwise disjoint collars $M_1, \ldots, M_m$ for the boundary components $\Sigma_1, \ldots, \Sigma_m$, respectively. Precisely, for each $i = 1, \ldots, m$, $M_i \subset M$ is an $n$-manifold, PL homeomorphic to $\Sigma_i \times [0,1]$ and having one boundary component $\Sigma_i$ and the other, $\Sigma'_i$, contained in the interior of $M$. Let $M'$ be the closure of $M \setminus (M_1 \cup \cdots \cup M_m)$ in $M$.

We fix a simplicial structure $Q$ on $M$ for which $M'$ is the space of a subcomplex, and let $M_{m+1}, \ldots, M_q$ be the collection of $n$-simplices in $M'$. Then

$$ \mathcal{M} = \{M_1, \ldots, M_m, M_{m+1}, \ldots, M_q\} $$

a manifold partition of $M$, whose pairwise common boundary $\partial_{i,j}\mathcal{M}$ is the union

$$ \partial M_{m+1} \cup \cdots \cup \partial M_q. $$

Note that

$$ \mathcal{M}' = \{M_{m+1}, \ldots, M_q\} $$

is a manifold partition of $M'$.

To construct the branched cover $f: M \to N$ in the theorem, we first fix an Alexander sketch.

On the target side, let $X$ be a branched $(n-1)$-sphere of rank $p + 1$ in the interior of $N$, positioned in such a way that the essential partition $E = (E_1, \ldots, E_{p+1})$ of $S$ induced by the branched sphere $X$ is PL and admits a labeling satisfying $B_j \subset \text{int} E_j$ for $j = 1, \ldots, p$. Let $K_X$ be the standard complex on the branched sphere $X$. So the restriction $K_X|_{\partial E_j}$ is isomorphic to $K_{S_{n-1}}$ for each $j = 1, \ldots, p + 1$.

On the domain side, by passing to a subdivision of $Q|M'$, we fix a simplicial complex $R$ on $M$ with the property that $R|_{M'}$ is a simplicial subcomplex, $R|_{\partial M}$ admits an Alexander map for each $i = m + 1, \ldots, q$, and $R|_{\Sigma_i}$ is isomorphic to $R|_{\Sigma'_i}$ for $i = 1, \ldots, m$.

Fix, for each $i = 1, \ldots, m$, a product map $f_i: M_i \to E_{c(i)} \setminus \text{int} B_{c(i)}$ for which the restriction $f_i|_{\Sigma_i}: \Sigma_i \to \partial B_{c(i)}$ is an $R|_{\Sigma'_i}$-Alexander map, and he restriction $f_i|_{\Sigma'_i}: \Sigma'_i \to \partial B_{c(i)}$ is an $R|_{\Sigma'_i}$-Alexander map. For each $i = m + 1, \ldots, q$, we set $f_i: M_i \to E_{p+1}$ to be a branched cover for which $f_i|_{\partial M_i}: \partial M_i \to \partial E_{p+1}$ is an $R|_{\partial M_i}$-Alexander map.

We may further assume that maps $f_1, \ldots, f_q$ are orientation preserving and that they agree on the $(n-2)$-skeleton of $R_{\partial_i\mathcal{M}}$. Hence

$$ \mathcal{F} = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_q\} $$

is an Alexander sketch on $\mathcal{M}$. We extend the surjection $c$ to a function $c: \{1, \ldots, q\} \to \{1, \ldots, p + 1\}$ by setting $c(i) = p + 1$ for $m + 1 \leq i \leq q$.

Observe that the maps in $\mathcal{F}$ need not agree on their common boundaries and that even if they do, the map obtained by naively gluing them together need not be open. Sphericalization and partial weaving are used below to target these issues.
Indeed, since \( \partial \mathcal{M} \) is simplicially connected, the weaving theorem (Theorem 10.5) yields an essential partition \( \mathcal{M}' = \{ \mathcal{M}_1', \ldots, \mathcal{M}_q' \} \) of \( \mathcal{M} \), a branched covering map \( f: \mathcal{M} \to \mathbb{S}^n \) and, for each \( i = 1, \ldots, q \), a map \( \lambda_i: \mathcal{M}_i \to \mathcal{M}_i' \) which is a homeomorphism \( \text{int}\mathcal{M}_i \to \text{int}\mathcal{M}_i' \), satisfying the following properties:

1. \( f|_{\partial \mathcal{M}'} \) is an Alexander-Rickman map,
2. for \( i = 1, \ldots, m \), \( f|_{\Sigma_i} = f_i|_{\Sigma_i} \), and \( f \circ \lambda_i|_{\Sigma_i'} \) is \( f_i|_{\Sigma_i'} \) expanded by simple covers, and
3. for \( i = m + 1, \ldots, q \), \( f \circ \lambda_i|_{\partial \mathcal{M}_i} \) is \( f_i|_{\partial \mathcal{M}_i} \) expanded by simple covers.

Since the collar \( M_i'' \) of \( \Sigma_i \) is homeomorphic to \( \Sigma_i \times [0, 1] \) and \( E_c(i) \setminus \text{int}B_{c(i)} \) is homeomorphic to \( \partial B_{c(i)} \times [0, 1] \) for \( 1 \leq i \leq q \), we may replace the branched covering map \( f|_{M_i''}: M_i'' \to E_c(i) \setminus \text{int}B_{c(i)} \) by a product map \( f|_{M_i''}: M_i'' \to E_c(i) \setminus \text{int}B_{c(i)} \) with the property that \( f|_{\Sigma_i}: \Sigma_i \to \partial B_{c(i)} \) is an \( R|_{\Sigma_i} \)-Alexander map expanded by simple covers.

Let \( K = R|_{\partial \mathcal{M}} \). The claim follows. \( \square \)

**Remark 11.1.** The degree \( \deg f \) of the map \( f \) in Theorem 1.5 satisfies, for each \( j = 1, \ldots, p \), the condition

\[
\deg f = \sum_{c(i) = j} \deg (f|_{\Sigma_i}: \Sigma_i \to \partial B_j).
\]

For manifolds, which are cubical complexes, we have also a version of Bernstein-Edmonds extension theorem for all dimensions. Heuristically, it asserts that, after a suitable collection of simple covers being added, we may extend an Alexander map on \( \partial \mathcal{M} \), associated to the cubical structure, to a branched cover on the entire manifold \( \mathcal{M} \).

**Theorem 1.6** (Branched cover extension for stabilized Alexander maps). Let \( n \geq 3 \), \( m \geq p \geq 2 \), and let \( c: \{1, \ldots, m\} \to \{1, \ldots, p\} \) be a surjection. Let \( K \) be a cubical complex for which \( M = |K| \) is an \( n \)-manifold with boundary components \( \Sigma_1, \ldots, \Sigma_m \), and let \( N = S \setminus \text{int}(B_1 \cup \cdots \cup B_p) \) be a PL \( n \)-sphere with \( p \) mutually disjoint \( n \)-cells \( B_1, \ldots, B_p \) removed.

For each \( i = 1, \ldots, m \), let \( g_i: \Sigma_i \to \partial B_{c(i)} \) be an orientation preserving \( (K|_{\Sigma_i})^\Delta \)-Alexander map. Then there exists a branched covering map \( g: \mathcal{M} \to N \) for which the image \( g(B_g) \) of the branch set \( B_g \) is an \( (n-2) \)-sphere, and each restriction \( g|_{\Sigma_i}: \Sigma_i \to \partial B_{c(i)} \) is the map \( g_i \) expanded by free simple covers.

**Proof.** We note first that the Alexander maps \( g_i \) are uniquely determined, topologically, by the surjection \( c \), the complex \( K \), and the fact that they preserve the orientation.

We assume as we may that there exist pairwise disjoint subcomplexes \( K_1, \ldots, K_m \) which are products \( K|_{\Sigma_i} \times [0, 1] \), \( K|_{\Sigma_m} \times [0, 1] \) and whose spaces \( M_i = |K_i| \) are collars of the boundary components.

Since \( K \) is a cubical complex, its canonical triangulation \( K^\Delta \) has the property that, for each \( \sigma \in K^{(n)} \), the restriction \( K^\Delta|_{\partial \sigma} \) admits an Alexander map. Let \( K' \) be the subcomplex of \( K \) which is obtained by removing the collars \( K_1, \ldots, K_m \), and label the \( n \)-cubes in \( (K')^{(n)} \) as \( M_{m+1}, \ldots, M_q \). We now follow the proof of Theorem 1.5 with \( K^\Delta \) in place of the simplicial complex \( R \). This argument yields a new essential partition \( \mathcal{M}' = \{ M_1', \ldots, M_q' \} \).
of $M$, together with maps $\lambda_i: M_i \to M''_i$, and a branched cover $f'': M \to N$ with the property that, on the boundary component $\Sigma'_i$ of $M_i$ which is not $\Sigma_i$, the map $f'' \circ \lambda_i|_{\Sigma'_i}: \Sigma'_i \to \partial E_{c(i)}$ is $K^\Delta$-Alexander expanded by free simple covers.

For each $i = 1, \ldots, m$, the restrictions of the cubical structure $K$ on the two boundary components of $M_i$ are isomorphic. Hence we may extend $f''|_{\bigcup_{i=m+1}^qM}$ to the map $g: M \to N$ claimed in the theorem, by employing product maps $M_i \to \text{cl}(E_{c(i)} \setminus B_{c(i)})$ for all $i = 1, \ldots, m$. □

Remark 11.2. Branched covering maps $f: M \to N$ in Theorems 1.5 and 1.6 can be chosen to be a BLD-map. However, as discussed in Remark 10.7, the BLD-constant depends on the choices made in the proof.

Since the choices depends in an essential way on the the collars $M_1, \ldots, M_m$ of the boundary components and the submanifold $M'$ outside these collars, we study, in Section 16, quasiregular branched covers with uniformly controlled distortion under the assumption that $M'$ is degenerate.

Part 3. Quasiregular Extension

In this part, we introduce a technique for building quasiregular extensions of the so-called two-level Alexander maps. We apply this technique to iteratively construct quasiregular mappings having a large local index in a Cantor set; see Theorems 1.8 and 1.9 in Part 4.

The results in this part are akin to a quasiregular extension theorem for Alexander maps over dented molecules proved as a part of [14, Proposition 5.25].

Theorem. Let $\mathcal{D}$ be a dented $n$-molecule and $f: \partial|\mathcal{D}| \to \mathbb{S}^{n-1}$ a standard Alexander map of any degree. Then $f$ has a $K_{\mathcal{D}}$-quasiregular extension

$$F: |\mathcal{D}| \to \mathbb{B}^n$$

over the dented molecule $\mathcal{D}$, where $K_{\mathcal{D}} \geq 1$ depends only on the dimension $n$ and the atom length in the dented molecule $\mathcal{D}$.

The strategy in [14] is to prove that the space $|\mathcal{D}|$ of a dented molecule $\mathcal{D}$ is bilipschitz homeomorphic to a Euclidean cube $Q = [0,1]^n$ through a homeomorphism $g: |\mathcal{D}| \to Q$ whose bilipschitz constant depends only on $n$ and the geometry of $|\mathcal{D}|$. Since the Alexander map $f$ on $|\mathcal{D}|$ is standard, $f \circ g^{-1}|_{|\mathcal{D}|}$ is bilipschitz equivalent, with a bilipschitz constant depending only on the dimension $n$, to the restriction of a power map $p: \mathbb{R}^n \to \mathbb{R}^n$ on $\partial Q$. See [49, I.3.2] for the description of quasiregular power maps in $\mathbb{R}^n$. The map $F = p \circ g$ is the desired extension.

We adopt here a slightly simplified and slightly more general definition for dented molecules than in [14]. A cubical $n$-complex $A$ is an atom if its space $|A|$ is an $n$-cell and the adjacency graph of the $n$-cubes in $A$ is a tree. A molecule $M$ is a complex whose space admits an essential partition into spaces each of which supports an atom, and a dented molecule $\mathcal{D}$ is a complex which is obtained from a molecule by adding and removing smaller molecules; see Figure 34 for an illustration. These complexes are geometric cells in the sense that spaces of atoms, molecules, and dented molecules
are John domains with a John constant depending only on the data of the dented molecule.

The extension theorem in [14] mentioned above is not sufficient for establishing the Mixing Theorem (Theorem 1.7) and its applications (Theorems 1.8 and 1.9). To achieve this, we need a quantitative extension theorem for two-level Alexander maps

\[ \partial|\mathcal{D}| \to (S^{n-1} \times \{1, R\}) \cup (S^{n-2} \times [1, R]), \]

each of which is Alexander on two separate \((n-1)\)-cells on \(\partial|\mathcal{D}|\), and is radial on the remaining part of \(\partial|\mathcal{D}|\). Here, \(R\) is comparable to the side length of a maximal cube in \(\mathcal{D}\). The extension we obtain is a BLD-map \(|\mathcal{D}| \to S^{n-1} \times [1, R]\) with a constant depending only on the data of the dented molecule \(\mathcal{D}\).

In the last section of this part, we discuss the notion of separating complex, and prove the Mixing Theorem using the extension theorems for two-level Alexander maps.

12. Metric complexes and quasiregular maps

In this section, we discuss the notion of metric complexes and the so-called standard representatives of Alexander maps and simple covers.

12.1. Length metric on cubical complexes. Given a cubical \(n\)-complex \(K\), we define a metric \(d_K\) on \(|K|\) as in [21] Section 1.B. Suppose \(|K|\) is connected and let \(\text{Paths}(|K|)\) be the collection of all paths in \(|K|\). Fix, for each \(n\)-cube \(Q\) in \(K\), a simplicial homeomorphism \(\iota_Q: Q \to [0, 1]^n\) such that the maps \(\iota_{Q'} \circ \iota_Q^{-1}|_{\iota_{Q''}(Q' \cap Q'')}\) are isometries for all adjacent \(Q'\) and \(Q''\) in \(K^{(n)}\).

We define length \(\ell_K: \text{Paths}(|K|) \to [0, \infty]\) by taking \(\ell_K(\gamma)\) to be the supremum of

\[ \sum_{i=1}^k |\iota_{Q_i}(\gamma(t_i)) - \iota_{Q_i}(\gamma(t_{i-1}))| \]

over all partitions \(\gamma(t_0), \ldots, \gamma(t_k)\) of path \(\gamma\) with the condition that \(\gamma(t_{i-1})\) and \(\gamma(t_i)\) belong to the same \(Q_i\).

Let now \(d_K\) be the length metric associated to \(\ell_K\), that is, for any \(x, y \in |K|\),

\[ d_K(x, y) = \inf_{\gamma} \ell_K(\gamma), \]
where \( \gamma \) is a path connecting \( x \) to \( y \) in \( |K| \). Note that, in the length metric \( d_K \), all \( n \)-cubes are isometric to the Euclidean unit cube \([0, 1]^n\) through the fixed homeomorphisms \([0, 1]^n \to Q \) for \( Q \in K^{(n)} \) in the definition of \( d_K \).

For Cartesian cubical complexes, i.e. cubical complexes \( K \) for which \( |K| \subset \mathbb{R}^n \) and whose \( n \)-cubes are translates of \([0, 1]^n\), the metric \( d_K \) is the standard Euclidean path metric.

**Convention 12.1** (Metrics). Let \( K \) be a cubical \( n \)-complex. Unless otherwise mentioned, we assume that the space \( |K| \) is provided with the length metric \( d_K \). We assume also that the sphere \( \mathbb{S}^n \) is endowed with the ambient Euclidean metric in \( \mathbb{R}^{n+1} \).

**12.2. Metric refinements.** We first discuss the underlying cubical complexes of the Alexander maps and the metric refinements.

For the definition of \((1/3)-\)refinement of a complex, we fix a Euclidean model. Let \( R_\odot \) be the cubical complex obtained from the subdivision of \([0, 1]^n \) into Euclidean \( n \)-cubes of side length \( 1/3 \), i.e. the cubical complex in \( \mathbb{R}^n \) having \( n \)-cubes \( \{v + [0, 1/3]^n : v \in \{0, 1/3, 2/3\}^n\} \).

**Definition 12.2.** A complex \( \text{Ref}(K) \) is a \((1/3)-\)refinement of the cubical \( n \)-complex \( K \) if, for each \( n \)-cube \( Q \subset K \), the complex \( \text{Ref}(K)|_Q \) is a well-defined cubical complex isometrically isomorphic to \( R_\odot \), that is, there exists an isometry \( \iota : [0, 1]^n \to Q \) for which \( \text{Ref}(K)|_Q = \iota(R_\odot) \).

We set \( \text{Ref}_0(K) = K \), and call the complex
\[
\text{Ref}_k(K) = \text{Ref}(\text{Ref}_{k-1}(K)) \quad \text{for } k \geq 1,
\]
the \((1/3^k)-\)refinement of \( K \); and we call \( k \) the refinement index of \( P = \text{Ref}_k(K) \) in \( K \) and write \( k = \rho_K(P) \).

**Remark 12.3.** We observe that, for each \( k \in \mathbb{Z}_+ \), we have \( |K| = |\text{Ref}_k(K)| \), and that the identity map
\[
id : (K, d_K) \to (\text{Ref}_k(K), d_{\text{Ref}_k(K)})
\]

is a similarity which scales distances by the factor \( 3^k \).

We now introduce the notions of the core and the buffer of a cubical complex.

**Definition 12.4.** An \( n \)-cube \( Q \subset K \) is a core cube of \( K \) if \( Q \cap \partial|K| = \emptyset \). The minimal subcomplex \( \text{Core}(K) \) of \( K \) containing all core cubes of \( K \) is the core of \( K \). The minimal subcomplex of \( K \) containing all cubes not in the core of \( K \) is the buffer \( \text{Buffer}(K) \) of \( K \).

For each \( k \geq 1 \), we denote also
\[
\text{Core}_k(K) = \text{Core}(\text{Core}_{k-1}(K)),
\]
where \( \text{Core}_0(K) = K \). Similarly, for each \( k \geq 1 \), let
\[
\text{Buffer}_k(K) = \text{Buffer}(\text{Core}_{k-1}(K))
\]
be the \( k \)-th buffer layer of \( K \).

For all cubical \( n \)-complexes \( K \), the space \( |\text{Core}(\text{Ref}(K))| \) is homeomorphic to \( |K| \) and \( |\text{Buffer}(\text{Ref}(K))| \) is a collar of \( \partial|K| \) in \( |K| \); see Figure 35 for an illustration.
Given a cube $Q$, we call $c(Q) = \text{Core(Ref}(Q))$ the center cube of $Q$, and the set $w(Q) = \text{cl}(Q \setminus c(Q)) = |\text{Buffer(Ref}(Q))|$ the rim of cube $Q$.

12.3. Quasiregular maps on cubical complexes. A homeomorphism $f: X \to Y$ between metric spaces is $(\lambda, L)$-quasi-similar if there exist constants $\lambda > 0$ and $L \geq 1$ such that

$$\frac{\lambda}{L} d(x, x') \leq d(f(x), f(x')) \leq L\lambda d(x, x')$$

for all $x, x' \in X$; in this case, we say $Y$ is $(\lambda, L)$-quasi-similar map to $X$. In particular, an $(1, L)$-quasi-similar map is $L$-bilipschitz.

We also recall from the introduction that a discrete and open mapping $f: X \to Y$ between metric spaces is a mapping of $L$-bounded length distortion (or $L$-BLD for short) if

$$\frac{1}{L} \ell(\gamma) \leq \ell(f \circ \gamma) \leq L\ell(\gamma)$$

for each path $\gamma$ in $X$, where $\ell(\cdot)$ is the length of a path. Note that, in particular, BLD-mappings are branched covering maps.

A continuous map $f: M \to N$ between oriented Riemannian $n$-manifolds is $K$-quasiregular for $K \geq 1$ if $f$ is in the Sobolev space $W^{1,n}_{\text{loc}}(M, N)$ and satisfies the quasiconformality condition

$$\|Df\| \leq KJ_f$$

almost everywhere in $M$, where $\|Df\|$ is the operator norm of the weak differential $Df$ of $f$ and $J_f$ is the Jacobian determinant of $f$. In particular, BLD-mappings between Riemannian manifolds are quasiregular. We refer to Rickman’s book [49] for the local theory of quasiregular mappings, and to Martio–Väisälä [33] and Heinonen–Rickman [25] for discussion of a relation between BLD-maps and quasiregular mappings.

In forthcoming sections, we consider extensions of Alexander mappings which are quasiregular and locally BLD-mappings. In fact, locally these maps are uniformly BLD with a constant depending only on the data if we admit a re-scaling of the metric in the target. Thus these extensions are quasiregular mappings with a uniform distortion constant depending only on the data.

A typical setting for our considerations is a cubical complex $K$ embedded in a Riemannian manifold $(M, g)$, which in most cases is the Euclidean space $\mathbb{R}^n$. Unless explicitly stated, we tacitly assume that the metric $d_K$ of the
space $|K|$ of the cubical complex $K$ is locally uniformly bilipschitz equivalent to the length metric $d_g$ induced by the Riemannian metric $g$ on $M$.

For such an embedded metric complex $K$, we say that a continuous map $f: |K| \to \mathbb{S}^n$ is $K$-quasiregular if $f$ has an extension to a neighborhood of $|K|$ which is $K$-quasiregular in the Riemannian metric of $M$. We also say that a continuous map $f: |K| \to \mathbb{S}^n$ is interior $K$-quasiregular if $f|_{\text{int}|K|}$ is $K$-quasiregular with respect to the Riemannian metric of $M$. Note that, by Reshetnyak’s theorem [49, Theorem I.4.1], an interior $K$-quasiregular mapping is an interior branched covering map.

12.4. Standard expansions of Alexander maps. In this section, $(K, d_K)$ is a cubical $n$-complex $K$ with a length metric $d_K$, and $(E_0, E_\circ, e_\circ; \rho_\circ)$ is the reference package in $\mathbb{R}^n$ in the definition of cell-packages; see Definition 3.3.

Definition 12.5. Let $(K, d_K)$ be a cubical $n$-complex, and $\mathcal{E} = (\tilde{E}, E, e; \rho)$ a cell-package in $|K|$. The package $\mathcal{E}$ is called a standard cell-package if its support $\tilde{E}$ is contained in the interior of an $n$-simplex in $K^\Delta$ and there exists a similarity map $\theta_\Delta: \tilde{E} \to E_\circ$ for which $\rho = \theta_\Delta^{-1} \circ \rho_\circ \circ \theta_\Delta$.

As a reference, we fix a BLD simple cover

$$f_\circ: E_\circ \to \mathbb{S}^n$$

which maps $\partial E_\circ \cup e_\circ$ onto $S^{n-1} \subset \mathbb{S}^n$, and $\text{int} E_\circ \cap (\mathbb{R}^{n-1} \times (0, \infty))$ onto $\mathbb{S}^n \cap (\mathbb{R}^n \times (0, \infty))$. Fix a constant $L_\circ \geq 1$ so that $f_\circ$ is $L_\circ$-BLD.

Definition 12.6. Let $f: |K| \to \mathbb{S}^n$ be a $K^\Delta$-Alexander map and let $\mathcal{E} = (\tilde{E}, E, e; \rho)$ be a standard cell-package in $K$. A simple cover $f|_E$ associated to the package $\mathcal{E}$ is a standard simple cover if there exists a similarity map $\iota: \tilde{E} \to E_\circ$ and a M"obius transformation $g: \mathbb{S}^n \to \mathbb{S}^n$ for which $\iota(E) = E_\circ$ and $f|_E = g \circ f_\circ \circ \iota|_E$.

We may define the standard expansion of an Alexander map as follows.

Definition 12.7. A map $f: |K| \to \mathbb{S}^n$ is a standard expansion of an Alexander map $\hat{f}: |K| \to \mathbb{S}^n$, if $\hat{f}$ is $L_0$-BLD, and $f$ is an expansion of $\hat{f}$ by standard simple covers $f|_{E_1}, \ldots, f|_{E_m}$.

By reparametrizing an Alexander map $f: |K| \to \mathbb{S}^n$ in each simplex of $K^\Delta$ if necessary, we may assume that $f$ is $L_0$-BLD with a constant $L_0 = L_0(n) \geq 1$. We formulate this observation as follows.

Lemma 12.8. For each $n \geq 2$, there exists a constant $L_0 = L_0(n) \geq 1$ with the property that, for any cubical $n$-complex $(K, d_K)$ and any $K^\Delta$-Alexander map $f: |K| \to \mathbb{S}^n$, there exists a $K^\Delta$-Alexander map $f': (|K|, d_K) \to \mathbb{S}^n$ which is $L_0$-BLD and satisfies $f'|(\sigma) = f(\sigma)$ for all $\sigma \in K$.

Each expansion of an Alexander map by free simple covers is branched cover homotopic to a quasiregular map, which is a standard expansion of an Alexander map.

Lemma 12.9. Let $\hat{f}: |K| \to \mathbb{S}^n$ be an Alexander map and $f: |K| \to \mathbb{S}^n$ be an expansion of $\hat{f}$ by mutually disjoint free simple covers. Then there exists $K_0 = K_0(n) \geq 1$ and a $K_0$-quasiregular mapping $f': |K| \to \mathbb{S}^n$, branched
cover homotopic to \( f \), which is a standard expansion of an Alexander map \( |K| \to S^n \).

Proof. Since \( \hat{f} \) is a branched cover homotopic to an Alexander map \( \hat{f}' \) which is affine on \( n \)-simplices, and this branched cover homotopy may be expanded to a branched cover homotopy from \( f \) to an expansion of \( \hat{f}' \) by mutually disjoint free simple covers, we may assume that \( \hat{f} \) is already affine on \( n \)-simplices of \( K^\Delta \).

Let \( m = \deg f - \deg \hat{f} \). Let \( \{(E_i, E_i', e_i; \rho_i)\}_{i=1,\ldots,m} \) be a cell-packages associated to simple covers \( f|_{E_i} \) of \( f \). We fix a family \( \{(E_i', E_i'; e_i'; \rho_i')\}_{i=1,\ldots,m} \) of mutually disjoint standard cell-packages. Then by moving simple covers \( f|_{E_i} \) one by one, we obtain a branched covering map \( f' : |K| \to S^n \) which is a standard expansion of \( f \). The claim follows.

**Convention 12.10 (Standard maps).** Let \( K \) be a cubical \( n \)-complex \( K \) with a length metric \( d_K \). We assume, from now on, that all \( K^\Delta \)-Alexander maps are \( L_0 \)-BLD, all simple covers are standard, and all expansions of Alexander maps are standard and \( K_0 \)-quasiregular.

13. Atoms and molecules

In this section, we discuss the quasiregular extension of Alexander maps over geometric \( n \)-cells, called atoms and molecules.

13.1. Atoms and molecules. We first introduce the geometric cells called atoms.

**Definition 13.1.** A finite \( n \)-dimensional cubical complex \( A \) is an \( n \)-atom if \( |A| \) is an \( n \)-cell, and its adjacency graph \( \Gamma(A) \) is a tree. A subcomplex \( A' \subset A \) is a subatom of \( A \) if \( A' \) is an atom.

Note that under a length metric \( d_A \), all \( n \)-cubes in an atom \( A \) are isometric to a Euclidean unit cube.

A molecule \( M \) is a cubical complex which has a natural decomposition into atoms; the cubes in different atoms are allowed to have different sizes.

**Definition 13.2.** A cubical \( n \)-complex \( M \) is a molecule if its space \( |M| \) is an \( n \)-cell and if there is a collection \( \mathcal{A}_M = \{(A, d_A)\} \) of atoms with the following properties:

1. \( \{|A|: A \in \mathcal{A}_M\} \) is an essential partition of \( |M| \).
2. For each \( A \in \mathcal{A}_M \), there exists an integer \( \rho_M(A) \geq 0 \) for which \( M|_{|A|} = \text{Ref}_{\rho_M(A)}(A) \) and \( d_M|_{|A|} = 3^{\rho_M(A)} d_A \).
3. Two atoms \( A \) and \( A' \) in \( \mathcal{A}_M \) may meet only if they have different refinement indices \( \rho_M(A') \) and \( \rho_M(A) \). If \( A \) and \( A' \) meet and \( \rho_M(A') < \rho_M(A) \), then \( |A| \cap |A'| \) is an \( (n-1) \)-face of \( A' \), and is not a face of \( A \) but an \( (n-1) \)-cube in an iterated 1/3-refinement of an \( (n-1) \)-cube in \( A \).
4. Let \( A \in \mathcal{A}_M \), \( Q \in A \), and let \( q \) be a face of \( Q \). Then there is at most one atom in \( \mathcal{A}_M \setminus \{A\} \) that meets \( q \).
5. There is a unique atom \( A^+ \) in \( \mathcal{A}_M \) which has the largest refinement index, and atom \( A^+ \) contains at least one \( (n-1) \)-cube \( q^+ \) which is a face of an \( n \)-cube \( Q^+ \in A^+ \) for which \( q^+ \subset \partial |M| \).
We fix a triple \((A^+_M, Q^+_M, q^+_M)\) which satisfies the property \([5]\) in the definition, and call the components, respectively, the leading atom, the leading cube and the leading face of the molecule. We also denote by

\[ \partial_-|M| = \text{cl}(\partial|M| \setminus q^+_M) \]

the boundary of the molecule with the leading face removed.

The collection \(A_M\) is uniquely determined by the properties of the molecule, and it is finite by \([3]\) and \([5]\).

From the definition of a molecule, it follows that the adjacency graph \(\Gamma(Q_M)\) for the family of \(n\)-cubes in the atoms of a molecule \(M\)

\[ Q_M = \{ Q \in A^{(n)} : A \in A_M \} \]

has a tree structure. Here we say that cubes \(Q\) and \(Q'\) are adjacent if \(Q \cap Q'\) is a face of either \(Q\) or \(Q'\).

The tree \(\Gamma(Q_M)\) induces a (unique) partial ordering \(\ll_M\) on \(Q_M\) by taking the leading cube \(n\)-cube \(Q^+_M\) of the molecule, in Definition \([13.2]\), as the maximal element.

In this hierarchy, each atom \(A \neq A^+_M\) is attached to a unique atom \(A' \in A_M\) through a face \(q^+_A\) of a cube \(Q^+_A \in A^{(n)}\); we call \(Q^+_A\) and \(q^+_A\) the leading cube and the leading face of the atom \(A\), respectively. In this case, the face \(q'\) of \(Q'\) in \(A'\) that meets \(Q^+_A\) is called a back face of the second kind of \(A'\) (or of \(Q'\)). For completeness, we also call \(Q^+_M\) and \(q^+_M\) the leading cube and the leading face of \(A^+_M\), respectively.

Furthermore, for each \(A \in Q_M\), every \(n\)-cube \(Q \neq Q^+_A\) in \(A\) is adjacent to a unique \(Q' \in A^{(n)}\) satisfying \(Q \ll_M Q'\). In this case, \(q'^+_Q = |Q| \cap |Q'|\) is a face of both cubes, and we call it the leading face of \(Q\) and a back face of the first kind of \(Q'\). Observe that a cube may have several back faces of the first kind.

From now on, we denote by \(Q^+_Q\) the unique leading face of an \(n\)-cube \(Q \in Q_M\). Note that the faces of a cube \(Q \in Q_M\) belong to some or all of the following categories:

(i) the leading face \(Q^+_Q\);
(ii) exterior faces that are contained entirely in \(\partial|M|\);
(iii) back faces of the first kind, to each of which a cube from the same atom is attached; and
(iv) back faces of the second kind, to each of which an atom is attached.

### 13.2. Geometry of molecules

Let \(M\) be a molecule. We denote by \(\ell(M)\) the maximum atom length in \(A_M\), that is,

\[ \ell(M) = \max\{#A^{(n)} : A \in A_M\} \]

and by \(\varrho(M)\) the maximum refinement index in \(A_M\), that is,

\[ \varrho(M) = \max\{\rho_M(A) : A \in A_M\} \]

**Remark 13.3.** The maximum atom length \(\ell(M)\) prescribes the geometry of the molecule \(M\). The number \(3^{\ell(M)}\) captures the size of \(|M|\). The number \(3^{(n-1)\varrho(M)}\) quantifies the degree of an Alexander map \((M|_{\partial|M|})^\Delta \to \mathbb{R}^{n-1}\).
For each $Q \in \mathcal{Q}_M$, we define the tail complex $\tau_M(Q)$ of $Q$ in $M$ to be the subcomplex of $M$ whose space $|\tau_M(Q)|$ is the union of $Q$ and those $n$-cubes $Q'$ in $\mathcal{Q}_M$ satisfying $Q' \ll_M Q$.

Remark 13.4. Since there is at most one atom attached to a face of a cube in $\mathcal{Q}_M$, there exists a number $\beta = \beta(n, \ell(M))$ satisfying

$$\frac{\#(M|_{\partial \tau_M(Q)}\cap \partial M)^{(n-1)}}{\#(M|_{\tau_M(Q)}^{n-1})} \leq \beta,$$

for each $Q \in \mathcal{Q}_M$. In particular, the number $\beta$ is independent of $q(M)$.

Define next, for each $Q$ in $\mathcal{Q}_M$, an expansion index $\nu(q_Q^\pm)$ on its leading face $q_Q^\pm$ by

$$\nu(q_Q^\pm) = \# \left( (M|_{\partial \tau_M(Q)}\cap \partial M)^\Delta \right)^{(n-1)} - \# \left( (M|_{\tau_M(Q)}^{n-1})^\Delta \right)^{(n-1)}.$$

Remark 13.5. Note that $\# \left( (M|_{\tau_M(Q)}^{n-1})^\Delta \right)^{(n-1)} = \# \left( (\text{Ref}(M)|_{c(q_Q^\pm)}^\Delta \right)^{(n-1)}$. Thus the number $\nu(q_Q^\pm)$ may be interpreted as the difference of the degree of an orientation preserving $(M|_{\partial \tau_M(Q)}\cap \partial M)^\Delta$-Alexander map on $\partial M$ and that of an orientation preserving $(\text{Ref}(M)|_{c(q_Q^\pm)}^\Delta$-Alexander map on $|c(q_Q^\pm)|$.

13.3. Properly located simple covers. For a systematical placement of simple covers in molecules, we introduce the notion of properly located simple covers.

Let $M$ be an $n$-molecule with a length metric $d_M$. Let $\mu$ be a fixed large multiple of $n 3^{n-1} \ell(M)$ and let $c_0 = \frac{1}{10} \mu^{1/(n-1)}$. We fix in the unit cube $[0,1]^{n-1}$ a collection of $(n-1)$-balls

$$\mathcal{B} = \{B_1, \ldots, B_\mu\},$$

contained in $[\frac{1}{10}, \frac{9}{10}]^{n-1}$, of radius $c_0$ each, and having pairwise distance greater than $c_0$, to be used as a template. Also for each $q \in M^{(n-1)}$, we fix a copy of $\mathcal{B}_q$ of $\mathcal{B}$ identified by an isometry $q \to [0,1]^{n-1}$.

Definition 13.6. Let $M$ be an $n$-molecule with an atom decomposition $\mathcal{A}_M$ and a length metric $d_M$, and let $\mathcal{Q}_M$ be the collection of $n$-cubes in $\mathcal{A}_M$. Let $P$ be a subcomplex of $M^{(n-1)}$ and let $f: |P| \to S^{n-1}$ be a $B^\Delta$-Alexander map expanded by simple covers. We define the following:

(a) Simple covers of $f$ are properly located in an $(n-1)$-cube $q_0 \in P$, if the supports of these simple covers are contained in mutually disjoint balls in $\mathcal{B}_{q_0}$.

(b) If $q$ is a face of a cube $Q$ in $\mathcal{Q}_M$ that is contained in $|P|$, we say that the simple covers of $f$ contained in $q$ are properly located in $q$ if they are supported in mutually disjoint balls in

$$\bigcup \left\{ \mathcal{B}_{q'} : q' \in M^{(n-1)} \text{ and } q' \subset \text{cl}(q \setminus c(q)) \right\}.$$

(c) Suppose that $q_1, \ldots, q_m$, are mutually disjoint faces of cubes in $\mathcal{Q}_M$ each of which is contained in $|P|$. We say that simple covers of $f$ contained in $q_1 \cup \cdots \cup q_m$ are properly located in $q_1 \cup \cdots \cup q_m$, if, for
each $i = 1, \ldots, m$, the simple covers of $f$ in $q_i$ are properly located in $q_i$.

14. Extension over molecules

In this section we study quasiregular extensions of two-level Alexander maps. To introduce a two-level Alexander map, we set, for $0 < a < b < \infty$,

$$
\mathcal{T}(S^{n-1}; \{a, b\}) = (S^{n-1} \times \{a, b\}) \cup \left( (K_{S^{n-1}}(n-2)) \times [a, b] \right).
$$

Heuristically, we view $\mathcal{T}(S^{n-1}; \{a, b\})$ as a tower having floors $S^{n-1}$ at levels $a$ and $b$, and a supporting structure $S^{n-2} \times [a, b]$. The space $\mathcal{T}(S^{n-1}; \{a, b\}) \subset S^{n-1} \times [a, b]$ is homeomorphic to $S^{n-1} \cup (B^{n-1} \setminus B^{n-1}(1/2)) \cup S^{n-1}(1/2) \subset \mathbb{R}^n$, see Figure 36. Note that, the annulus $S^{n-2} \times [a, b]$ has a natural cubical structure induced by the CW $\Delta$-structures on the spheres $S^{n-1}(a)$ and $S^{n-1}(b)$.

![Figure 36. An embedded copy of $\mathcal{T}(S^2; \{a, b\})$ in $\mathbb{R}^3$.](image)

**Definition 14.1.** Let $M$ be a molecule, $q_M^+$ be the leading face of $M$, and $\rho \geq 2$ an integer. For $0 \leq a < b < \infty$, a map

$$
f : \partial|\ M| \to \mathcal{T}(S^{n-1}; \{a, b\})
$$

is a two-level Alexander map associated to $(M, q_M^+, \text{Ref}_\rho(M))$ if

1. $f|_{\partial|\ M|} : c(\partial|\ M|) \to S^{n-1} \times \{a\}$ is a $(M|_{\partial|\ M|})^\Delta$ Alexander map,
2. $f|_{c(q_M^+)} : c(q_M^+) \to S^{n-1} \times \{b\}$ is an $(\text{Ref}(M)|_{c(q_M^+)})^\Delta$ Alexander map, and
3. $f$ is radial in the rim $w(q_M^+)$. 

Our goal is to construct, for an $n$-molecule $M$ and a two-level Alexander map

$$
\partial|\ M| \to \mathcal{T}(S^{n-1}; \{1, 3e(M)\}),
$$

a $K$-quasiregular extension

$$
|M| \to S^{n-1} \times [1, 3e(M)],
$$

where the distortion $K \geq 1$ is independent of the degree of the two-level Alexander map.

We now state a brief version of the extension theorem, and give the full statement in Theorem 14.3.
Theorem 14.2. Let $M$ be an $n$-molecule having maximal atom length $\ell \geq 1$ and maximal refinement index $\varrho \geq 1$, and let $f : \partial_- |M| \to S^{n-1}$ be a standard $(\partial_- |M|)^{\Delta}$-Alexander map. Then there exist $L = L(n, \ell) \geq 1$ and an $L$-BLD-map $\tilde{f} : |M| \to S^{n-1} \times [1, 3\varrho]$ extending $f : \partial_- |M| \to S^{n-1} \times \{1\}$ such that $\tilde{f}|_{\partial |M|}$ is a two-level Alexander map.

Following Convention 12.10 we assume all Alexander maps are $L_0$-BLD and $K_0$-quasiregular. We begin the discussion on the terminologies related to the multi-level extensions.

### 14.1. Multi-level extensions

Let $M$ be a molecule and $A_M$ be its atom decomposition. Fix as in Definition 13.2 a leading cube $Q^+_M$, a leading face $q^+_M$, and a partial order $\ll_M$ for the cubes $Q_M$ in $A_M$. Let

$$\Sigma(M) = \{q^+_Q : Q \in Q_M\}$$

be the collection of all leading faces of the cubes in $Q_M$, and

$$\Sigma_c(M) = \{c(q^+_Q) : Q \in Q_M\}$$

be the collection of the centers of all leading faces.

For simplicity, we denote, from here on,

$$\ell = \ell(M), \quad \text{and} \quad \varrho = \varrho(M).$$

Denote

$$Z_+ = \{n \in \mathbb{Z} : n \geq 0\}.$$  

We fix a level function

$$\lambda : \{\partial_- |M|\} \cup \Sigma_c(M) \to Z_+ + \{0, 1/\ell, \ldots, (\ell - 1)/\ell\}$$

which satisfies

1. $\lambda(\partial_- |M|) = 0$;
2. if $Q$ is the leading cube of an atom $A \in A_M$ then $\lambda(c(q^+_Q)) = \rho_M(A)$; and
3. if $Q$ and $Q'$ are two adjacent cubes, satisfying $Q' \ll_M Q$, in the same atom $A$, then

$$\lambda(c(q^+_Q')) = \lambda(c(q^+_Q)) - 1/\ell.$$  

Note that the level function is uniquely determined by these properties. Heuristically, the level function $\lambda$ is used to lift an Alexander map $|c(q^+_Q)| \to S^{n-1}$, whose domain $c(q^+_Q)$ is on the level $\lambda(c(q^+_Q))$, to a map $|c(q^+_Q)| \to S^{n-1} \times \{3^{\lambda(c(q^+_Q))}\}$ of a new height.

Set

$$L = \{3^{\ell} : x \in Z_+ + \{0, 1/\ell, \ldots, (\ell - 1)/\ell\}\},$$

and

$$T(S^{n-1}; L) = (S^{n-1} \times L) \cup ((K_{S^n-1})^{(n-2)} \times \mathbb{R}),$$

recalling that $K_{S^n-1}$ is the CW$_\Delta$-complex on $S^{n-1}$ in Convention 2.6. We may view $T(S^{n-1}; L)$ as an infinite tower with floors at levels $L$.

We are now ready to state the main theorem of this section. Note that complexes $M|_{\partial(\partial_- |M|)}$ and $\text{Ref}(M)|_{\partial(c(q^+_Q))}$ are isomorphic.
Theorem 14.3. Let $M$ be an $n$-molecule having maximal atom length $\ell$ and maximal refinement index $\varrho$, and let $f: \partial_\varrho \mid M \rightarrow S^{n-1}$ be a standard $(M \mid \partial_\varrho \mid M)_{\Delta}$-Alexander map. Then there exist $L = L(n, \ell) \geq 1$ and an $L$-BLD-map $\tilde{f}: \mid M \rightarrow S^{n-1} \times [1, 3^\varrho]$ extending $f: \partial_\varrho \mid M \rightarrow S^{n-1} \times \{1\}$ such that $\tilde{f}|_{\partial_\varrho \mid M}$ is a two-level Alexander map:

1. $\tilde{f}|_{\partial_\varrho \mid M}$ is an $(M \mid \partial_\varrho \mid M)_{\Delta}$-Alexander map onto $S^{n-1} \times \{1\}$;
2. $\tilde{f}|_{c(q^+_\Delta)}: c(q^+_\Delta) \rightarrow S^{n-1} \times \{3^\varrho\}$ is a standard $(\text{Ref}(M)|_{c(q^+_\Delta)})_{\Delta}$-Alexander map expanded by $\nu(q^+_\varrho)$ standard simple covers, and
3. $\tilde{f}|_{w(q^+_\varrho)}: w(q^+_\varrho) \rightarrow |(K_{S^{n-1}})^{(n-2)}| \times [1, 3^\varrho]$ is radial.

Furthermore, the restriction

$$\tilde{f}|_{\Sigma(M)}: \Sigma(M) \rightarrow T(S^{n-1}; L)$$

to the leading faces satisfies the condition that, for each $q^+_\varrho \in \Sigma(M)$,

4. $\tilde{f}|_{c(q^+_\varrho)}: c(q^+_\varrho) \rightarrow S^{n-1} \times \{3^{\lambda(c(q^+_\varrho))}\}$ is a $(\text{Ref}(M)|_{c(q^+_\varrho)})_{\Delta}$-Alexander map expanded by $\nu(q^+_\varrho)$ standard simple covers, and
5. the restriction $\tilde{f}|_{w(q^+_\varrho)}: w(q^+_\varrho) \rightarrow |(K_{S^{n-1}})^{(n-2)}| \times [1, 3^{\lambda(c(q^+_\varrho))}]$ is radial.

The parameter $\nu$ in the theorem above is defined before Remark 14.3.

Remark 14.4. Instead of a lengthy definition, we call map $\tilde{f}|_{\partial_\varrho \mid M \mid \Sigma(M)}$ in Theorem 14.3 and, in what follows, maps having like properties, multi-level Alexander towers expanded by simple covers.

Remark 14.5. Before moving further, we make a comment on the BLD-constant of $f$. Let $q^+_\varrho \in \Sigma(M)$ with $Q \in A \in A_M$. Then

$$\rho(A) - 1 < \lambda(c(q^+_\varrho)) \leq \rho(A),$$

where $\rho(A)$ is the refinement index of $A$ in $M$. Since $c(q^+_\varrho) = \text{Core}(\text{Ref}(q^+_\varrho))$, we have

$$1 < \frac{3^{\lambda(c(q^+_\varrho))}}{\text{dist}(c(q^+_\varrho), \partial q^+_\varrho)} \leq 3.$$

Thus the radial map $\tilde{f}|_{w(q^+_\varrho)}$ is BLD with a constant depending only on $n$. Hence $\tilde{f}$ is a BLD-map with a constant depending only on $n$ and the placement of the simple covers in $\Sigma_c(M)$.

14.2. Extension over a single cube. In this section, we consider the special case when the molecule $M$ is the $\rho$-th refinement

$$M = \text{Ref}_\rho(Q)$$

of a single $n$-cube $Q$, where $\rho \geq 2$ is an integer.

Proposition 14.6. Let $Q$ be an $n$-cube, $\rho \geq 2$ an integer, and $M = \text{Ref}_\rho(Q)$ a molecule with space $Q$ and endowed with a length metric $d_M$. Fix a face $q$ of $Q$ designated as the leading face of molecule $M$. Let

$$f: \partial Q \rightarrow (S^{n-1} \times \{1, 3^\varrho\}) \cup ((K_{S^{n-1}})^{(n-2)} \times [1, 3^\varrho])$$
be a two-level Alexander map associated to $(Q, q, \text{Ref}_\rho(Q))$. Then there exist a constant $L = L(n) \geq 1$ and an $L$-BLD-map

$$
\tilde{f} : Q \to \mathbb{S}^{n-1} \times [1, 3^\rho],
$$
such that $\tilde{f}|_{\partial Q}$ is $f$ expanded by simple covers in $c(q)$.

To justify the lengthy proof that follows, we emphasize that it requires a series of local deformations and some judicious arrangements of the simple covers in order for the BLD-constant of $f$ not to go up with the number of steps in the deformation, in other words, to be independent of $\rho$.

Since, in metric $d_M$, each $n$-cube in $M$ has unit side length, we may identify $Q = |M|$ with the Euclidean cube $[-3^\rho/2, 3^\rho/2]^{n-1} \times [-3^\rho, 0]$.

We fix in $Q$ a nested sequence of cubes $C_m = C_m, \rho = [-3^\rho/2 + m, 3^\rho/2 - m]^{n-1} \times [-3^\rho + 3m, 0]$ for $m = 0, \ldots, 3^\rho - 1$. Then $C_0 = Q$ and $C_{3^\rho-1} = [-3^{\rho-1}/2, 3^{\rho-1}/2] \times \{0\} = c(q)$.

The construction of $\tilde{f}$ over $Q$ starts from the outermost layer $\text{cl}(C_0 \setminus C_1)$. In the $m$-th step, the BLD-extension to $\text{cl}(C_{m-1} \setminus C_m)$ is a BLD branched cover homotopy, which deforms an Alexander map $\tilde{f}|_{\text{cl}(\partial C_m \setminus (\mathbb{R}^{n-1} \times \{0\}))}$ expanded by simple covers to an Alexander map $\tilde{f}|_{\text{cl}(\partial C_m \setminus (\mathbb{R}^{n-1} \times \{0\}))}$ expanded by simple covers.

To prove Proposition 14.6, let $0 \leq m \leq 3^\rho - 1$. For the proof, we write $h_m = 3^\rho - 3m$, $s_m = 3^\rho/2 - m$, and

$$
C_m = [-s_m, s_m]^{n-1} \times [-h_m, 0].
$$

Denote by

$$
b_m = C_m \cap (\mathbb{R}^{n-1} \times \{-h_m\})
$$

the base of $C_m$, by

$$
U_m = \partial C_m \setminus ((-s_m, s_m)^{n-1} \times \{0\})
$$

the part of $\partial C_m$ with the top removed, and by

$$
S_m = \partial C_m \setminus ((-s_m, s_m)^{n-1} \times \{0, -h_m\})
$$

the part of $\partial C_m$ with both the top and the base removed.

We fix, for each $0 \leq m \leq 3^\rho - 1$, a new cubical structure $P_m$ on $U_m$ as follows. Let $P_0 = M|_{U_0}$. For $m \geq 1$, the complex $P_m$ on the side $S_m$ is an
affine bilipschitz copy of $P_0|_{S_0 \cap (\mathbb{R}^{n-1} \times [-h_0,0])}$, and on the base $b_m$ is a scaling of $P_0|_{-s_0,s_0}^{n-1} \times \{-h_0\}$. More precisely, let

$$\varsigma_m: U_0 \setminus (S_0 \cap (\mathbb{R}^{n-1} \times (-h_0, -h_m))) \to U_m$$

be the map given by

$$(z,t) \mapsto \left( \frac{s_m}{s_0} z, t \right) \quad \text{on} \quad S_0 \cap (\mathbb{R}^{n-1} \times [-h_m,0]),$$

$$(z,-h_0) \mapsto \left( \frac{s_m}{s_0} z, -h_m \right) \quad \text{on} \quad [-s_0,s_0]^{n-1} \times \{-h_0\};$$

these maps $\varsigma_m$ are uniformly bilipschitz with a constant depending only on $n$. The complex $P_m$ on $U_m$ is the copy of the complex $P_0|_{U_0 \setminus (S_0 \cap (\mathbb{R}^{n-1} \times (-h_0, -h_m)))}$ carried over by the map $\varsigma_m$.

For $1 \leq m \leq 3^p-1$, consider a piecewise linear function $\theta_m: [-h_m,0] \to [-h_{m+1},0]$ which satisfies

1. $\theta_m(-h_m) = -h_{m+1}$ and $\theta_m(-h_{m+1} + 1) = -h_{m+1} + 1$,
2. linear on $[-h_m, -h_{m+1} + 1]$, and
3. $\theta_m(t) = t$ for $t \in [-h_m + 1,0]$.

Then the map $\varphi_m: C_m \to C_{m+1}$ given by

$$(z,t) \mapsto \left( \frac{s_{m+1}}{s_m} z, \theta_m(t) \right)$$

is $L_1$-bilipschitz with a constant $L_1 = L_1(n) \geq 1$ depending only on $n$. Set

$$\psi_m = \varphi_m \circ \cdots \circ \varphi_0,$$

and observe that $\psi_m = \varsigma_m$ on $U_0 \setminus (S_0 \cap (\mathbb{R}^{n-1} \times (-h_0, -h_m + 1)))$ for each $m$.

For each $m = 0, \ldots, 3^p-1$, the complex $P_m$ is a refinement of $\varphi_{m-1}(P_{m+1})$ and agrees with $\varphi_{m-1}^{-1}(P_{m+1})$ in the complement of

$$\tilde{S}_m \equiv S_m \cap (\mathbb{R}^{n-1} \times (-h_m, -h_{m+1} + 1));$$

we call $\tilde{S}_m$ the lowest band on the side $S_m$.

Note that each $(n-1)$-cube in the complex $\varphi_{m-1}^{-1}(P_{m+1}|_{\tilde{S}_{m+1}})$ connects $\mathbb{R}^{n-1} \times \{-h_m\}$ to $\mathbb{R}^{n-1} \times \{-h_{m+1} + 1\}$. Furthermore, the complex $P_m|_{\tilde{S}_m}$ has a natural partition into atoms, each of which consists of four unit $(n-1)$-cubes connecting $\mathbb{R}^{n-1} \times \{-h_m\}$ to $\mathbb{R}^{n-1} \times \{-h_{m+1} + 1\}$ and has space that of a cube in $\varphi_{m-1}^{-1}(P_{m+1}|_{\tilde{S}_{m+1}})$. We call these atoms strips.

After these preliminaries, we now discuss the extension.

1. Initial Deformation. To begin, we reduce the simplicial complex $P_0^\Delta$ to the simplicial complex $(\varphi^{-1}_0(P_1))^\Delta$ by deforming the $P_0^\Delta$-Alexander map $f\vert_{U_0}: U_0 \to S^{n-1}$ to a $(\varphi^{-1}_0(P_1))^\Delta$-Alexander map $f_0: U_0 \to S^{n-1}$ expanded by simple covers. For this, we deform $f\vert_{U_0}$ simultaneously on all strips of four cubes in the band $\tilde{S}_0$ to the corresponding Alexander stars on cubes in $\varphi^{-1}_0(P_1|_{\tilde{S}_0})$ expanded by simple covers; Figure 37 illustrates the deformation on a strip of four cubes.

We omit the technical details that the deformation may be chosen to agree on the pairwise common boundary of any two strips and to fix all the...
simplices in $P_0^\Delta \cap (\mathbb{R}^{n-1} \times \{-h_0, -h_1 + 1\})$. Thus we obtain a deformation of $P_0^\Delta$ which leaves all simplices in $P_0^\Delta \cap (\mathbb{R}^{n-1} \times [-h_1 + 1, 0])$ unchanged.

The obtained complex is isomorphic and uniformly bilipschitz equivalent to $P_1^\Delta$ on $U_1$. We may identify the map $f'_0$ carried by this complex with a $P_1^\Delta$-Alexander map $f_1: U_1 \to S^{n-1} \times \{1\}$ expanded by simple covers through a second homotopy. These deformations may be chosen to be BLD. Their composition yields a BLD-extension $\hat{f}: \text{cl}(C_0 \setminus C_1) \to S^{n-1} \times [0, 1]$ of $f|_{U_0}$ and $f_1|_{U_1}$, such that $\hat{f}$ is radial on $\text{cl}(C_0 \setminus C_1) \cap (\mathbb{R}^{n-1} \times \{0\})$. The number of simple covers created depends only on $n$; the BLD-constant of the extension depends only on $n$ and the placement of simple covers.

![Figure 37. Reduction of a strip of four cubes to one cube.](image)

Below, we discuss how to place the newly created simple covers and how to regulate their subsequent moves.

2. Placement of Simple Covers. We now discuss the schemes for placing simple covers in the first step, and the rearrangement and the placement of simple covers in subsequent steps.

*Initial Placement.* In the initial deformation, the mutually disjoint simple covers in $f_1|_{U_1}$, created by reducing each strip $\sigma$ of four cubes in $P_0$ to a cube $q(\sigma) \in P_1$, are properly placed in $q(\sigma)$, i.e., placed on mutually disjoint balls in $B_{q(\sigma)}$ as in Definition 13.6. This rule is applied to all strips in the band $S_0$.

Suppose that there exist already, for some $m \geq 1$, a constant $L \geq 1$ depending only on $n$ and an $L$-BLD-extension $\tilde{f}: \text{cl}(C_0 \setminus C_m) \to S^{n-1} \times [1, 3^m]$ of $f|_{U_0}$ which is radial on $\text{cl}(C_0 \setminus C_m) \cap (\mathbb{R}^{n-1} \times \{0\})$ and whose restriction $f_m|_{U_m}: U_m \to S^{(n-1)} \times \{3^m\}$ is a $P_m^\Delta$-Alexander expanded by simple covers. Suppose also that the simple covers on $U_m$ consist of the newly created ones located in the lowest layer of cubes on the side $S_m$, and others located in the outermost $m - 1$ layers of the buffers $\text{Buffer}_j(P_m|_{b_m})$, $j = 1, \ldots, m - 1$. 


of $P_m|_{b_m}$ in the base. Assume further that all simple covers are properly placed in the sense of Definition 13.6.

First Rearrangement. To prepare for the extension to $\text{cl}(C_m \setminus C_{m+1})$ for $m \geq 2$, we move simultaneously every simple cover of $f_m$ in the base $b_m$ one step towards the center of $b_m$. More precisely, a simple cover supported in $B_q$ for some $q$ in $\text{Buffer}_j(b_m)$ is moved to $B_{q'}$, where $q'$ is any cube in $\text{Buffer}_{j+1}(b_m)$ that meets $q$. This move – a BLD branched cover homotopy from $f_m$ to $f'_m$ – is a translation on each simple cover, and the trajectories of these simple covers can be chosen to be mutually disjoint. Since $\#C_0(n-1)/\#C_{3^m-1} \leq cn3^n$, this rearrangement can be achieved if the number $\mu$ of balls in $B$ is a sufficiently large multiple of $3^n$.

Second Rearrangement. We now move the simple covers of $f'_m$ located in the lowest band $S_m$ on the side $S_m$ to the newly vacant outermost buffer in $b_m$. During this move, simple covers located in a cube $q$ are moved into a cube in the base that is adjacent to $q$. Again, this can be achieved if the number $\mu$ is a sufficiently large multiple of $n3^n$. As in the previous rearrangement, the homotopy from $f'_m$ to $f''_m$ is a translation on each simple cover, the trajectories of these simple covers are mutually disjoint, and the branched cover homotopy is BLD with a constant depending only on $n$.

It is also understood that all simplices in $P_m \cap (\mathbb{R}^{n-1} \times [-h_m, 0])$ remain fixed during this process.

3. Subsequent Deformations. Having all simple covers in $S_m$ stored in $b_m$, we may, as before, apply the initial deformation procedure to reduce each strip $\sigma$ of four $(n-1)$-cubes in $S_m$ to an $(n-1)$-cube $q(\sigma) \in \sigma$ in the lowest band $S_{m+1}$ on the side of $C_{m+1}$. We have now obtained a BLD-extension,

$$\hat{f}: \text{cl}(C_0 \setminus C_{m+1}) \to \mathbb{S}^{n-1} \times [1, 3^{m+1}]$$

of $f|_{U_0}$, which is radial on $\text{cl}(C_0 \setminus C_{m+1}) \cap (\mathbb{R}^{n-1} \times \{0\})$ and whose restriction $f_{m+1} = \hat{f}|_{U_{m+1}}: U_{m+1} \to \mathbb{S}^{n-1} \times \{3^{m+1}\}$ is a $P_m^{\Delta}$-Alexander map expanded by simple covers.

Continuing this process for $m = 1, \ldots, 3^{\rho-1}$, we obtain a BLD-map $\hat{f}: Q \to \mathbb{S}^{n-1} \times [1, 3^{\rho-1}]$ which, on $\partial Q$, is an expansion of $f$ by properly placed simple covers in $c(q)$.

4. Constants. With the exception of the initial steps during which simple covers are created, simple covers are moved in each subsequent step within a controlled distance by similarities whose trajectories have mutually disjoint neighborhoods. Thus, the BLD-constants of the extensions at intermediate steps stay uniformly bounded. As a consequence, the BLD-constant of the final extension $\hat{f}$ depends only on $n$, in particular, it is independent of the number of steps which is in the magnitude of $3^\rho$.

This completes the proof. \hfill \Box

Remark 14.7. Proposition 14.6 remains true if the given boundary map $f$ contains simple covers which are properly placed in the $(n-1)$-cubes in $\text{Ref}_\rho(Q)$. We omit the details.
14.3. Proof of Theorem 14.3 Let \( f : \partial_-|M| \to \mathbb{S}^{n-1} \) be the \( (M|_{\partial_-|M|})^{\Delta} \)-Alexander map in the statement of Theorem 14.3.

We first extend \( f \) to a multi-level Alexander tower

\[
\tilde{f}|_{(\partial_-|M|) \cup \Sigma(M)} : (\partial_-|M|) \cup |\Sigma(M)| \to \mathcal{T}(\mathbb{S}^{n-1}; L)
\]
as follows. For each \( Q \in \Sigma(M) \), let \( \iota_{\partial Q} : q^+_Q \to c(q^+_Q) \) be a radial similarity having a scaling factor \( 1/3 \), and define \( f|_{c(q^+_Q)} : c(q^+_Q) \to \mathbb{S}^{n-1} \times \{ 3^j(c(q^+_Q)) \} \) to be the \((\text{Ref}(q^+_Q)|_{c(q^+_Q)})^{\Delta}\)-Alexander map

\[
x \mapsto (\tilde{f} \circ \iota_{\partial Q}^{-1}(x), 3^j(c(q^+_Q))).
\]

Let \( f|_{w(q^+_Q)} \) to be the natural radial map that interpolates the maps \( f|_{\partial_-|M|} \) and \( f|_{c(q^+_Q)} \) between two-levels.

For \( Q \in \mathcal{QM} \), let \( \delta(Q) = \delta_{\Gamma(QM)}(Q) \) be the maximal combinatorial distance, on tree \( \Gamma(QM) \), between \( Q \) and the leaves in the tail \( \tau_Q \); a leaf \( Q \) in \( \Gamma(QM) \) has \( \delta(M)(Q) = 0 \).

The BLD-extension \( \tilde{f} : |M| \to \mathbb{S}^{n-1} \times [1, 3^m] \) is constructed by induction, on \( \mathcal{QM} \), in the order of the values of the function \( Q \mapsto \delta(Q) \).

In the initial step, for each \( Q \in \mathcal{QM} \) with \( \delta(Q) = 0 \), we extend \( f|_{\partial_-M} \) to \( Q \) by Proposition 14.6. The restriction of the resulting map \( \tilde{f}|_{QQ} \) is a two-level Alexander tower expanded by \( \nu(Q) \) simple covers properly placed in \( c(q^+_Q) \).

In the induction step, we assume that \( f \) has been extended to all cubes \( Q \in \mathcal{QM} \) with \( \delta(Q) \leq m \), the extension \( \tilde{f} \) satisfies conditions (1) and (2) in Theorem 14.3 for all leading faces \( q^+_Q \) of cubes \( Q \) with \( \delta(Q) \leq m \), and the extension is BLD with a constant depending only on \( n \) and \( \ell \).

Let now \( Q \in \mathcal{QM} \) be a cube for which \( \delta(Q) = m + 1 \). The goal is to extend the map \( \tilde{f} : c(\tau_M(Q) \setminus Q) \to \mathcal{T}(\mathbb{S}^{n-1}; L) \) defined on the tail to the remaining part of \( Q \).

To this end, we study first the already defined values of \( \tilde{f} \) on \( \partial Q \). Suppose that \( Q \) is a cube of atom \( A \) in the atom decomposition of \( M \), which has refinement index \( \rho_0 \). Then \( Q \) is isometric to a Euclidean cube of side length \( 3^{\rho_0} \). By the definition of levels, \( \lambda(c(q^+_Q)) = \rho_0 + j(Q)/\ell \), where \( c(q^+_Q) \) is the center of the leading face of \( Q \), and \( j(Q) \), \( 0 \leq j(Q) \leq \ell - 1 \), is the combinatorial distance between \( Q \) and the leading cube \( Q^+_A \) of \( A \) on the tree \( \Gamma(QM) \). We observe that

(i) on the leading face \( q^+_Q \), \( f|_{c(q^+_Q)} \) is an Alexander map at height \( 3^{\rho_0 - j(Q)/\ell} \) without simple covers, \( f|_{w(q^+_Q)} \) is radial, but \( \tilde{f}|_{q^+_Q} \) has not been defined;

(ii) on an exterior face of \( Q \), if it exits, \( \tilde{f} \) is Alexander at height 1;

(iii) if \( q \) is a back face of \( Q \) to which a cube \( Q' \) from the same atom is attached, then both \( Q \) and \( Q' \) have refinement index \( \rho_0 \), \( q = q^+_Q \) is the leading face of \( Q' \), \( \tilde{f}|_{c(q)} \) is a \((\text{Ref}(c(q))^{\Delta})\)-Alexander map at height \( 3^{\rho_0 - (1 + j(Q))/\ell} \) expanded by properly placed simple covers in \( c(q) \), and \( f|_{w(q^+_Q)} \) is radial;
(iv) if $q$ is a back face of $Q$ to which a cube $Q'$ from a different atom is attached, then $\rho(Q') \leq \rho_0 - 1$ and the leading face $q' = qQ'$ of $Q'$ is contained in the interior of $q$. Hence $f|_{\text{cl}(q\setminus q')} \upharpoonright_{\partial \text{cl}(q')}$ is Alexander at height 1, $f|_{\text{cl}(q')}$ is (Ref($c(q')$))$\Delta$-Alexander expanded by properly placed simple covers at height 3$\rho(Q')$, and $f|_{\text{w}(q)}$ is radial. Because $f|_{\partial \text{cl}(q)}$ is multi-leveled, Proposition 14.6 is not readily applicable; we need to transform $f|_{\partial \text{cl}(q)}$ to a two-level Alexander map on the boundary of $c(Q) = \text{Core(Ref}(Q))$. We now give the details.

Step 1. Reducing the number of levels. We transform in $\text{cl}(Q \setminus c(Q))$, which is bi-Lipschitz equivalent to $\partial Q \times [0, 1/3]$, the multi-level Alexander tower $f|_{\partial \text{cl}(q)}$ expanded by simple covers to a two-level Alexander tower $f|_{\partial \text{cl}(q)} : \partial \text{cl}(Q) \to \mathcal{T}(\mathbb{S}^{n-1}; L)$ expanded by simple covers. Heuristically, this BLD-extension $f : \text{cl}(Q \setminus c(Q)) \to \mathcal{T}(\mathbb{S}^{n-1}; L)$ seals up the rims in the back faces of $Q$, and arrives at a map $f|_{\partial \text{cl}(q)}$ that is suitable for the application of Proposition 14.6.

We begin by defining tents over faces. For each face $q$ of $Q$, denote by $q'$ the face of $c(Q)$ that is opposite to $q$ and let $T(q)$ be the convex hull of $q$ and $q'$ in $Q$; we call $T(q)$ the tent over $q$ in $Q$.

Associated to each back face $q$ of $Q$ of the second kind, we define also a smaller tent. Let $\bar{Q}$ be the unique cube in $Q_{\lambda} \setminus \{Q\}$ which meets $q$ in a face; write $\bar{q} = Q \cap \bar{Q}$. Let $Q'$ be the unique cube in some iterated $(1/3)$-refinement of $Q$ that has $c(\bar{q})$ as a face, and let $T(q)$ be the convex hull of $Q'$ and $\bar{q}$; we call $T(q)$ the tent over $\bar{q}$ in $Q$.

For a tent $T$ of either kind, we call $T \cap \partial Q$ the base of the tent $T$, and the face of the tent opposite to the base the top of the tent; see Figure 38.

![Figure 38. Tents over faces.](image)

Observe that the level function on $(\partial M \cup \Sigma_c(M)) \cap \partial Q$ takes its values in the set $\{\rho_0 - j(Q)/\ell, \rho_0 - (j(Q) + 1)/\ell, \rho_0 - 1, \ldots, 1, 0\}$, and

$$\lambda(c(q)) = \rho_0 - j(Q)/\ell.$$

We extend $\lambda$ to $(n - 1)$-cubes in $\partial(c(Q))$ by setting

$$\lambda(\partial(c(Q)) \setminus q_{c(Q)}) = \lambda(c(q^+)) - \frac{2}{\delta \ell},$$
Recall that if \( q \) is a back face of \( Q \) of the second kind then the level \( \lambda(c(\tilde{q})) \) of the center of the base of the tent \( T(\tilde{q}) \) is the refinement index \( \rho(\tilde{q}) = \rho(\tilde{Q}) \); in this case we set the level of the top of \( T(\tilde{q}) \) to be \( \rho(\tilde{q}) + \frac{1}{10} \).

For the extension of \( f \), we fix a nested sequence of cubes \( Q_0, Q_1, \ldots, Q_{\rho_0} \) as follows: \( Q_0 = Q, Q_{\rho_0} = c(Q) \), and, for \( k = 1, \ldots, \rho_0 - 1 \),

\[
Q_k = \left| \text{Core}(\text{Ref}_{\rho_0-(k-1)}(Q)) \right|.
\]

We denote \( \lambda_k = \rho_0 - (k-1) \) for \( k = 0, \ldots, \rho_0 - 1 \), and \( \lambda_{\rho_0} = \rho_0 - (j(Q) + \frac{1}{3})/\ell \).

Let \( F_Q \) be the collection of back faces of \( Q \) of the second kind, and let \( \Omega(Q) \) be the domain

\[
\Omega(Q) = \text{cl} \left( Q \setminus \left( c(Q) \cup \bigcup_{q \in F_Q} T(\tilde{q}) \right) \right) \subset Q,
\]

which is the complement of the center cube and the small tents in \( Q \).

The construction below follows a close-to-level-preserving principle: **Principle A.** For each \( k = 0, \ldots, \rho_0 \), the extension \( \tilde{f} \) maps \( \Omega(Q) \cap (Q_{k-1} \setminus \text{int} Q_k) \) into \( S^{n-1} \times [3^{\lambda_k} - 1, 3^{\lambda_k}] \), even when this rule is not specifically mentioned.

We now extend \( f \) to a map \( \tilde{F} \) on \( \text{cl}(Q \setminus c(Q)) \) by defining its values on the tents over faces. We consider each type of face separately.

(i) For the leading face \( q_0^+ \), we define \( \tilde{F}|_{T(q_0^+)}: T(q_0^+) \to S^{n-1} \times [1, 3^{\lambda(c(q_0^+))}] \) as follows. On the top of the tent, we set \( \tilde{F}|_{T(q_0^+) \cap c(Q)} \) to be a copy of \( f|_{q_0^+} \) for which the domain is scaled by \( 1/3 \) and the image is lifted to between levels \( 3^{\lambda(c(q_0^+))} - \frac{2}{3} \) and \( 3^{\lambda(c(q_0^+))} - \frac{1}{3} \). In the rest of \( T(q_0^+) \), \( \tilde{F}|_{T(q_0^+)} \) interpolates between boundary functions on the base \( q_0^+ \) and on the top \( T(q_0^+) \cap c(Q) \), respecting Principle A.

We call attention to that fact that \( \tilde{F}|_{T(q_0^+) \cap c(Q)} \) is not expanded by simple covers, because \( f|_{q_0^+} \) is not expanded by simple covers.

(ii) For an exterior face \( q \), we define \( \tilde{F}|_{T(q)} \) following case (i) with the exception that the top of \( T(q) \cap c(Q) \) of the tent is mapped onto a single level \( S^{n-1} \times \{3^{\lambda(c(q_0^+))} - \frac{2}{3}\} \).

(iii) Suppose that \( q \) is a back face of \( Q \) of the first kind. We extend \( f|_{qQ} \) to the tent \( T(q) \) by setting

\[
\tilde{F}|_{T(q)}: T(q) \to S^{n-1} \times [1, 3^{\lambda(c(q^+_0))} - \frac{1}{3}]
\]
to be the map which

(a) agrees with \( \tilde{f} \) on the base \( q \);
(b) translates \( f|_q \) to the top \( T(q) \cap c(Q) \) of the tent, and lifts the target from \( S^{n-1} \times \{3^{\lambda(c(q))}\} \) to \( S^{n-1} \times \{3^{\lambda(c(q^+_0))} - \frac{1}{3}\} \); and
(c) is the natural radial map on the remaining part of \( T(q) \), interpolating already defined values on different levels, respecting Principle A.
Finally, let $q$ be a back face of $Q$ of the second kind. Recall that, in this case, a cube $Q'$ with a smaller refinement index from another atom is attached to $q$, and observe that $\rho(\tilde{q}) = \lambda(c(Q'))$. We first extend $f|_{\partial Q}$ to the small tent $T(\tilde{q})$,

$$\tilde{F}|_{T(\tilde{q})}: T(\tilde{q}) \to S^{n-1} \times [1, 3^{\rho(\tilde{q})} + \frac{1}{10n}],$$

following the extension steps in case (iii), except that the images of the base and the top now are $S^{n-1} \times \{3^{\rho(\tilde{q})}\}$ and $S^{n-1} \times \{3^{\rho(\tilde{q})} + \frac{1}{10n}\}$, respectively.

For the extension to $\text{cl}(T(q) \setminus T(\tilde{q}))$, note that the top $t(T(\tilde{q}))$ of $T(\tilde{q})$ lies in $\partial Q_{\rho(\tilde{q})+1}$ and that $\text{cl}(T(q) \cap \partial Q_{\rho(\tilde{q})+1}) \setminus \partial Q(q)$ and $\text{cl}(q \setminus \tilde{q})$ are bilipschitz equivalent with a constant depending only on $n$. We extend $\tilde{F}|_{t(T(\tilde{q}))}$ to a map $\tilde{F}: T(q) \cap \partial Q_{\rho(\tilde{q})+1} \to S^{n-1} \times 3^{\rho(\tilde{q})} + \frac{1}{10n}$ for which the restriction $\tilde{F}|_{\text{cl}(T(q) \cap \partial Q_{\rho(\tilde{q})+1}) \setminus t(T(\tilde{q}))}$ is a bilipschitz copy of $f|_{\text{cl}(q \setminus \tilde{q})}$. In the rest of $Q \setminus c(Q)$, we interpolate the boundary values in accordance with Principle A.

The combination of all $\tilde{F}|_{T(q)}$ yields an extension $\tilde{F}$ of $f|_{\partial Q}$ over the ring domain $\text{cl}(Q \setminus c(Q))$. By the construction, $\tilde{F}$ is BLD with a constant depending only on $n$, and its restriction $\tilde{F}|_{\partial c(Q)}$ is a two-level Alexander tower expanded by properly located simple covers.

In the above discussion, we omitted the details regarding the following facts: (a) the construction can be made in accordance with Principle A; (b) the extension can be made to agree on the boundaries of the tents; and (c) the BLD-constant of the extension is independent of $\rho$. This completes the reduction of levels in the multi-level Alexander tower on $\partial Q$.

**Step 2.** We construct, following Proposition 14.6 and Remark 14.7 a BLD map $\tilde{f}$ on $c(Q)$ whose restriction $\tilde{f}|_{\partial c(Q)}$ on the boundary is the map $\tilde{F}|_{\partial c(Q)}$ expanded by simple covers properly located on the top $T(q^+_{c(Q)}) \cap c(Q)$ of the tent $T(q^+_{c(Q)})$.

**Step 3.** It remains to define $\tilde{f}$ on $\text{cl}(Q \setminus c(Q))$ by modifying $\tilde{F}$ on $\text{cl}(Q \setminus c(Q))$.

Since $\tilde{f}|_{T(q^+_{c(Q)}) \cap c(Q)}$ is $\tilde{F}|_{T(q^+_{c(Q)}) \cap c(Q)}$ expanded by simple covers, maps $\tilde{f}|_{\partial c(Q)}$ and $\tilde{F}|_{\partial c(Q)}$ agree except on the top $T(q^+_{c(Q)}) \cap c(Q)$ of tent $T(q^+_{c(Q)})$.

We set $\tilde{f} = \tilde{F}$ on $\text{cl}(Q \setminus c(Q) \cup T(q^+_{c(Q)}))$. For the tent $T(q^+_{c(Q)})$, we set $\tilde{f}|_{q^+_{c(Q)}}$ to be a scaled copy of $\tilde{f}|_{T(q^+_{c(Q)}) \cap c(Q)}$ with image $S^{n-1} \times \lambda(c(q^+_{c(Q)}))$ and $\tilde{f}|_{T(q^+_{c(Q)})}$ to be a natural BLD-extension of its boundary map.

This completes the definition of $\tilde{f}$ in the induction step, and therefore the proof of the theorem.

**14.4. Variations of Theorem 14.3** Theorem 14.3 has a corollary on quasiregular extension over ring domain $|M| \setminus \text{int } c(Q^+_M)$, where $Q^+_M$ is a leading cube of $M$.

**Corollary 14.8.** Let $M$ be an $n$-molecule having maximal atom length $\ell$ and maximal refinement index $\rho$; let $Q^+_M$ be its leading cube, and $c(Q^+_M) = \text{Core}(\text{Ref}(Q^+_M))$ the center of $Q^+_M$. Suppose that $f: \partial |M| \to S^{n-1}$ is an
(M|\partial M)^\Delta -Alexander map. Then there exist L = L(n, \ell) \geq 1 and an L-BLD-map
\[ \hat{f}: |M| \setminus \text{int } c(Q^+_M) \to S^{n-1} \times [1, 3^\ell] \]
extending \( f: \partial |M| \to S^{n-1} \times \{1\} \), and for which
\[ \hat{f}|_{\partial c(Q^+_M)}: \partial c(Q^+_M) \to S^{n-1} \times \{3^\ell\} \]
is a (Ref(M)|\partial c(Q^+_M))\(^\Delta\)-Alexander map expanded by properly located simple covers in \( \partial c(Q^+_M) \).

Proof. Let \( \{Q_1, \ldots, Q_m\} \) be the collection of cubes in \( Q_M \) which meet \( Q^+_M \) in an \( (n-1) \)-cell. Thus for each \( j = 1, \ldots, m \), the intersection \( Q_j \cap Q^+_M \) is the leading face of \( Q_j \), and is also, by definition, the leading face \( q^+_j = q^+_M \) of the tail complex \( Q_j = \tau_M(Q_j) \).

Applying Theorem \[14.3\] to all triples \( (M_j, q^+_j, f|_{\partial c(M_j)}) \), \( j = 1, \ldots, m \), we obtain a map \( \hat{f}: |M| \setminus \text{int } Q^+_M \to S^{n-1} \times [1, 3^\ell-1] \) for which

1. \( \hat{f}|_{\partial c(M)} = f \),
2. each \( \hat{f}|_{\text{int } M_j} \) is \( L'(n, \ell) \)-BLD for some constant \( L'(n, \ell) \geq 1 \),
3. each restriction
\[ \hat{f}|_{c(q^+_j)}: c(q^+_j) \to S^{n-1} \times \{3^\ell-1\} \]
is a (Ref(M)|q^+_j)\(^\Delta\)-Alexander map expanded by \( \nu(q^+_j) \) properly located simple covers.

It remains to extend map \( \hat{f} \) to \( Q^+_M \setminus \text{int } c(Q^+_M) \). For this, we adapt Step 1 in the proof of Theorem \[14.3\] to transform the multi-level Alexander tower \( \hat{f}|_{\partial Q^+_M} \) expanded by simple covers to a single-level (Ref(M)|\partial c(Q^+_M))\(^\Delta\)-Alexander map expanded by simple covers. Indeed, we set \( \partial c(Q^+_M) \) at level \( q-1 \), and extend the map \( \hat{f}|_{q} \) on the faces \( q \) of \( Q \) to the corresponding tents \( T(q) \) following Step 1. Since the leading face \( q^+_j \) of \( Q^+_M \) is now an exterior face, a slight modification is needed for the extension over the tent \( T(q^+_j) \).

Since \( \hat{f}|_{q^+_j} \) is Alexander map on the base of tent \( T(q^+_j) \), the extension is also Alexander on the top. Therefore the extended map \( \hat{f}|_{\partial c(Q^+_M)} \) is an Alexander map expanded by simple covers on a single level.

Corollary \[14.8\] gives an alternative proof, in case of molecules, for the quasiregular extension theorem in \[14\] stated in the beginning of this part.

**Corollary 14.9.** Let \( M \) be an \( n \)-molecule having maximal atom length \( \ell \), \( d_M \) a length metric on \( M \), and \( f: \partial |M| \to S^{n-1} \) a standard \((M|\partial |M|)^\Delta\)-Alexander map. Then \( f \) has a \( K_M \)-quasiregular extension
\[ F: |M| \to B^n \]
over the molecule \( M \) for some \( K_M = K_M(n, \ell) \geq 1 \).

Proof. By Corollary \[14.8\] there exists an \( L(n, \ell) \)-BLD-extension
\[ \tilde{f}: |M| \setminus \text{int } c(Q^+_M) \to S^{n-1} \times [1, 3^\ell] \]
of \( f : \partial |M| \to S^{n-1} \times \{1\} \) for which \( \hat{f} |_{\partial c(Q^+)} : \partial c(Q^+_M) \to S^{n-1} \times \{3^p\} \) is a \( \text{Ref}(M)|_{\partial c(Q^+_M)} \)-Alexander map expanded by simple covers. We now postcompose \( \hat{f} \) with a quasiregular map expanded by simple covers. Then the map \( F = g \circ \hat{f} : |M| \setminus c(Q^+_M) \to B(0,1) \setminus \text{int} B(0,e^{-3^p}) \). Then the map \( F = g \circ \hat{f} : |M| \setminus c(Q^+_M) \to B(0,1) \setminus \text{int} B(0,e^{-3^p}) \) is \( \mathcal{K}' \)-quasiregular for some \( \mathcal{K}' = \mathcal{K}(n, \ell) \). A canonical radial extension of \( F \) over \( c(Q^+_M) \) yields, as in Rickman [49, I.3], a \( \mathcal{K} \)-quasiregular map \( F : |M| \to B(0,1) \) for some \( \mathcal{K} = \mathcal{K}(n, \ell) \). This completes the proof. \( \square \)

15. Extension over dented molecules

The extension principles of two level Alexander maps over molecules yield also extension of two level Alexander maps over a more general class of complexes, called dented molecules. To define this class of complexes, we begin with some auxiliary definitions.

15.1. Properly embedded molecule in a cube. Recall that associated to each molecule \( \hat{M} \) and after the identification of a leading cube \( Q^+_M \) and a leading face \( q^+_M \) of \( \hat{M} \), there is a partial ordering \( \ll_M \) of the \( n \)-cubes \( Q_M \) in \( M \). With respect to this partial ordering, the adjacency graph \( \Gamma(Q_M) \) is a tree.

**Definition 15.1.** A molecule \( \hat{M}, \ll_M \) is properly embedded in a cube \( Q \) if

1. for each cube \( Q' \) in \( Q_M \), there is a (unique) integer \( \beta_Q(Q') \geq 1 \) for which \( Q' \in \text{Ref}_\beta(Q')(Q) \);
2. the leading cube \( Q^+_M \) of \( \hat{M} \) has either one or two \((n-1)\)-face contained in \( \partial Q \), and the leading face \( q^+_M \) is contained in \( \partial Q \cap Q^+_M \); and
3. each cube \( Q' \in Q_M \) if \( Q' \neq Q^+_M \), has exactly one \((n-1)\)-face contained in \( \partial Q \).

Suppose that \( \hat{M} \) is a properly embedded molecule in a cube \( Q \). We call the number

\[ \beta_Q(\hat{M}) = \max \{ \beta_Q(Q') : Q' \in Q_M \} \]

the refinement index of the embedded \( \hat{M} \) in \( Q \).

We introduce now the 'base-roof-wall' partition of the boundary \( \partial |M| \) of a properly embedded molecule \( M \) in a cube \( Q \).

**Definition 15.2.** Let \( \hat{M} \) be a properly embedded molecule in a cube \( Q \). An \((n-1)\)-face \( q \) of a cube \( Q' \in \Gamma(Q_M) \) is

1. a base face of \( \hat{M} \) if \( q \) is not the leading face \( q^+_M \), and \( q \subset \partial Q \), and
2. a roof face of \( \hat{M} \) if there exists a \( n \)-cube \( Q \) in \( \Gamma(Q_M) \) for which \( q \) is a face of \( Q \) opposite to a base face in \( Q \).

We say that \( q \) is a wall if \( q \) is neither a base face, a roof face, nor the leading face \( q^+_M \).

Note that each cube \( Q' \in \Gamma(Q_M), Q' \neq Q^+_M \), has exactly one base face and exactly one roof face, and that all other \((n-1)\)-faces of \( Q' \) contained
in $\partial |\hat{M}|$ are walls. Note also that each roof face is essentially contained in the interior of $Q$.

We denote $\text{Base}(\hat{M}; Q)$, $\text{Roof}(\hat{M}; Q)$, and $\text{Wall}(\hat{M}; Q)$ the collections of base, roof, and wall faces of $\hat{M}$, respectively. Since these collections of are disjoint, their spaces $|\text{Base}(\hat{M}; Q)|$, $|\text{Roof}(\hat{M}; Q)|$, and $|\text{Wall}(\hat{M}; Q)|$, together with the leading face $q^+_M$, form an essential partition of $\partial \hat{M}$. We note also that sets $|\text{Base}(\hat{M}; Q)|$ and $|\text{Wall}(\hat{M}; Q)| \cup |\text{Roof}(\hat{M}; Q)|$ are $(n-1)$-cells.

**Convention 15.3.** In what follows, we assume that the leading face $q^+_M$ of a molecule $M$ is chosen so that it meets the base $|\text{Base}(\hat{M}; Q)|$ in an $(n-2)$-cell, that is, $|\text{Base}(\hat{M}; Q) \cup \{q^+_M\}|$ is $(n-1)$-cell.

15.2. Dented atoms.

**Definition 15.4.** Let $A$ be an atom and $D$ be a subcomplex of an iterated refinement $\text{Ref}_k(A)$ of $A$. The complex $D$ is a dented atom in $A$ if, for each $n$-cube $Q \in A$, either (a) $|Q| \subset |D|$, or (b) there exists a properly embedded molecule $\hat{M}_Q$ in $Q$ for which $\text{cl}(Q \setminus |\hat{M}_Q|) = |D| \cap |Q|$ and $|\hat{M}_Q|$ does not meet any other $n$-cube in $A$.

We call the atom $A$ in Definition 15.4 the hull $\mathcal{H}(D)$ of the dented atom $D$, and the properly embedded molecules $\hat{M}_Q$ the dents of $D$, or the dents of $\mathcal{H}(D)$. We denote

$$\beta_{\text{dents}}(D) = \max\{\beta_Q(\hat{M}_Q) : \hat{M}_Q \text{ is a dent of } D\}.$$

Let $D$ be a dented atom having atom $A$ as its hull. The dented cubes in $D$ may be ordered following the ordering of the undented cubes in $A$.

In future applications, we typically consider a dented atom $D$, which is a subcomplex of a larger complex $P$. In these cases, the hull $\mathcal{H}(D)$ need not be a subcomplex of $P$, but there exists a refinement index $k(D) \in \mathbb{N}$ for which $D \subset \text{Ref}_{k(D)}(\mathcal{H}(D)) \subset P$.

**Remark 15.5.** The definition of the dented atom is [13] is more general, which allows, in each cube, multiple dents satisfying a criterion on relative separation. Our applications in Part 3 do not require this more general setting.

We now reduce the problem of extension over dented atoms to the problem of extension over atoms.

Let $D$ be a dented atom, $A = \mathcal{H}(D)$ its hull, and $f : \partial |D| \to \mathbb{S}^{n-1}$ a $(D|_{\partial |D|})^D$-Alexander map, possibly expanded by standard simple covers. We show that the extension problem for $f$ over $|D|$ is equivalent to an extension problem for an $(A|_{\partial A})^A$-Alexander map, expanded by standard simple covers, over $|A|$. This is a simple consequence of the following flattening lemma; the proof is a combination of [13] Proposition 3.12 and the extension procedures over molecules in Section 14.

**Lemma 15.6** (Flattening Lemma). Let $D$ be a dented atom and $A = \mathcal{H}(D)$ be the hull of $D$. Then there exist $L = L(n) \geq 1$ and an $L$-bilipschitz map $\varphi : |D| \to |A|$ which satisfy the conditions:

1. $\varphi$ is the identity on $\partial |D| \cap \partial |A|$;
(2) \( \varphi \) is the identity on each \( n \)-cube \( Q \in A \) which satisfies \( Q \subset |D| \), and on each \((n-1)\)-cube \( q \) in \( A \) which does not meet any of the dents of \( D \);

(3) for each dent \( \hat{M} \) of \( D \),
\[
\varphi([\text{Roof}(\hat{M};Q) \cup \text{Wall}(\hat{M};Q)]) = [\text{Base}(\hat{M};Q)] \cup q^+_\hat{M},
\]
and

(4) for each standardly expanded \((D|\partial|D|)\)-Alexander map \( f: \partial|D| \to S^{n-1} \) that is \( L'\)-BLD for some \( L' \geq 1 \), there exist \( L'' = L''(L, L') \geq 1 \), a standardly expanded \((A|\partial|A|)\)-Alexander \( L''\)-BLD-map \( f_A: \partial|A| \to S^{n-1} \), and an \( L'\)-BLD branched cover homotopy
\[
F: \partial|A| \times [1, 3^{\text{dents}(D)}] \to S^{n-1} \times [1, 3^{\text{dents}(D)}]
\]
from \( f \circ \varphi^{-1} \) to \( f_A \).

Proof. Since each dent \( \hat{M} \) in \( D \) meets only one cube in \( A \) and each cube in \( Q_M \) belongs to an iterated \((1/3)\)-refinement of \( A \), we conclude that the distance between each \( Q' \in Q_M \) and any \( Q'' \in A \) not intersecting \( Q' \) is at least the side length of \( Q' \). Thus it suffices to consider the case that \( A \) consists of a single cube \( Q \) with a single dent \( \hat{M} \).

We define \( \varphi: |D| \to Q \) following the folding process introduced in [14 Proposition 3.12]; the folding collapses the cubes in \( Q_M \) one by one along the tree \( \Gamma(Q_M) \) starting at the leaves.

We may list the elements in \( Q_M \) in a linear order \((Q_1, Q_2, \ldots, Q_m)\), where

(1) \( Q_m \) is the leading cube \( Q^+_M \), and

(2) if \( 1 \leq i < j \leq m \), then either \( Q_i \preceq_M Q_j \), or \( Q_i \) and \( Q_j \) are not ordered by \( \preceq_M \).

For a cube \( Q' \in Q_M \) which is the \( j \)-th on the list, we take
\[
G(Q') = \bigcup \{Q_i: 1 \leq i \leq j\}.
\]

Let \( Q' \in Q_M \) be a cube. We consider two cases.

Case I. Suppose that \( Q' \) is not the leading cube \( Q^+_M \). Let \( Q'' \in Q_M \) be the unique predecessor of \( Q' \) in the tree \( \Gamma(Q_M) \), and \( s_{Q'} = Q' \cap Q'' \). Let \( b_{Q'} \) be the base face of \( Q' \), and \( r_{Q'} \) the roof face of \( Q' \). Then there exists a homeomorphism
\[
\varphi_{Q'}: \text{cl}(Q \setminus \hat{M}) \cup G(Q') \to \text{cl}(Q \setminus \hat{M}) \cup G(Q') \cup Q'
\]
which satisfies the conditions:

(1) \( \varphi_{Q'}(\partial Q' \cap \partial|\hat{M}|) = \partial Q' \cap \partial|\hat{M}| \), and \( \varphi_{Q'} \) is bilipschitz;

(2) \( \varphi_{Q'} \) is the identity on \( \partial|D| \cap \partial|A| \), and

(3) \( \varphi_{Q'} \) is the identity in the complement of the set \( U(Q') \), where \( U(Q') \) is the union of \( Q' \) and the \( n \)-cubes adjacent to \( Q' \) in \( \text{Ref}_{B|Q(M)}(Q) \).

In particular, we may assume that \( \varphi_{Q'} \) is \( L_0\)-bilipschitz with a constant \( L_0 \geq 1 \) depending only on \( n \). Note that each point in \( Q \) can belong to at most \( k \) sets in the collection \( \{U(Q')\} \), where \( k = k(n) \geq 2 \) is a constant depending only on \( n \).
Case II. Suppose that \( Q' = Q^+_M \). Set \( bQ' = \partial Q' \cap \partial Q \), which is the union of one or two faces of \( Q' \), and let \( \varphi_{Q'} : cl([D] \setminus Q') \to [D] \) be any \( L_0 \)-bilipschitz homeomorphism that maps \( cl(\partial Q' \setminus bQ') \) onto \( bQ' \), and satisfies the conditions (S2), (S3), and (S4) above.

The homeomorphism \( \varphi : [D] \to [A] \) may now be defined as the composition of maps \( \varphi_{Q'} \) over all cubes in \( Q^+_\hat{M} \) following the given ordering \((Q_1, Q_2, \ldots, Q_m)\). We conclude that \( \varphi \) is \( L \)-bilipschitz homeomorphism, with \( L = L(n, L_0) \), which maps \( |\text{Wall}(\hat{M}; Q) \cup \text{Roof}(\hat{M}; Q)| \) affinely, locally on cubes, onto the \((n - 1)\)-cell \(|\text{Base}(\hat{M}; Q)| \cup \{q^+_M\}\).

It remains to show that the map \( f' = f \circ \varphi^{-1} : \partial |A| \to S^{n-1} \) is BLD-homotopic to a standardly expanded \((A|_{\partial |A|})^\Delta\)-Alexander map \( \partial |A| \to S^{n-1} \). The proof is given inductively with respect to the partial ordering of the atoms in \( \hat{M} \).

Let \( \hat{A} \) be the maximal atom in the (partially ordered) atom decomposition of \( \hat{M} \), and denote by \( \text{Base}(\hat{A}; Q) \), \( \text{Roof}(\hat{A}; Q) \), and \( \text{Wall}(\hat{A}; Q) \), respectively, the subcollection of the elements in \( \text{Base}(\hat{M}; Q) \), \( \text{Roof}(\hat{M}; Q) \), and \( \text{Wall}(\hat{M}; Q) \), respectively, which are contained in \( \hat{A} \).

Since the number of steps is finite, it suffices to assume that we have already found a BLD-homotopy \( F' : \partial |A| \times [1, 3^\beta] \to S^{n-1} \times [1, 3^\beta] \) from \( f' \) to a BLD-map \( f'_{\hat{A}} : \partial |A| \to S^{n-1} \), where \( f'_{\hat{A}} \) is an \((A|_{\partial |A|})^\Delta\)-Alexander map on \( \partial |A| \setminus (\text{Base}(\hat{A}; Q) \cup q^+_M) \) and \( f_{\hat{A}} \) is a standardly expanded Alexander map on \( |\text{Base}(\hat{A}; Q)| \cup q^+_M \) with respect to the complex \( P_M \) induced by \( \varphi \) from \( |\text{Roof}(\hat{M}; Q)| \cup |\text{Wall}(\hat{M}; Q)| \). In view of folding, \( P_M |_{|\text{Base}(\hat{A}; Q)| \cup q^+_M} \) is indeed a well-defined subcomplex of \( P_M \). Furthermore, we assume that the simple covers expanding the underlying Alexander map are standard, and properly located in the sense of Definition 14.6.

We proceed to deform \( f_{\hat{A}} \) on \(|\text{Base}(\hat{A}; Q)| \cup q^+_M\) following the proof of Proposition 14.6. Since the walls in \( \partial |\hat{M}| \) have a natural partition into \((n - 1)\)-atoms, i.e. rows of unit \((n - 1)\)-cubes, connecting the base to the roof, we conclude that the same holds for rows of affine \((n - 1)\)-cubes in \( P_M \). Following Steps 1 and 3 in the proof of Proposition 14.6 we may iteratively reduce, in \( \beta_Q(\hat{A}) \) steps, the complex in \( \varphi(|\text{Wall}(\hat{A}; Q)|) \) to the outermost layer of \(|\text{Base}(\hat{A}; Q)| \cup q^+_M \), while simultaneously expanding the image \( \varphi(|\text{Roof}(\hat{A}; Q)|) \) to the remaining part of \(|\text{Base}(\hat{A}; Q)| \cup q^+_M \) with uniformly controlled distortion. During this process, we also make sure that the simple covers, created by the reduction of the complex \( P|_{\varphi(|\text{Wall}(\hat{A}; Q)|)} \), are properly located.

This reduction of the complex \( P \) to \( A|_{|\text{Base}(\hat{M}; Q)|} \) and the placement of standard simple covers yield the desired BLD-homotopy.

15.3. **Dented molecules.** The definition of a dented molecule is analogous to that of a molecule in Definition 13.2.

**Definition 15.7.** An \( n \)-complex \( \mathcal{D} \) is a dented molecule if \( |\mathcal{D}| \) is an \( n \)-cell and there exist a dented atom decomposition \( \mathcal{D} \) of \( \mathcal{D} \) and a partial ordering \( \ll_{\mathcal{D}} \) such that...
for adjacent dented atoms $D$ and $D'$ in $\mathcal{D}$ satisfying $D \ll_D D'$, their intersection $|D| \cap |D'|$ is a face of a cube in $\mathcal{D}$, but not a face of a cube in $D'$.

(2) for each $D \in \mathcal{D}$ and a dent $\hat{M}$ in its hull $\mathcal{H}(D)$, and for each $n$-cube $Q \in Q_{\hat{M}}$, there exists at most one other dented atom $D' \in \mathcal{D}$ that is contained in $Q$ and meets $|D|$.

The heuristic idea behind Definition 15.7 is that a dented molecule $\mathcal{D}$ has an essential partition $\mathcal{D}$ into dented atoms and that dents of dented atoms in $\mathcal{D}$ may contain other dented atoms in $\mathcal{D}$. In other words, we do not assume that the hulls of dented atoms in $\mathcal{D}$ are mutually essentially disjoint. The combinatorial structure of a dented molecule is, however, otherwise similar to that of a molecule, that is, in both cases the complex may be subdivided into simpler pieces (atoms or dented atoms) which are arranged into a monotone tree in terms of the side lengths. We refer to [14, Section 3] for a related discussion.

We note that, as in the atom decomposition of a molecule, condition (1) in Definition 15.7 makes the dented atom decomposition of a dented molecule unique.

Quasiregular extensions of Alexander maps over dented molecules are now obtained similarly as the extensions over molecules using methods in the proofs of Proposition 14.6 and Theorem 14.3, together with the Flattening Lemma (Lemma 15.6). Since the hulls of dented atoms in a dented molecule are not mutually essentially disjoint, three cases arise in the extension. We call the different cases as classes, and outline the idea in extension for each class.

Let $\mathcal{D}$ be a dented molecule, and $\mathcal{D}$ its dented atom decomposition with a partial ordering $\ll_{\mathcal{D}}$.

Class I: Atoms. The simplest class of elements in $\mathcal{D}$ is the atoms. We emphasize that there is no difference, from the point of view of adjacency, in having an atom $A$ in a dented atom decomposition of $\mathcal{D}$ or in an atom decomposition of a molecule. Therefore, given an atom $A$ in $\mathcal{D}$, we may extend an Alexander map $|A| \cap \partial |A| \to S^{n-1}$ quasiregularly over $|A|$ as in the case of molecules in Theorem 14.3.

Class II: Dented atoms with empty dents. Suppose that $D \in \mathcal{D}$ is a dented atom whose hull $\mathcal{H}(D)$ does not contain any other elements of $\mathcal{D}$; thus $|\mathcal{H}(D)| \cap |\mathcal{D}| = |D|$. In this case, we apply the Flattening Lemma (Lemma 15.6) to reduce the extension problem to the case of atoms.

Class III: General dented atoms. In the general case, $D \in \mathcal{D}$ is a dented atom whose hull $\mathcal{H}(D) = A$ contains other elements of $\mathcal{D}$. These dented atoms $\text{Dents}(D)$ in $\mathcal{D}$ are contained in the dents of $D$, that is, in the components of $\text{cl}(A \setminus D)$. Elements in $\text{Dents}(D)$ may be grouped into dented molecules $\mathcal{D}_A, \mathcal{D}_{A,1}, \ldots, \mathcal{D}_{A,m_A}$, which have natural dented atom decompositions $\mathcal{D}_{A,1}, \ldots, \mathcal{D}_{A,m_A}$ respectively, induced by the partial ordering $\ll_{\hat{M}}$. Based on this hierarchy, we may assume, before extending an Alexander map $|D| \cap \partial |\mathcal{D}| \to S^{n-1}$ over $|D|$, that we have already extended the Alexander map over the dented molecules contained in the dents of $D$. Thus the mapping to extend from the boundary $\partial |D|$ to $|D|$ is, in fact, a multi-level Alexander map; note that, since $\partial |D|$ is not contained in $\partial |\mathcal{D}|$, there is no
pre-defined Alexander map on $\partial |D|$. After these preparations the extension problem is almost verbatim to the extension problem for multi-level Alexander maps for dented atoms. A similar induction step is formalized in [14] (see especially the Machine in Section 5), and we omit the further details of the induction here.

There are several possible formulations for theorems for dented molecules. A model case is stated in Theorem 15.8. Since the proof of this statement is almost verbatim to that of Theorem 14.3 modulo the application of the flattening lemma, we omit the details.

For the statement, we fix some notations. Let $D^+$ the maximal dented atom in the decomposition $(D, \ll_D)$ of $D$ having hull $\mathcal{H}(D^+) = A$. Let $Q^+$ be the leading $n$-cube of the atom $A$, and $Q = Q^+ \cap D^+$ be the leading (dented) $n$-cube in $D^+$; we also designate $\hat{Q}$ to be the leading (dented) $n$-cube in $D$. By Definition 15.7, $\partial \hat{Q} \cap \partial Q^+$ contains at least one $(n-1)$-face of $Q^+$. We assign one such face, $q^+$, as the leading face of $D^+$, and also call $q^+ = q^+(D)$ the leading face of $D$.

**Theorem 15.8.** Let $D$ be a dented $n$-molecule with a dented atom decomposition $D$, $f : c(\partial |D| \setminus q^+(D^+)) \rightarrow S^{n-1}$ be a $(\mathcal{H}|_{\partial |D|})^{\Delta}$-Alexander map, and

$$\ell = \max_{D \in D} \#(\mathcal{H}(D)^{(n)}).$$

Then there exist $L = L(n, \ell) \geq 1$, $\lambda = \lambda(n, \ell) \in \mathbb{N}$, and an $L$-BLD-map

$$F : |D^+| \rightarrow S^{n-1} \times [1, 3^\lambda]$$

extending $f$, for which the restriction

$$F|_{c(q^+(D))} : c(q^+(D^+)) \rightarrow S^{n-1} \times \{3^\lambda\}$$

is a $\text{Ref}(c(q^+(D)))^{\Delta}$-Alexander map expanded by properly placed standard simple covers.

**Remark 15.9.** Theorem 15.8, together with an argument analogous to the proof of Corollary 14.9, yields a new proof for the quasiregular extension theorem, from [14], stated in the beginning of Part 3.

15.4. Dented molecules in an ambient complex. We finish this section by formulating a dented molecule extension theorem over an ambient complex. There are several variations of such results. The version below is used in the Mixing Theorem (Theorem 1.7).

The starting point is a product complex $P = K \times I$, where $I$ is a 1-dimensional complex for which $|I| = [0, 1]$ and in which all 1-cubes have the same length. The complex $K \times I$ is now obtained by

$$K \times I = \{\sigma \times \tau : \sigma \in K, \tau \in I\}.$$ 

Denote by

$$P_0 = K \times \{0\} \quad \text{and} \quad P_1 = K \times \{1\},$$

respectively, subcomplexes of $P$; their spaces $|P_0|$ and $|P_1|$ are the boundary components of $|P|$. 
We modify the complex $P$ by carving out properly embedded mutually disjoint dented molecules from $P$ along $P_0$, and by adding properly attached mutually disjoint dented molecules to $P_0$.

**Definition 15.10.** A dented molecule $\mathcal{D} \subset P$ is properly embedded into $P$ if each $n$-cube $Q$ in the hull $\mathcal{H}(D^+) \subset P$ of the maximal dented atom $D^+$, in the dented atom decomposition, of $\mathcal{D}$ has a face in $|P_0|$.

In this case, we say that the subcomplex $P - \mathcal{D} \subset P$ satisfying

$$|P - \mathcal{D}| = \text{cl}(|P| \setminus |\mathcal{D}|)$$

is obtained by subtracting $\mathcal{D}$ from $P$.

**Definition 15.11.** A dented molecule $\mathcal{D}$ is a properly attached to $P$ if the space $|P|$ and the hulls $\mathcal{H}(D)$ of the dented atoms $D_i$ in the dented atom decomposition of $\mathcal{D}$, are essentially disjoint, and if the leading cube $Q^+$ in the hull $\mathcal{H}(D^+)$ of the maximal dented atom $D^+$ in the dented decomposition of $\mathcal{D}$ has a face in $|P_0|$. We say that the complex $P \cup \mathcal{D}$ is obtained by adding $\mathcal{D}$ to $P$.

When a dented molecule $\mathcal{D}$ is either added to, or subtracted from $P$, we assume that the leading face $q^+ \mathcal{D}$ is contained in $|P_0|$.

**Theorem 15.12.** Let $K$ be a cubical complex on a closed $(n-1)$-manifold $\Sigma$, I be a complex on $[0,1]$ which contains at least three 1-simplices, and $P = K \times I$ be the product cubical complex on $\Sigma \times [0,1]$. Let $P''$ be a cubical complex which is obtained from a refinement $\text{Ref}_k(P)$ of $P$ by properly adding and properly subtracting mutually disjoint dented n-molecules $\mathcal{D}_1^a, \ldots, \mathcal{D}_r^a$ and $\mathcal{D}_1^s, \ldots, \mathcal{D}_s^s$, respectively, to the boundary component $\Sigma \times \{0\}$. Let $\ell \geq 1$ be the maximum of the atom lengths of these attached and removed dented molecules.

Then, given a $(P''|_{\partial P''\setminus(\Sigma \times \{1\})})^\Delta$-Alexander map $f_0 : \partial |P''| \setminus (\Sigma \times \{1\}) \to S^{n-1}$, possibly expanded by standard simple covers, there exist a constant $K = K(n, \ell), \geq 1$, a number $0 < r = r(n, \ell, k) > 0$, and a $K$-quasiregular extension

$$F : |P'| \to B^n \setminus \text{int}B^n(r)$$

of $f_0$, for which the restriction

$$F|_{\Sigma \times \{1\}} : \Sigma \times \{1\} \to \partial B^n(r)$$

is a $(P_1)^\Delta$-Alexander map expanded by properly placed standard simple covers.

**Sketch of a proof.** We outline the steps in the proof. First, since the dented molecules $\mathcal{D}_1^a, \ldots, \mathcal{D}_r^a$ and $\mathcal{D}_1^s, \ldots, \mathcal{D}_s^s$ are mutually disjoint, we may assume that $\mu = 1$ and $\nu = 1$ and denote $\mathcal{D}_1^a = \mathcal{D}_1^a$ and $\mathcal{D}_1^s = \mathcal{D}_1^s$.

We consider first the dented molecule $\mathcal{D}_s^s$ which is subtracted from $P$. Let $H^s$ be the union of the hulls of dented atoms in the dented atom decomposition of $\mathcal{D}_s^s$. Then $H^s$ is a union of mutually disjoint molecules in $P$, and the components of $\text{cl}(H^s \setminus \mathcal{D}_s^s)$ are dented molecules. Thus we may extend $f_0$ by Theorem 15.8 over each component of $\text{cl}(H^s \setminus \mathcal{D}_s^s)$ and obtain a standardly expanded multi-level Alexander tower on $\text{cl}(\partial |H^s| \setminus |P_0|)$. The method used in the proof of the Flattening Lemma (Lemma 15.6) allows us
to pass from a standardly expanded multi-level $\text{Ref}_k(P_0)\Delta$-Alexander tower on $\text{cl}(\partial[H^*] \setminus |P_0|)$ to an Alexander tower on $|H^*| \cap |P_0|$.

Similarly, for $\mathcal{D}^a$, we may extend $f$ to the union $H^*$ of hulls of dented atoms in the atom decomposition of $\mathcal{D}^a$. Again, we may extend $f$ over the dented molecules in $\text{cl}(H^* \setminus \mathcal{D}^a)$ and the Flattening Lemma allows us to pass the extension to a standardly expanded multi-level $\text{Ref}_k(P_0)\Delta$-Alexander tower on $\text{cl}(\partial[H^*] \cap |P_0|)$.

The extension of the obtained expanded multi-level Alexander tower over the product complex $P$ is now trivial. \hfill \Box

16. Separating complex and quasiregular branched covers

In this section we apply the dented molecule extension theorems in Section 15 to prove the Mixing Theorem (Theorem 1.7). The Mixing Theorem is a geometric version of Theorem 1.5, which establishes the existence of quasiregular branched covers between manifolds of arbitrarily large degree with distortion independent of the degree, under the assumptions: (a) the existence of a separating complex in the domain manifold, and (b) the domain and the target having the same number of boundary components.

16.1. Separating complexes. The notion of the separating complex and the method of extension have their origin in Rickman’s Picard construction [48], and Heinonen and Rickman in [24] and [25]. See Figure 39 for a simple example of a separating complex.

For the definition, we say that an $(n-1)$-dimensional subcomplex $L$ of a cubical $n$-complex $K$ is cubically connected if for any two $(n-1)$-simplices $q$ and $q'$ in $L$, there exists a sequence $q = q_0, q_1, \ldots, q_k = q'$ of adjacent $(n-1)$-simplices in $L$. Further, we say that $L$ is locally separating in $K$ if, for each point $x \in |L|$ and each sufficiently small neighborhood $U$ of $x$ in $|K|$, the set $U \setminus |L|$ is not connected.

Definition 16.1. Let $K$ be a cubical complex on an $n$-manifold with boundary components $\Sigma_1, \ldots, \Sigma_m$. A cubically connected and locally separating $(n-1)$-dimensional subcomplex $Z$ of $K$ is a separating complex if there exist subcomplexes $K_1(Z), \ldots, K_m(Z)$ of $K$ for which

(1) $|K_1(Z)|, \ldots, |K_m(Z)|$ is an essential partition of $|K|$,  
(2) $|K_i(Z)| \cap |K_j(Z)| \subset |Z|$ for all $i \neq j$, and  
(3) $|K_i(Z)| \cap \partial|K| = \Sigma_i$ and the set $|K_i(Z)| \setminus |Z|$ is homeomorphic to $\Sigma_i \times [0,1)$ for each $i = 1, \ldots, m$.

Note that, in the definition of a separating complex, we require neither that $|K_i(Z)|$ be homeomorphic to $\Sigma_i \times [0,1]$ nor that $|Z| \subset \bigcup_{i=1}^m \partial|K_i(Z)|$. In fact, in our applications, neither of these stronger properties holds; see Figure 39.

If $Z$ is a separating complex in $K$, we denote $\mathcal{N}_K(Z) = \{Q \in K: Q \cap |Z| \neq \emptyset\}$, and note that $|Z| \subset \text{int}|\mathcal{N}_K(Z)|$. We call $\mathcal{N}_K(Z)$ and $|\mathcal{N}_K(Z)|$ the cubical neighborhoods of $Z$ and $|Z|$, respectively.

In the following proposition, we show that, apart from very special cases, all cubical complexes of interest to us have separating complexes. In some of
the forthcoming applications, separating complexes are specifically chosen in concatenation with the topology of the manifolds in question in order to obtain control of the distortion based on the initial data only.

We thank Gaven Martin for asking us a question on the existence of separating complexes, which led us to consider this proposition.

Proposition 16.2. Let \( K \) be a cubical \( n \)-complex for which \(|K|\) is an \( n \)-manifold with boundary. If the boundary components of \(|K|\) have pairwise disjoint collars, each of which is the space of a subcomplex of \( K \), then \( K \) has a separating complex \( Z \).

**Proof.** Let \( \Sigma_1, \ldots, \Sigma_m \) be the boundary components of \(|K|\). Let \( K_1, \ldots, K_m \) be the pairwise disjoint subcomplexes of \( K \) whose spaces \(|K_i|\) are collars of \( \Sigma_i \). Let \( K' \) be the subcomplex of \( K \) for which \(|K'| = |K| \setminus \text{int}(|K_1| \cup \cdots \cup |K_m|)\).

Let \( \Gamma \) be a maximal tree in the adjacency graph of the \( n \)-cubes in \( K' \), and let \( q_1 \) be a common \((n-1)\)-face of an \( n \)-cube in \( K' \) and an \( n \)-cube in \( K_1 \).

We now let \( Z \) be a subcomplex of \( K' \) which is obtained from \( K' \) by removing \( q_1 \) and the \((n-1)\)-faces in \( \Gamma \). Since the \((n-2)\)-skeleton of \( K' \) is connected, so is the complex \( Z \). Also, \( Z \) is locally separating because \( \Gamma \) is a tree. Moreover, \(|K_i| \subset |K_i(Z)|\) for each \( i \), and \(|K'| \subset |K_1(Z)|\). Thus \((K_1(Z), \ldots, K_m(Z))\) is an essential partition of \(|K'|\). Furthermore, for each \( i \), \(|K_i(Z)| \setminus |Z|\) is homeomorphic to \( \Sigma_i \times [0,1) \). Since \(|K_i(Z)| \cup |K_j(Z)| \subset |Z|\) for \( i \neq j \), the complex \( Z \) is a separating complex in \( K \).

**Remark 16.3.** Note that, given a cubical complex \( K \) with boundary collars, we may always subdivide the cubes in the collars so that the obtained complex \( K' \) has pairwise disjoint boundary collars. Thus as a corollary we obtain that each cubical complex \( K \) has a cubical refinement \( K' \) which admits a cubical structure and \( K| \Sigma = K'| \Sigma \).

16.2. The Mixing Theorem. We now statement of the main theorem of this section. The word ‘mixing’ in the title of the next theorem refers to repeated trading of pieces of adjacent regions near the boundary.

**Theorem 1.7** (Mixing). Let \( n \geq 3 \) and \( m \geq 2 \). Suppose that \( K \) is a cubical complex on an \( n \)-manifold with boundary components \( \Sigma_1, \ldots, \Sigma_m \) and that \( K \) has a separating complex.

Then there exists a constant \( K = K(n,K) \geq 1 \) for the following. For any \( k' \in \mathbb{N} \), there exist \( k \geq k' \) and a \( K \)-quasiregular map

\[
f : |K| \to \mathbb{S}^n \setminus \text{int}(B_1 \cup \cdots \cup B_p),
\]
where $B_1, \ldots, B_p$ are pairwise disjoint Euclidean balls, such that each restriction $f|_{\Sigma_i}: \Sigma_i \to \partial B_i$ is a $(\text{Ref}_k(K)|_{\Sigma_i})^\Delta$-Alexander map expanded by free simple covers.

**Remark 16.4.** Although the distortion $K$ of $f$ is independent of the refinement index $k$, the degree $\deg f \geq c(n, K)3^k$ for a constant $c(n, K) > 0$ depending only on the dimension $n$ and the chosen separating complex for $K$.

Let $Z$ be a separating complex of $K$, $K_i$ be the subcomplexes of $K$ in Definition 16.1, and
\[
U_i = |K_i(Z)| \setminus |Z|.
\]
Note that $\overline{U_i}(Z) = |K_i(Z)|$. In what follows, we also work with the (abstract) metric completions $\overline{U_i}(Z)$ of domains $U_i(Z)$. Note that, since $Z$ is locally separating, for each $i = 1, \ldots, m$, the completion of the path metric in $U_i(Z)$ doubles the boundary points of $U_i(Z)$ in $|Z| \cap \text{int}|K_i(Z)|$. This metric completion $\overline{U_i}(Z)$ is an $n$-manifold with boundary, homeomorphic to $\Sigma_i \times [0, 1]$. We denote by
\[
\mathcal{U}(Z) = (\overline{U_1}(Z), \ldots, \overline{U_m}(Z))
\]
the ordered sequence of these completions. Formally, $\mathcal{U}(Z)$ is not an essential partition of $|K|$. However, we consider $\mathcal{U}(Z)$ as an essential partition of $|K|$ via the natural projection $\pi_Z: \overline{U_i}(Z) \to \overline{U_i}(Z)$ from the metric completion $\overline{U_i}(Z)$ to $|K|$. We also extend the terminology of Alexander sketches as follows. A sequence $\mathcal{F} = (F_1, \ldots, F_m)$ is called an $(\mathcal{U}(Z), \mathcal{E})$-Alexander sketch if $\mathcal{F}$ is an Alexander sketch on the metric completion $\mathcal{U}(Z)$ and each $F_i$ is a composition $\tilde{F}_i \circ \pi_Z$ of the natural projection $\pi_Z$ and a branched cover $\tilde{F}_i: |K_i(Z)| \to E_i$. As before, $\mathcal{E} = (E_1, \ldots, E_m)$ is a cyclic cell partition of $\mathbb{S}^n$.

**16.3. Strategy of the proof.** To a given separating complex $Z$ of $K$, we associate a $(\mathcal{U}(Z), \mathcal{E})$-Alexander sketch $\mathcal{F} = (F_1, \ldots, F_m)$. We then weave the maps in the sketch $\mathcal{F}$ into a branched cover $|K| \to \mathbb{S}^n$. This procedure is topological. But, if the mappings in the Alexander sketch are BLD, the resulting map $|K| \to \mathbb{S}^n$ is also BLD. The BLD-constant of this resulting map depends on two factors: (a) the bilipschitz constants of the chosen homeomorphisms $U_i \to \Sigma_i \times [0, 1]$, and (b) the connected trees in the neighborly graph $\Gamma(\mathcal{M}(Z), \mathcal{F})$ used in weaving. See Theorem 10.3 for weaving, and Remark 10.7 for discussions of BLD-constants.

In order to obtain quasiregular branched covers of arbitrarily large degree, we pass to refinements $\text{Ref}_k(K)$ of $K$ of high orders. We modify the given separating complex $Z$ iteratively by a method in 14 to obtain separating complexes $Z_k$ in $\text{Ref}_k(K)$ having suitable geometric properties. This type of the iterated modification has its origin in Rickman [48] under the term caving.

The modification of $Z$ is done globally in the initial step and locally in the subsequent steps. The goal is to obtain a sequence $(Z_k)$ of separating
complexes \( Z_k \) for which the corresponding sequence of manifold partitions 
\( (\overline{U}_1(Z_k), \ldots, \overline{U}_m(Z_k)) \), for \( k \geq 1 \), have the following properties:

(S1) the components \( U_1(Z_k), \ldots, U_m(Z_k) \) of \( |K| \setminus |Z_k| \) are bilipschitz equivalent to the domains \( U_1(Z), \ldots, U_m(Z) \), respectively, with bilipschitz constants depending only on \( K \) and \( Z \),

(S2) the components \( U_1(Z_k), \ldots, U_m(Z_k) \) of \( |K| \setminus |Z_k| \) are uniformly bilipschitz equivalent to the domains \( U_1(Z_1), \ldots, U_m(Z_1) \) for all \( k \geq 2 \), with the bilipschitz constants depend only on \( n \) and \( m \),

(S3) the Hausdorff distance, associated to the metric \( d_K \), between \( |Z_k| \) and \( \overline{U}_1(Z_k) \cap \cdots \cap \overline{U}_m(Z_k) \) is bounded above by \( C(n)3^{-k} \), for each \( k \geq 1 \), and

(S4) for each \( n \)-cube \( Q \in \mathcal{N}_{\text{Ref}_k(K)}(Z_k) \), the complex \( Z_k|\partial Q| \) is connected, where \( \mathcal{N}_Q = \{ Q' \in \mathcal{N}_{\text{Ref}_k(K)}(Z_k) : Q' \cap Q \neq \emptyset \} \).

The complex \( Z_k \) is deduced from \( Z_{k-1} \) by adding and subtracting mutually disjoint dented molecules to the domains \( (U_1(Z_{k-1}), \ldots, U_m(Z_{k-1})) \). This procedure produces, for each \( k \), an essential partition \( (\overline{U}_1(Z_k), \ldots, \overline{U}_m(Z_k)) \) whose components satisfy the conditions in Theorem 15.12. We call this process as mixing of domains, hence the name of the theorem.

16.4. Modification of separating complexes. We now state the technical part of the proof as a proposition from which Theorem 17 follows.

**Proposition 16.5.** Let \( n \geq 3 \), \( m \geq 2 \), \( K \) be a cubical complex on an \( n \)-manifold with boundary components \( \Sigma_1, \ldots, \Sigma_m \), and let \( Z \) be a separating complex in \( K \). Let \( \mathcal{E} = (E_1, \ldots, E_m) \) be a cyclic partition of \( S^n \).

Then there exists a constant \( K = K(n, K) \geq 1 \) for the following: for each \( k' \in \mathbb{N} \), there exist \( k \geq k' \) and a separating complex \( Z_k \) of \( \text{Ref}_k(K) \) contained in \( |\mathcal{N}_K(Z)| \) satisfying the following condition:

\( (M1) \) there exists a \( K \)-quasiregular map \( f : |K| \to S^n \setminus (B_1 \cup \cdots \cup B_m) \), where for each \( i \), \( B_i \) is a Euclidean ball contained in the interior of \( E_i \), and the restriction \( f|_{\Sigma_i} : \Sigma_i \to \partial B_i \) is a \( (\text{Ref}_k(K)|_{\Sigma_i})^A \)-Alexander map expanded by free simplex covers.

**Proof.** As an initial step, we consider the first refinement \( \text{Ref}(K) \) of \( K \). By following a spanning tree in the adjacency graph \( \Gamma(\mathcal{N}_K(Z)) \) of \( \mathcal{N}_K(Z) \), we construct an atom \( A \subset \text{Ref}(K) \) having the following properties:

1. the atom \( A \) meets each \( Q \in \mathcal{N}_K(Z) \) in an \( n \)-cube, that is, every \( n \)-cube \( Q \) in \( \mathcal{N}_K(Z) \) contains an \( n \)-cube \( Q' \in A \), and
2. \( |A| \cap |Z| \) is an \((n-1)\)-cell.

By subdividing each \( n \)-cube in \( A \) into (skewed) \( n \)-cubes, we obtain a subdivision of \( A \) into atoms \( A_1, \ldots, A_m \) for which \( \partial A_1 \cup \cdots \cup \partial A_m \) is a branched sphere, that is,

(i) the \((m+1)\)-tuple \( (|A|, |A_1|, \ldots, |A_m|) \) is homeomorphic to \( (E_1 \cup \cdots \cup E_m, E_1, \ldots, E_m) \) for any cyclic partition \( (E_1, \ldots, E_m, E_{m+1}) \) of \( S^n \), and in particular, \( |A_i| \cap |A_{i+1}| \) is an \((n-1)\)-cell and \( \cap |A_i| \) is an \((n-2)\)-sphere,

(ii) \( |A| \cap |A_1| = |A| \cap |Z| \),

(iii) for each \( Q \in A^{(n)} \) that is not a leaf of \( \Gamma(A) \), every \( Q \cap |A_i| \) is an \( n \)-cell, and
(iv) each \( n \)-cell \(|A_i|\) is bilipschitz equivalent to \(|A|\) with a constant depending only on \( n \) and \( m \).

We refer to [14] Section 8 for the terminology and the construction of the skewed atoms \( A_1, \ldots, A_m \).

To construct the first refined separating complex \( Z_1 \), we remove parts of \( |A| \cap |K_i(Z)| \) from domains \( U_i(Z) \), whenever appropriate, and modify the skewed atoms \( A_1, \ldots, A_m \) locally, so that the modified atoms \( A_1', \ldots, A_m' \) are subcomplexes of \( \text{Ref}_2(K) \) and each \( A_i' \cap |K_i(Z)| \) contains an \((n-1)\)-cube \( q_i \) in \( \text{Ref}_2(K) \). We then attach each modified atom \( A_i' \) back to \( U_i(Z) \) through the passage \( q_i \), and denote the new complexes obtained this way \( K_1', \ldots, K_m' \).

In this process, we require \( \text{int} q_i \subseteq \text{int}|K_i'| \). Note that \(|A|\) has an essential partition \((|K_1'|, \ldots, |K_m'|)\). We set \( Z_1 \) to be the separating complex induced by this new partition, which has the property that \(|Z_1| \subset \partial|K_1'| \cup \cdots \cup \partial|K_m'|\), and \((K_1(Z_1), \ldots, K_m(Z_1)) = (|K_1'|, \ldots, |K_m'|)\).

From the construction, \((\overline{U_1}(Z_1), \ldots, \overline{U_m}(Z_1)) = (|K_1'|, \ldots, |K_m'|)\) and the partition satisfies \(\{S1\} \) \(\{S2\}\) and \(\{S3\}\).

To construct separating complexes \( Z_k \) of \( \text{Ref}_k(Z) \) inductively for \( k \geq 2 \), we assume that \( Z_1, \ldots, Z_{k-1} \) have been constructed to satisfy \(\{S1\} \) to \(\{S4\}\). We now follow the local repartitioning scheme in [14] Section 8 to modify the complexes \((K_1(Z_{k-1}), \ldots, K_m(Z_{k-1}))\) by adding and subtracting mutually disjoint atoms, in the \((1/3)\)-refinement of \( \text{Ref}_{k-1}(K) \), with uniformly bounded atom lengths. We set \( Z_k \) to be the separating complex in \( \text{Ref}_k(K) \) induced by this modification, and \((K_1(Z_k), \ldots, K_m(Z_k))\) the resulting new complexes. As discussed in [14] Sections 5 and 8, the atom length of these dented atoms may be chosen to be uniformly bounded and that each complex \( K_i(Z_k) \) enters each cube in \( N_{\text{Ref}_{k-2}(K)}(Z_{k-2}) \). Furthermore, we may arrange \( Z_k \) to have the properties \(\{S1\} \) to \(\{S4\}\).

Thus, the complex \( K_i(Z_k) \) is obtained from \( K_i(Z) \) by an iterated refinement and by adding and subtracting mutually disjoint dented molecules with uniformly bounded atom length.

We show now that separating complexes \( Z_k \) satisfy the condition \(\{M1\}\). Let \( k \geq 1 \) and fix an \((U(Z_k), E)\)-Alexander sketch \( F_k = (f_{k,1}, \ldots, f_{k,m}) \) associated to \( U(Z_k) = (|K_1(Z_k)|, \ldots, |K_m(Z_k)|) \). In view of \(\{S3\} \) and \(\{S4\}\), each complex \( K_i(Z_k) \) meets each cube in \( N_{\text{Ref}_{k-2}(K)}(Z_{k-2}) \) in at least one \( n \)-cube. Hence we may fix a neighborhood forest \( \Gamma(U(Z_k), F) \) consisting of trees of size depending only on the dimension \( n \). Guided by these trees, we obtain, by the weaving theorem (Theorem 10.5), an essential partition \( M''_k = (M''_{k,1}, \ldots, M''_{k,m}) \) of \( K \) and a branched covering map \( F_k : |K| \to S^n \) of the sketch \( F \), so that the restrictions \( F_k | M''_{k,i} : M''_{k,i} \to E_i \) are uniformly BLD-mappings and that each domain \( \text{int} M''_{k,i} \) is uniformly bilipschitz to \( \text{int}|K_i(Z_k)| \) for all \( k \geq 1 \) and \( 1 \leq i \leq m \).

Now, as discussed in the proof of Theorem 10.5 there exist, for \( i = 1, \ldots, m \), bilipschitz homeomorphisms \( \lambda_i : \text{int}|K_i(Z_k)| \to \text{int} M''_{k,i} \), which allow us to pass from the maps \( F_k | M''_{k,i} \), to a corresponding family of branched covers \( f''_{k,i} : |K_i(Z_k)| \to E_i \) for which the restrictions \( f''_{k,i} | |Z_k| \cap \partial |K_i(Z_k)| : |Z_k| \cap \partial |K_i(Z_k)| \to \partial E_i \) and \( f''_{k,i} | |\Sigma| : \Sigma \to \partial E_i \) are Alexander maps expanded by...
standard simple covers. Furthermore, these Alexander maps are uniformly BLD with a constant depending only on the dimension $n$.

Since each complex $K_i(Z_k)$ is obtained from $Rf_i(K)|_{U_i(Z)}$ by adding and subtracting dented molecules of uniformly bounded atom length to only one of the boundary components of $U_i(Z)$, we conclude by Theorem 15.12 that the restrictions $f''|_{Z_k|\cap K_i(Z_k)}: |Z_k| \cap K_i(Z_k) \to \partial E_i$ admit $K$-quasiregular extensions $f_i: |K_i(Z_k)| \to E_i \setminus \text{int} B_i$, where $B_i$ is a Euclidean ball in the interior of $E_i$, and $K = K(n, m, A) \geq 1$ and hence $K = K(n, K)$.

Let now, for a large $k \in \mathbb{N}$, $f: |K| \to \mathbb{S}^n \setminus (B_1, \ldots, B_m)$ be the map satisfying $f|_{M''_k} = f_i \circ \lambda_i$ for each $i = 1, \ldots, m$, where $\lambda_i$ is the extension of $\lambda_i$ in Theorem 10.5. The proof is complete. \qed

Part 4. Applications

17. Quasiregular mappings with assigned preimages

An immediate consequence of Theorem 1.5 is an observation that, for $n \geq 3$, there exists quasiregular maps $\mathbb{S}^n \to \mathbb{S}^n$ with arbitrarily assigned preimages. This is topological in the sense that there is no control of the distortion.

The corresponding statement is false in dimension $n = 2$.

**Theorem 1.13.** Let $n \geq 3$, $p \geq 2$, and let $z_1, \ldots, z_p$ be distinct points in $\mathbb{S}^n$ and let $Z_1, \ldots, Z_p$ be mutually disjoint finite non-empty sets in $\mathbb{S}^n$. Then there exists a quasiregular map $f: \mathbb{S}^n \to \mathbb{S}^n$ satisfying $f^{-1}(z_i) = Z_i$ for each $i = 1, \ldots, p$.

**Proof.** Let $B_1, \ldots, B_p$ be mutually disjoint $n$-cells containing points $z_1, \ldots, z_p$ in their interiors, respectively. We fix, for each $x \in \bigcup_i Z_i$, a PL $n$-cell $G_x \subset \mathbb{S}^n$ with $x \in \text{int} G_x$ so that cells in $\{G_x: x \in \bigcup_i Z_i\}$ are pairwise disjoint. For each $i = 1, \ldots, p$, let $G_i = \bigcup_{x \in Z_i} G_x$.

By Theorem 1.5 there exists a PL branched covering map

$$f: \mathbb{S}^n \setminus \text{int} \bigcup_{i=1}^p G_i \to \mathbb{S}^n \setminus \text{int} \bigcup_{i=1}^p B_i,$$

which maps $\partial G_x$ onto $\partial B_i$ for each $x \in Z_i$. We now extend this branched cover to a map $f: \mathbb{S}^n \to \mathbb{S}^n$ by taking the cone extension $G_x \to B_i$ from $f|_{\partial G_x}: \partial G_x \to \partial B_i$, for each $x \in Z_i$ and $i = 1, \ldots, p$. Since $f$ is PL with respect to the standard PL structure, it is quasiregular. The claim follows. \qed

**Remark 17.1.** By Picard constructions ([48], [14]), we may assign empty pre-images at finitely many points in $\mathbb{S}^n$ for quasiregular maps $\mathbb{R}^n \to \mathbb{S}^n$. Theorem 1.13 complements this result by showing that we may also prescribe finite pre-images at finitely many points for quasiregular maps $\mathbb{S}^n \to \mathbb{S}^n$.

**Remark 17.2.** Theorem 1.13 does not hold in dimension two, which we may see as follows.

There is no entire function $f: \mathbb{R}^2 \to \mathbb{R}^2$ whose preimages of three distinct points $z_1, z_2$, and $z_3$, consist of one, one, and two points, respectively. Indeed, if there were an entire $f$ with this property then, by the Big Picard Theorem, $f$ would be a polynomial of degree $p \geq 2$. Then the derivative $f'$ would
have at least $3p-4$ zeros counting multiplicities; hence $p-1 = \deg f' \geq 3p-4$, which is impossible.

In view of the Stoilow factorization theorem [4, Theorem 5.5.1], every quasiregular map $\mathbb{R}^2 \to \mathbb{R}^2$ is composition $h \circ \varphi$ of a quasiconformal homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ and an entire function $h : \mathbb{R}^2 \to \mathbb{R}^2$. Therefore, there is no quasiregular map $f : S^2 \to S^2$ whose preimages of four distinct points $z_1, z_2, z_3$, and $z_4$ consist of one, one, one, and two points, respectively.

18. **Rickman’s large local index theorem**

In this section we prove a version of Rickman’s large local index theorem in [47] for quasiregular mappings $S^n \to S^n$ in dimensions $n \geq 3$.

**Theorem 1.8** (Large local index). Let $n \geq 3$. Then there exists a constant $K = K(n) \geq 1$ having the property that for each $c > 0$ there exists a $K$-quasiregular mapping $F : S^n \to S^n$ of degree at least $c$ for which

$$E_F = \{ x \in S^n : i(x, F) = \deg(F) \}$$

is a Cantor set.

Here $\deg(F)$ is the degree of the map $F$ and $i(x, F)$ the local index of $F$ at the point $x \in S^n$. We refer to [49] for the formal definition of the local index and recall that $i(x, F)$ has the property that, each $x \in S^n$ has a neighborhood $U$ for which

$$i(x, F) = \max_y \#(V \cap F^{-1}(y))$$

for each neighborhood $V$ of $x$ contained in $U$; see [49, Section I.4.2].

The Cantor set $E_F$ in Theorem 1.8 is tame, i.e., there is a homeomorphism $h : S^n \to S^n$ for which $h(E_F)$ is the standard ternary Cantor set contained in $S^1 \subset S^n$. In Section 19, we construct a similar example in dimension $n = 4$ where the large local index set is a wild Cantor set. For the definition of wild Cantor sets, see Definition 19.1.

The outline of the proof of Theorem 1.8 follows the 3-dimensional theorem of Rickman in [47]. At the core of the construction, however, is the Mixing Theorem (Theorem 1.7).

The essential part of Theorem 1.8 is presented in the following proposition. Having this proposition at our disposal, a standard Schottky group argument yields Rickman’s large local index theorem in all dimensions $n \geq 3$.

**Proposition 18.1.** Let $n \geq 3$ and $p \geq 2$. Then there exists a constant $K = K(n, p) \geq 1$ for the following: for each $c > 0$, there exist pairwise disjoint balls $B^n(x_1, r), \ldots, B^n(x_p, r)$ in $S^n$ and a $K$-quasiregular mapping $f : S^n \to S^n$ for which $i(x_1, f) = \cdots = i(x_p, f) \geq c$ and which, for each $j = 1, \ldots, p$, satisfies $f^{-1}f(B^n(x_j, r)) = B^n(x_j, r)$.

**Sketch of a proof of Theorem 1.8.** Let $f : S^n \to S^n$ be a quasiregular mapping as in Proposition 18.1 and let, for each $j = 1, \ldots, p$, $\gamma_j : S^n \to S^n$ be the reflection with respect to the sphere $\partial B^n(x_j, r)$ in $S^n$, respectively. Let $\Gamma$ be the (Schottky) groups generated by the set $\{\gamma_1, \ldots, \gamma_p\}$. For each word $w = i_1 \cdots i_k$ with letters $i_j \in \{1, \ldots, p\}$, let $\gamma_w = \gamma_{i_1} \circ \cdots \circ \gamma_{i_k} \in \Gamma$.

Let $M_0 = S^n \setminus \bigcup_{j=1}^{p} B^n(x_j, r)$ and, for each $k \in \mathbb{Z}_+$, let $M_k = M_{k-1} \cup \bigcup_{i=1}^{k} \gamma_i M_{k-1}$. Let now $E_f$ be the Cantor set $E_f = \bigcap_{k=1}^{\infty} (S^n \setminus M_k)$. Then
there exists a unique map $F: S^n \to S^n$ for which $F|_{M_0} = f$ and, for each word $w = i_1 \cdots i_k$, we have $F \circ \gamma_w|_{M_0} = \gamma_w \circ F|_{M_0}$. The mapping $F$ is $K$-quasiregular and satisfies the conclusions of the claim; we refer to [47] for details.

**Proof of Proposition 18.7.** We reduce the proof to the construction of a separating complex $Z$, and an application of the Mixing Theorem (Theorem 1.7).

Let $c > 0$ and $p \geq 2$. We construct now a quasiregular mapping $f: S^n \to S^n$ which has local index $i(\cdot, f) \geq c$ at $x_1, \ldots, x_p \in S^n$.

**Step 1: Initial configuration.** Let $Q$ be an $n$-cube, $k \in \mathbb{N}$ with $3^k \geq p + 1$, and $\text{Ref}_k(Q)$ the $k$-th refinement of $Q$. Let $S$ be the topological sphere obtained by gluing two copies of $Q$ together along the boundary. In other words, $S$ is the quotient space $(Q \times \{1, 2\})/\sim$, where $\sim$ is the smallest equivalence relation satisfying $(x, 1) \sim (x, 2)$ for all $x \in \partial Q$.

Let $K$ be the natural double of $\text{Ref}_k(Q)$ in $S$, of which $\text{Ref}_k(Q)$ is a subcomplex of $K$. Let also $q \subset Q$ be a face of $Q$, and $Q_1, \ldots, Q_{p-1}$ be adjacent $n$-cubes in $\text{Ref}_k(Q)$ such that each $Q_j$ has a face $q_j$ contained in $q$ and their union $\bigcup_{j=1}^{p-1} Q_j$, in the metric $d_{\text{Ref}_k(Q)}$, is isometric to a Euclidean cube $[0,1]^{(n-1)} \times [0, p-1]$.

For each $j = 1, \ldots, p-1$, let $Q_j'$ be the unique $n$-cube in $K \setminus \text{Ref}_k(Q)$ for which $q_j = Q_j \cap Q_j'$ is a common face of $Q_j$ and $Q_j'$. Also, for each $j = 1, \ldots, p-1$, let $Q_j'' = Q_j$ if $j$ is odd, and $Q_j'' = Q_j'$ if $j$ is even, and let $\hat{Q}_j$ be the center cube of $Q_j''$ in the refinement $\text{Ref}(K)$. We fix an $n$-cube $Q_p \in \text{Ref}(K)$ which meets neither $q$ nor any one of the cubes $Q_1', \ldots, Q_{p-1}'$.

Let now $\hat{K} = \text{Ref}(K) \setminus \bigcup_{i=1}^{p-1} \hat{Q}_j$ be the subcomplex of $\text{Ref}(K)$ with $n$-cubes $\hat{Q}_1, \ldots, \hat{Q}_p$ removed, and $Z$ be the subcomplex of $\hat{K}$ for which

$$|Z| = \bigcup_{i=1}^{p-1} \partial Q_j''.$$

Then $Z$ is a separating complex of $\hat{K}$.

**Step 2: Application of the Mixing Theorem.** We now apply the Mixing Theorem to the complex $\hat{K}$ and its separating complex $Z$. In view of Theorem 1.7 there exist a refinement index $k \in \mathbb{N}$, satisfying $3^k \geq p + 1$ and $3^{k(n-1)} \geq c$, a constant $K' = K'(n, p)$, and a $K'$-quasiregular map $\hat{f}: |\hat{K}| \to S^n \setminus (B_1 \cup \cdots \cup B_p)$ such that every restriction $\hat{f}|_{\partial \hat{Q}_j} : \partial \hat{Q}_j \to \partial B_j$ is a $\text{Ref}_k(\hat{K})$-Alexander map expanded by properly placed standard simple covers, where $B_1, \ldots, B_p$ are pairwise disjoint Euclidean balls, and $K' \geq 1$ is a constant depending only on the dimension $n$ and the number $p$.

Since the sphere $S = |K|$, in the metric $d_K$, is bilipschitz to the Euclidean sphere $S^n$, the sphere $S$ in the metric $d_{\text{Ref}_k(K)}$ is $(3^k, L)$-quasiregular to $S^n$ for a constant $L = L(n, p)$ independent of $k$. In particular, there exist pairwise disjoint Euclidean balls $B_j' = B^n(x_j, r), j = 1, \ldots, p$, of the same size, and a quasiconformal homeomorphism $F: (|\hat{K}|, d_{\hat{K}}) \to S^n \setminus \text{int}(B_1' \cup \cdots \cup B_p')$,
with a distortion constant depending only on \( n \) and \( p \).

**Step 3: Extension over \( \mathbb{S}^n \).** It remains to extend the mapping

\[
 f = \tilde{f} \circ F^{-1} : \mathbb{S}^n \setminus (B_1' \cup \cdots \cup B_p') \to \mathbb{S}^n \setminus (B_1 \cup \cdots \cup B_p)
\]

to a quasiregular map \( \mathbb{S}^n \to \mathbb{S}^n \). For simplicity, we assume that all balls are contained in \( \mathbb{R}^n \subset \mathbb{S}^n \). We may also assume that, for each \( j = 1, \ldots, p \), the mapping \( f|_{\partial B_j'} : \partial B_j' \to \partial B_j \) is an Alexander map expanded by properly placed standard simple covers.

Then, with \( \beta_j = \deg F|_{\partial Q_j} \), the radial extensions \( \beta_j : B_j' \to B_j \)

\[
 x \mapsto \left( \frac{|x - x_j|}{r} \right)^{\beta_j} \left( f \left( r \frac{x - x_j}{|x - x_j|} + x_j \right) - x_j \right) + x_j,
\]

of \( f \) into \( B_j' \), yield the desired \( K \)-quasiregular extension \( \mathbb{S}^n \to \mathbb{S}^n \) with \( K = K(K', n) \); see [49] Example 1.3.2 for the model case. Note that the degree \( \deg F|_{\partial Q_j} \) is independent on \( j \) and in fact is equal to the degree of \( f \).

By the choice of refinement index \( k \), the local index of \( f \) satisfies \( i(x_j, f) \geq 3^k(n-1) \geq c \) for each \( j = 1, \ldots, p \). This completes the proof. \( \Box \)

19. Heinonen–Rickman theorem on branching in dimension 4

In this section, we prove a 4-dimensional version of a theorem of Heinonen and Rickman on wild branching of quasiregular maps [24, Theorem 2.1].

**Theorem 1.9** (Wildly branching quasiregular map). Let \( n \) be either 3 or 4. Then there exist a wild Cantor set \( X \subset \mathbb{R}^n \) and constants \( K \geq 1, c_0 \geq 1, \) and \( m_0 \geq 1 \) for the following. For each \( c \geq c_0 \), there exist \( c' \geq c \) and a \( K \)-quasiregular mapping \( F : \mathbb{S}^n \to \mathbb{S}^n \) for which \( i(x, F) = c' \) for each \( x \in X \), and \( i(x, F) \leq m_0 \) for each \( x \in \mathbb{S}^n \setminus X \). Furthermore, given \( s_0 \geq 1 \), we may choose the mapping \( F \) to have the property: there exists \( s \geq s_0 \) for which

\[
 \frac{1}{C} \text{dist} (x, X)^s \leq J_F(x) \leq C \text{dist} (x, X)^s,
\]

for almost every \( x \in \mathbb{S}^n \).

**Definition 19.1.** A Cantor set \( X \subset \mathbb{S}^n \) is **tame**, if there is a homeomorphism \( h : \mathbb{S}^n \to \mathbb{S}^n \) for which \( h(X) \subset S^1 \subset S^n \). It is **wild** if there exists no such homeomorphism.

There are wild Cantor sets in all dimensions. The classical construction of Antoine [8] gives an example, the so-called Antoine’s necklace, in dimension three. The (topological) construction of Antoine’s necklace was generalized to dimensions \( n \geq 4 \) by Blankinship [8]. In the proof of Theorem 1.9, we use a quasi-self-similar version of Blankinship’s wild Cantor set; this (geometrical) construction in dimension 4 is discussed in the appendix. The dimension restriction in Theorem 1.9 stems from this dimension constraint. In fact, we are not aware of the existence of quasi-self-similar Cantor sets in dimensions \( n \geq 5 \).

We mention in passing, that the 3-dimensional result of Heinonen and Rickman in [24] is based on a self-similar Antoine’s necklace; see also Semmes [50].
Proof of Theorem 1.5. We prove the theorem by reducing the question to the Mixing Theorem (Theorem 1.7) and an iterative process in the spirit of the proof of Theorem 1.8. We begin by describing the construction of the Cantor set and the corresponding separating complex.

As described in more detail in the appendix, Blankinship’s Cantor set $X$ is obtained as the intersection

$$X = \bigcap_{k=0}^{\infty} X_k,$$

where $X_0 = T$ is homeomorphic to $B^2 \times S^1 \times S^1$ and for each $k \geq 1$, $X_k$ is a compact manifold with boundary, consisting of $m^k$ components $T_w$ each of which is homeomorphic to $B^2 \times S^1 \times S^1$, where $m \in \mathbb{N}$ is a fixed even integer and $w$ is a word of length $k$ in letters $\{1, \ldots, m\}$. For each $k \geq 0$, the pair $(X_k, X_{k+1})$ is a union of mutually disjoint pairs $(T_w, T_w \cap X_{k+1})$ each of which is homeomorphic to the pair $((B^2 \times S^1) \times S^1, A \times S^1)$, where $A$ is a union of $m$ mutually disjoint solid 3-tori $A_1, \ldots, A_m$ linked in $B^2 \times S^1$ as in the construction of Antoine’s necklace in $\mathbb{R}^3$.

From now on, we assume that $m$ is large and that the solid tori $A_1, \ldots, A_m$ are isometric to each other and are similar to an embedded solid 3-torus $A$ for which $X_0 = A \times S^1$. Set $A_0 = A$. We may further assume that the pairs $(T_w, T_w \cap X_{k+1})$ belong to two similarity classes for all words $w$ of length $k$ and all $k \geq 1$; see the appendix for more discussion.

The main part of the proof is to define a suitable cubical complex $K$ on $(A \times S^1, A \times S^1)$ which admits a separating complex $Z$. After that, we apply Theorem 1.7 to construct a quasiregular branched covering map $f$ on $(A \times S^1) \setminus \text{int}(A \times S^1)$ associated to $(K, Z)$, and then appeal to the quasi-similarity in the construction to obtain the map $\mathbb{S}^4 \to \mathbb{S}^1$ claimed.

**Step 1: Cubical 2-complexes** $\{R_i : 0 \leq i \leq m\}$ on 2-tori $\{\partial A_i : 0 \leq i \leq m\}$. We set

$$A_0 = A = B^2 \times S^1 \subset \mathbb{R}^3.$$

Given $\ell \geq 2$, let $C_\ell$ be a cubical complex on $S^1$ having $\ell$ 1-simplices of equal length, e.g. with 1-simplices

$$\sigma_{k,\ell} = \{e^{i\theta} \in S^1 : \theta \in [2\pi(k-1)/\ell, 2\pi k]\}, \quad \text{for } k = 1, \ldots, \ell.$$

Let now $R_0$ be a cubical 2-complex on $\partial A = \partial B^2 \times S^1$ isomorphic to $C_4 \times C_{m/2}$, to be fixed more precisely later. This structure subdivides the longitudinal direction of $\partial A$ into $m/2$ equal parts. For each $k = 1, \ldots, m/2$, we set

$$G_k = B^2 \times \sigma_{k,m/2}$$

to be a 3-cell.

We also identify (topologically) each solid 3-torus $A_i$ with $B^2 \times S^1$. For each even index $i \in \{1, \ldots, m\}$, let $R_i$ be a cubical 2-complex on $\partial A_i$ isomorphic to $C_2 \times C_2$, to be fixed later. We consider those $A_i$ with even indices as the vertical rings. For an odd index $i$, let $R_i$ be a 2-complex on $\partial A_i$ isomorphic to $C_2 \times C_4$. These rings are considered as horizontal rings.

Observe that, for each $i = 0, \ldots, m$, the cubical complex on the surface $\partial A_i$ is isomorphic to a shellable cubical refinement of the model complex $C_2 \times C_2$. 

As a preparation for the next step, we arrange the solid 3-tori $A_1, \ldots, A_m$ in such a way that each even-indexed torus $A_{2k}$ is contained in $G_k$ and the odd-indexed tori $A_{1+2k}$ are symmetric with respect to the 2-disk $G_k \cap G_{k+1}$ for $k = 1, \ldots, m/2$, where we identify $G_{m/2+1} = G_1$.

**Step 2:** Cubical 3-complexes $\{P_i\}$ on the collars $\{M_i\}$ for $\{\partial A_i\}$, and cubical 2-complexes $\{Y_i\}$ on the filling disks $\{D'_i\}$. We fix an inner collar $M_0$ for $\partial A$ in $A$ and mutually disjoint outer collars $M_1, \ldots, M_m$ for $A_1, \ldots, A_m$, respectively, in such a way that $A'_0 = A \setminus \text{int}M_0 = B^2(1-\varepsilon) \times S^1$ for some $\varepsilon > 0$ and that each solid 3-torus $A'_i = A_i \cup M_i$ is contained in the interior of $A'_0$. Note that configuration $(A'_0, A'_1, \ldots, A'_m)$ is homeomorphic to $(A, A_1, \ldots, A_m)$.

Since collars $M_0$ and $M_1, \ldots, M_m$ are product spaces, we may associate to each collar $M_i$, for $i = 0, \ldots, m$, a natural product structure $P_i$ isomorphic to $R_i \times [0, 1]$ for which $P_i|\partial A_i = R_i$. In particular, $P_i|\partial A'_i$ is also isomorphic to $R_i$.

By making the tori $A'_1, \ldots, A'_m$ uniformly bilipschitz to solid tori $B^2(r) \times S^1(t)$ for some parameters $r$ and $t$ if necessary and rearranging their placement, we may fix 2-cells $D_1, \ldots, D_m$, uniformly bilipschitz to the Euclidean disk $B^2(t-2r)$, with the properties that

1. $D_i \cap \partial A'_i = \partial D_i$, and
2. intersections $D_i \cap A'_{i-1}$ and $D_i \cap A'_{i+1}$ are 2-disks in the interior of $D_i$, and $D_i \cap A'_i = \emptyset$ for $j \neq i-1, i, i+1$ modulo $m$; and
3. $A'_0 \setminus (A'_1 \cup \cdots \cup A'_m \cup D_1 \cup \cdots \cup D_m)$ is homeomorphic to $\partial A'_0 \times [0, 1)$.

For each $i = 1, \ldots, m$, let

$$D'_i = D_i \setminus (A'_{i-1} \cup A'_{i+1}).$$

We give $D'_i$ a cubical structure $Y_i$ isomorphic to the cubical structure illustrated in Figure 40 for the even-indexed $D'_i$ on the left and and the odd-indexed $D'_i$ on the right.

**Figure 40.** Cubical 2-complexes for surfaces $D'_i$; the even-indexed on the left and the odd-indexed on the right.

Further, we may assume that the complexes $Y_i$ and $P_i$ agree on $D_i \cap A'_i$ whenever the intersection is non-empty. Now there is a well-defined 2-complex $Y$ on

$$\Sigma = \bigcup_{i=1}^m (D'_i \cup \partial A'_i),$$

which contains each $P_i|\partial A'_i$ and each $Y_i$ as a subcomplex. We note that at this stage $A \setminus \Sigma$ has $m+1$ components, each of which is homeomorphic to the product $(S^1 \times S^1) \times [0, 1)$. 
Step 3: Cubical 3-complex $P$ on $A_0 \setminus \text{int}(A_1 \cup \cdots \cup A_m)$. Having the preliminary cubical structures on $\mathcal{A}_0'$ and $\Sigma$, we now define a cubical structure on $A_0 \setminus \text{int}(A_1 \cup \cdots \cup A_m)$ as follows.

Recall that, for each $k = 1, \ldots, m/2$, the 3-cell $G_k$ contains the vertical ring $A_{2k}'$ and two horizontal half rings $A_{2k-1}' \cap G_k$ and $A_{2k+1}' \cap G_k$. Moreover, we may assume that the cubical structure $Y$ on $\Sigma$ has been chosen so that each subcomplex $X \times G_k$ is a well-defined cubical subcomplex of $Y$ isomorphic to the first refinement $\text{Ref}(P_0|_{\partial A'_0 \cap G_k})$ of the complex $P_0$ on $G_k \cap \partial A'_0$. Furthermore, these isomorphisms are induced by the maps $\pi_k: G_k \cap \partial A'_0 \rightarrow \Sigma \cap G_k$; see Figure 41 for an illustration. For this reason, we now replace the complex $R_0$ on $A_0$ by its first refinement $\text{Ref}(R_0)$, and change the cubical complex $P_0$ accordingly.

We may now define the cubical complex $P$ on $A_0 \setminus \text{int}(A_1 \cup \cdots \cup A_m)$ to be the unique cubical 3-complex (up to isomorphism) for which

1. $P|_{A_0 \setminus A_0'} = P_0$,
2. $P|_{A'_i \setminus \text{int} A_i} = P_i$ for each $i = 1, \ldots, m$,
3. $P|_{A'_0 \setminus \text{int}(A'_1 \cup \cdots \cup A'_m \cup D'_1 \cup \cdots \cup D'_m)}$ is isomorphic to the product $P_0|_{\partial A'_0} \times [0,1]$, and
4. $P|_{\Sigma} = Y$.

Then $Y$ is a separating complex for $P$.

![Figure 41. Cubical complex $\text{Ref}(P_0|_{\partial A'_0 \cap B_k})$ unfolded (above); cubical complex $X|_{\Sigma \cap G_k}$ unfolded and sliced (below).](image)

Step 4: Cubical 4-complex $K$ on $A \times S^1$, and the separating complex $Z$. We now consider the 4-dimensional pair $(A \times S^1, A \times S^1)$. Recall that $A = A_1 \cup \cdots \cup A_m$.

Let

$$K = P \times C_2$$
be a cubical 4-complex on $A \times S^1$ and set

$$Z = Y \times C_2 \subset K.$$ 

Then $Z$ is a separating complex for $K$.

**Step 5: Quasiregular branched cover on $(A \times S^1) \setminus \text{int}(A \times S^1)$** By Theorem 1.7, there exist a constant $K_1 = K(n, K) \geq 1$, a refinement $\text{Ref}_k(K)$ of $K$ for a sufficiently large $k \in \mathbb{N}$, and a $K_1$-quasiregular mapping

$$f: (A \times S^1) \setminus \text{int}(A \times S^1) \rightarrow B^4 \setminus \text{int}(B_1 \cup \cdots \cup B_m),$$

where $B_1, \ldots, B_m$ are pairwise disjoint Euclidean 4-balls in the interior of $B^4$, for which the restrictions $f|_{\partial(A \times S^1)}: \partial(A \times S^1) \rightarrow \partial B^4$ and $f|_{\partial A_i}: \partial A_i \rightarrow \partial B_i$, for $i = 1, \ldots, m$, are all of the same degree, call it $c_0$, and each of which is a $(\text{Ref}_k(K))^{A}$-Alexander map expanded by properly placed standard simple covers. Note that we may arrange for the multiplicity of the map $f$ to be greater than any given $c \geq c_0$ by further refinement.

**Step 6: Iterative construction.** To complete the proof, we set $X_0 = T = A \times S^1$, and index the components $\{T_w\}$ of $X_k$ by words $w$ in letters $\{1, \ldots, m\}$ of length $k$, in such a way that $T_{wi} \subset T_w$ for each word $w$ and $i \in \{1, \ldots, m\}$. In view of Remark 20.6 on the construction of the Cantor set $X$, we may fix, for each word $w$ of length $k$, a sense-preserving $(b^k, L)$-quasi-similarity map $\varphi_w: A \times S^1 \rightarrow T_w$, which is a homeomorphism between pairs $(A \times S^1, A \times S^1)$ and $(T_w, T_w \cap X_{k+1})$ and where $c > 0, b < 1$, and $L \geq 1$ are constants.

Recall that, after a partition based on the (model) complex $C_2 \times C_2$, the cubical complex on each surface $\partial A_i$, for $i = 0, \ldots, m$, is a totally shellable refinement of $C_2 \times C_2$. Thus, we may fix these homeomorphisms $\varphi_w$ inductively so that, for each word $w$ and letter $i$, the induced complex $\varphi_{wi}(K)|_{\partial T_{wi}}$ is a totally shellable refinement of the complex $K|_{\partial T_{wi}}$, and that each $\varphi_w$ is $K_2$-quasiconformal, with $K_2 \geq 1$ a constant independent of $w$ and $i$.

Fix for each $j = 1, \ldots, m$, a sense-preserving similarity $\lambda_j: B^4 \rightarrow B_j$, and for a given word $w = i_1 \cdots i_k$, write $\lambda_w = \lambda_{i_1} \circ \cdots \circ \lambda_{i_k}$.

Let $f: (A \times S^1) \setminus \text{int}(A \times S^1) \rightarrow B^4 \setminus \text{int}(B_1 \cup \cdots \cup B_m)$ be the map defined in Step 4. Since the cubical complexes on $\partial A_i$ for $i = 0, \ldots, m$, are totally shellable refinements of the same complex $C_2 \times C_2$, the Alexander maps $f|_{\partial T_i}$ and $\lambda_w \circ f \circ \varphi_{wi}^{-1}|_{\partial T_i}$ from $\partial T_i$ to $\partial B_i$ are in fact BLD branched cover homotopic, with a BLD-constant depending only on $n$ and $m$. Thus we may assume that the restrictions of

$$f: T \setminus \text{int}(T \cap X_1) \rightarrow B^4 \setminus \text{int}(B_1 \cup \cdots \cup B_m)$$

to boundary components are conjugate to each other by (uniformly) quasi-similarities of the domains and similarities of the targets.

Repeating the construction of the map $f$ inductively on the length of the word $w$, we obtain a constant $K' \geq 1$ depending only on $K_1, K_2$, and, for each $k \geq 1$ and a word $w$ of length $k$, a $K'$-quasiregular mapping

$$f_w: T_w \setminus \text{int}(T_w \cap X_{k+1}) \rightarrow \lambda_w(B^4) \setminus (\cup_{j=1}^m \text{int}\lambda_w(B^4))$$
having the property that, for each \( i = 1, \ldots, m \), the mappings \( f_w \) and \( f_{wi} \) agree on \( \partial T_w \) and the map

\[
f : X_0 \setminus X \to B^4,
\]
defined by the condition \( f\mid_{T_w \setminus \text{int}(T_w \cap X_{k+1})} = f_w \) for each \( w \), is well-defined and \( K' \)-quasiregular.

The quasiregular map \( f : X_0 \setminus X \to B^4 \) extends now over to the Cantor set \( X \) as a \( K' \)-quasiregular map.

**Step 7: Final extension.** A similar construction using a separating complex, for example, extends \( f\mid_{\partial X_0} : \partial X_0 \to \partial B^4 \) to a \( K'' \)-quasiregular map \( S^4 \setminus \text{int}X_0 \to S^4 \setminus \text{int}B^4 \) for some constant \( K'' \geq 1 \) depending only on \( n \) and the original complex \( K \). By combining these two parts, we obtain a \( K \)-quasiregular map \( f : S^n \to S^n \) with \( K = \max\{K', K''\} \).

By the construction, the local index \( i(x, f) = c' \) for each \( x \in X \) for some number \( c' \) which is at least as large as \( c_0 = \deg(f\mid_{\partial(A \times S^1)}) \). On the other hand, in the complement of the Cantor set \( X \), the local degree of \( f \) is determined by the cubical complex \( K \) on \( A \times S^1 \), and hence there exists \( m_0 \geq 1 \) for which \( i(x, f) \leq m_0 \) for \( x \notin X \). This completes the construction.

**Step 7: The Jacobian estimates.** To verify the Jacobian estimates in the theorem, we make the following observations. First, since the restriction \( F\mid_{X_0 \setminus \text{int}X_1} \) is BLD, the estimates hold in \( X_0 \setminus \text{int}X_1 \). Second, quasi-similarities \( \varphi_w : T \to T_w \), for words \( w \) of length \( k \), have scaling constant \( b^k \) by the construction of the Blankinship necklace. Third, after applying a quasiconformal mapping of \( \mathbb{R}^4 \) which is identity outside \( B^4 \), we may assume the balls \( B_1, \ldots, B_m \) are centered on the axis \( \{(0, 0, 0)\} \times \mathbb{R} \) and have the same diameter \( 1/(2m) \). Thus, similarities \( \lambda_1, \ldots, \lambda_m \) have the same scaling constant \( 1/(2m) \).

It is now easy to observe that, for each \( k \geq 1 \) and a word \( w \) of length \( k \), the Jacobian \( J_f\mid_{T_w \setminus X_{k+1}} \) is comparable to \( ((1/(2mb))^{k})^4 \). On the other hand, for \( x \in T_w \setminus X_{k+1} \), the distance \( d(x, X) \) is comparable to \( b^k \). Let

\[
s = -4 \log(2mb)/\log(b).
\]

Then \( d(x, X)^s \) and \( J_f(x) \) are comparable at each point of differentiability in \( T_w \setminus X_{k+1} \) for the mapping \( f \).

20. A UQR mapping with a wild Julia sets in dimension 4

Recall that a quasiregular self-mapping \( f : M \to M \) of a Riemannian manifold \( M \) is uniformly quasiregular (UQR), if there exists a constant \( K > 1 \) for which \( f \) and all its iterates are \( K \)-quasiregular. There are UQR maps in \( S^3 \) whose Julia sets are wild Cantor sets \([13]\).

Wild Cantor sets are known to exist in all dimensions \( n \geq 3 \). However, geometrically self-similar wild Cantor sets are known only in dimension three. In order for a Cantor set to be a potential candidate for the Julia set of a UQR map, it needs to be at least quasi-self-similar (Definition 20.1). We are able to construct such a Cantor set which satisfies a sharper condition (Remark 20.6) sufficient for this purpose in \( \mathbb{R}^4 \). The Hopf theorem (Theorem 1.3) is then used to build a UQR map, in dimension 4, whose Julia set is the above wild Cantor set.
Theorem 1.14 (Wild Julia set). For each $k \in \mathbb{N}$, there exists a uniformly quasiregular map $S^4 \to S^4$ of degree at least $k$, whose Julia set is a wild Cantor set.

We refer to [18] for more discussion on the role of wild Julia sets in complex dynamics. In the following proof, we identify, using stereographic projection, $S^4$ with the one point compactification $\mathbb{R}^4$ of $\mathbb{R}^4$.

Proof of Theorem 1.14. Let

$$X = \bigcap_{k=0}^{\infty} X_k$$

be the quasi-self-similar wild Cantor set constructed in Appendix, associated to the parameters $b$, $m$, and $\rho$ subject to the relations [1] and [2]. We retain, from here on, all notations from this particular construction.

We assume that the initial 4-tube $X_0$ is chosen to have pattern $T$ and that the number of linked 4-tubes in $X_1$ is $m = 48d^3$ for some even integer $d$. Denote by $B(r)$ the closed ball $B^4(0, r)$ in $\mathbb{R}^4$, and note from the construction that

$$X_1 = \bigcup_{j=1}^{m} X_{1,j} \subset X_0 \subset B(2) \subset \mathbb{R}^4.$$

We have now the essential partitions

$$\mathbb{R}^4 = (\mathbb{R}^4 \setminus B(4)) \cup (B(4) \setminus B(3)) \cup (B(3) \setminus X_1) \cup X_1$$

and

$$\mathbb{R}^4 = (\mathbb{R}^4 \setminus B(4^d)) \cup (B(4^d) \setminus B(3)) \cup (B(3) \setminus X_0) \cup X_0$$

of $\mathbb{R}^4$. As in [18], we construct a UQR map $f: \mathbb{R}^4 \to \mathbb{R}^4$ for which $f(X_1) = X_0$, $f(B(3) \setminus X_1) = B(3) \setminus X_0$, $f(B(4) \setminus B(3)) = B(4^d) \setminus B(3)$, and $f(\mathbb{R}^4 \setminus B(4)) = \mathbb{R}^4 \setminus B(4^d)$.

Step 1: Let $\varphi_j: X_0 \to X_{1,j}$ be the homeomorphisms in Appendix, and let $f|_{X_1}: X_1 \to X_0$ be the $m$-fold covering map satisfying

$$f|_{X_{1,j}} = \varphi_j^{-1}$$

for each $j = 1, \ldots, m$.

Step 2: We define next $f|_{B(3)\setminus X_1}: B(3) \setminus X_1 \to B(3) \setminus X_0$ to be the composition of two winding maps $\omega$ and $\omega'$, and bilipschitz homeomorphisms of $\mathbb{R}^4$ as follows.

Let $\omega: \mathbb{R}^4 \to \mathbb{R}^4$ be the degree $m/2$ winding map

$$(x_1, x_2, r, \theta) \mapsto (x_1, x_2, r, \theta m/2),$$

where $(r, \theta)$ are the polar coordinates in $\mathbb{R}^2$.

The winding map $\omega$ is a BLD-map, which maps the triple $(B(3), X_0, X_1)$ onto the triple $(B(3), X_0, \omega(X_{1,1}) \cup \omega(X_{1,2}))$. Sets $\omega(X_{1,1})$ and $\omega(X_{1,2})$ remain linked inside $X_0$ and we may straighten them by a bilipschitz homeomorphism $\xi: \mathbb{R}^4 \to \mathbb{R}^4$, which is identity on $\mathbb{R}^4 \setminus \text{int}X_0$, in such a way that the involution $\iota: X_0 \to X_0$,

$$(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, -x_4),$$
interchanges the images $\omega(X_{1,1})$ and $\omega(X_{1,2})$. In particular, 

\[ \iota(\xi \circ \omega(X_0)) = X_0, \quad \iota(\xi \circ \omega(X_{1,1})) = \xi \circ \omega(X_{1,2}), \quad \iota(\xi \circ \omega(X_{1,2})) = \xi \circ \omega(X_{1,1}). \]

Let now $\omega': \mathbb{R}^4 \to \mathbb{R}^4$ to be the winding map

\[(x_1, r' \cos \theta', r' \sin \theta', x_4) \mapsto (x_1, r' \cos(2\theta'), r' \sin(2\theta'), x_4),\]

where $(r', \theta')$ are the polar coordinates in $\mathbb{R}^2$. So $\omega'$ is a degree 2 sense preserving BLD-map under which

\[ \omega' \circ \xi \circ \omega(X_{1,1}) = \omega' \circ \xi \circ \omega(X_{1,2}), \quad \omega' \circ \xi \circ \omega(B(3)) = B(3), \]

and

\[ (\omega' \circ \xi \circ \omega(X_0), \omega' \circ \xi \circ \omega(X_{1,1})) \approx (B^3 \times S^1, \tau \times S^1), \]

where $\tau \approx B^2 \times S^1$ is a solid torus contained in the interior of $B^3$.

Let now $\eta: \mathbb{R}^4 \to \mathbb{R}^4$ be a bilipschitz homeomorphism of $\mathbb{R}^4$ which is identity outside $B(3)$ and for which

\[ \eta(\omega' \circ \xi \circ \omega(X_{1,1})) = \eta(\omega' \circ \xi \circ \omega(X_{1,2})) = X_0. \]

Thus $\eta \circ \omega' \circ \xi \circ \omega_{X_{1,j}}: X_{1,j} \to X_0$ for all $j = 1, \ldots, m$. We may further adjust the bilipschitz map $\eta$ in $\text{int}B(3) \setminus X_1$ and find a 4-manifold $P \approx B^3 \times S^1$ satisfying $X_0 \subset \text{int}P \subset P \subset \text{int}B(2)$, and $P = \eta(\omega' \circ \xi \circ \omega(X_0))$.

The composition

\[ f|_{B(3)} = \eta \circ \omega' \circ \xi \circ \omega|_{B(3)}: (B(3), X_1) \to (B(3), X_0) \]

is a BLD-map of degree $m$ which extends the already defined BLD-map $f|_{X_1}: X_1 \to X_0$, and maps

\[(B(3) \setminus P, P \setminus X_0, X_0 \setminus X_1) \mapsto (B(3) \setminus B(2), B(2) \setminus P, P \setminus X_0).\]

Step 3: Before defining the map $f: \overline{\mathbb{R}^4} \setminus B(4) \to \overline{\mathbb{R}^4} \setminus B(4^d)$, we recall first so-called the Mayer’s power map. Mayer constructed, for any $n \geq 3$ and $d \geq 2$, a UQR map $p: \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ of degree $d^{n-1}$, whose Julia set is the unit sphere $S^{n-1}$. The restriction $p|_{\mathbb{R}^n \setminus \{0\}}: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ of $p$ is the higher dimensional counterpart of the degree $d^{n-1}$ power map. In fact, $p$ is derived from the Zorich map on $\mathbb{R}^{n-1} \times \mathbb{R}$ using a cylindrical structure $K \times \mathbb{R}$, where $K$ is the standard cubical partition of $\mathbb{R}^{n-1}$; see [33, Theorem 2] for the detailed construction.

As a preparation for the Hopf Theorem (Theorem 1.2), we consider a modified Mayer’s map $q: \overline{\mathbb{R}^4} \to \overline{\mathbb{R}^4}$ associated to a modified Zorich map given by a refined cylindrical structure $K^\Delta \times \mathbb{R}$, where $K^\Delta$ is the canonical triangulation of $K$. In this triangulation a unit cube is subdivided into 48 3-simplices. Thus our modified Mayer’s map $q$ has degree $m = 48d^3$. We leave details to the interested reader.

We define now $f: \overline{\mathbb{R}^4} \to \overline{\mathbb{R}^4}$ in the complement of $B(4)$ to agree with $q$. Note that the sphere $\partial B(4)$ is mapped onto $\partial B(4^d)$ under the map $q$, and that $f|_{\partial B(4)} = q|_{\partial B(4)}$ is a sense preserving cubical Alexander map of degree $m$.

Step 4: It remains to find a quasiregular mapping $B(4) \setminus B(3) \to B(4^d) \setminus B(3)$ extending the already defined parts of $f$. 
Recall that $f|_{\partial B(3)} : \partial B(3) \to \partial B(3)$ is the composition of a degree $m/2$ winding map $\omega$ and a degree $2$ winding map $\omega'$, modulo bilipschitz adjustments. In view of Theorem 1.1, we may deform $f|_{\partial B(3)}$ to a winding map on $\partial B(3)$ through a level preserving BLD-maps. This BLD-homotopy yields a BLD-extension $B(4) \setminus B(3) \to B(4^d) \setminus B(3)$ of the restrictions $f|_{\partial B(3)} : \partial B(3) \to B(3)$ and $f|_{\partial B(4)} : \partial B(4) \to \partial B(4^d)$. This completes the construction of the quasiregular map $f : \mathbb{R}^4 \to \mathbb{R}^4$.

**Final step:** The uniform quasiregularity of $f$ follows from two observations. First, by the construction of the Cantor set $X$, there exist $\lambda \geq 1$ and $L \geq 1$ for which the iterated compositions $f_k \circ \cdots \circ f_{k+\ell-1} : X_{j+\ell} \to X_j$ of the maps $f_j = f|_{X_{j+1}} : X_{j+1} \to X_j$ are $(\lambda^\ell, L)$-quasi-similarities. Second, the Mayer’s map $q$ is UQR. The fact that $f$ has the wild Cantor set $X$ as its Julia set follows from the argument in [18] almost verbatim. This completes the proof of Theorem 1.14.

**Appendix: A quasi-self-similar wild Cantor set in dimension 4**

A Cantor set $X$ in $\mathbb{R}^n$ is **tame** if there is a homeomorphism of $\mathbb{R}^n$ that maps $X$ onto the standard ternary Cantor set contained in a line. A Cantor set in $\mathbb{R}^n$ is **wild** if it is not tame. Antoine constructed the first wild Cantor set in $\mathbb{R}^3$, which is now known as Antoine’s necklace. Blankinship [8] extended Antoine’s construction to produce wild Cantor sets in $\mathbb{R}^n$ for every $n \geq 4$. Constructions of wild Cantor sets are abundant in geometric topology, see e.g. Bing [6] and Daverman and Edwards [12] for more examples.

While there exist geometrically self-similar Antoine’s necklaces allowing a sufficiently large number of tori at each stage [24], [62], [39], it is unknown whether geometrically self-similar wild Cantor sets exist in $\mathbb{R}^n$ for $n \geq 4$; see Garity and Repovš [40, pp. 675-679].

In this section we construct a wild Cantor set in $\mathbb{R}^4$ which is quasi-self-similar; we do not know whether quasi-self-similar wild Cantor sets exist in dimensions five or higher. The notion of quasi-self-similarity was introduced by McLaughlin [35]; see also [17].

**Definition 20.1.** A nonempty set $X$ in a metric space $(S, d)$ is **$L$-quasi-self-similar** for $L \geq 1$, if there exists a radius $r_0 > 0$ such that, given any ball $B$ of radius $r < r_0$, there exists a map $f_B : B \cap X \to X$ satisfying

$$\frac{1}{L} \frac{r_0}{r} d(y, z) \leq d(f_B(y), f_B(z)) \leq L \frac{r_0}{r} d(y, z)$$

for all $y, z \in B \cap X$. A metric space $(X, d)$ is **self-similar** if it is 1-quasi-self-similar.

**Theorem 20.2.** There exist geometrically quasi-self-similar wild Cantor sets in $\mathbb{R}^4$.

Topologically the Cantor set that we construct is an Antoine-Blankinship’s necklace. We recall first the terminologies of the construction. A solid $n$-tube in $\mathbb{R}^n$, $n \geq 3$, is a topological space homeomorphic to $B^2 \times (S^1)^{n-2}$. Consider the embedding of $m$ linked solid 4-tubes $T_1, T_2, \ldots, T_m$ in an unknotted solid tube $T$ in $\mathbb{R}^n$ as in Antoine [3] for $n = 3$, or as in Blankinship
for \( n \geq 4 \). The Cantor set in question is the intersection

\[
X = \bigcap_{k=0}^{\infty} X_k,
\]

where \( X_k \) is a collection of \( m^k \) disjoint solid 4-tubes and for each tube \( \tau \) in \( X_k \) the triple \((\mathbb{R}^n, \tau, \tau \cap X_{k+1})\) is homeomorphic to \((\mathbb{R}^n, T, \bigcup_{j=1}^{m_1} T_j)\). Chosen with care, the diameters of the components of \( X_k \) approach zero as \( k \to \infty \); hence \( X \) is a Cantor set in \( \mathbb{R}^n \). See also [38] for another description of Blankinship’s construction.

In dimension 3, geometric self-similarity of \( X \) can be reached by choosing \( m \) sufficiently large; see [39] for related discussion.

In dimension 4, we construct a wild Cantor set \( X = \bigcap_{k=1}^{\infty} X_k \) in \( \mathbb{R}^4 \) whose difference sets \( \{ \tau \setminus X_{k+1} : \tau \in X_k \text{ and } k \geq 0 \} \) belong to precisely two (geometric) similarity classes. Because our construction follows that of Blankinship topologically, the wildness of \( X \) follows from [8, Section 2].

### 20.1. The construction.

For the proof of Theorem 20.2, our construction differs geometrically from that of Blankinship.

We first give an overview of the construction. To fit the steps together, we need Proposition 20.3 below.

Let \( \Phi : \mathbb{R}^4 \to \mathbb{R}^4 \) and \( \Psi : \mathbb{R}^4 \to \mathbb{R}^4 \) be isometries defined by \( x \mapsto A_{\Phi} x + e_3 \) and \( x \mapsto A_{\Psi} x + e_3 \), respectively, where the linear mappings \( A_{\Phi} \) and \( A_{\Psi} \) are given by matrices

\[
A_{\Phi} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
A_{\Psi} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

in the standard basis, and \( e_3 \) is the vector \((0, 0, 1, 0)\). In particular, given \((x_1, x_2, x_3, x_4)\), we have

\[
\Phi(x_1, x_2, x_3, x_4) = (x_1, x_4, 1 - x_3, x_2)
\]

and

\[
\Psi(x_1, x_2, x_3, x_4) = (x_3, x_4, 1 + x_1, x_2),
\]

and that both isometries \( \Phi \) and \( \Psi \) are orientation preserving. Heuristically, \( A_{\Phi} \) is essentially an exchange of coordinates in \( \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \), and \( A_{\Psi} \) exchanges the \( \mathbb{R}^2 \) factors of \( \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \).

Let \( \varrho : \mathbb{R}^4 \to \mathbb{R}^4 \) be the rotation

\[
(x_1, x_2, r, \theta) \mapsto (x_1, x_2, r, \theta + 2\pi/m)
\]

where \((r, \theta)\) are the polar coordinates in \( \mathbb{R}^2 \), that is,

\[
(x_1, x_2, x_3, x_4) \mapsto \left( x_1, x_2, x_3 \cos\left(\frac{2\pi}{m}\right) - x_4 \sin\left(\frac{2\pi}{m}\right), x_3 \sin\left(\frac{2\pi}{m}\right) + x_4 \cos\left(\frac{2\pi}{m}\right) \right)
\]

in Cartesian coordinates. Let also, for \( j \in \mathbb{Z}_+ \), \( \varrho^j : \mathbb{R}^4 \to \mathbb{R}^4 \) be the \( j \)th iterate of \( \varrho \), that is, the map

\[
(x_1, x_2, r, \theta) \mapsto (x_1, x_2, r, \theta + 2j\pi/m).
\]
Let $b_0, b_1, c_0,$ and $c_1$ be constants in $(0, 1)$, whose values will be fixed later in Lemma 20.4. We denote $\rho = \min\{c_0, c_1\}/10$, and fix constants $b \in (0, 1)$ and $m \in 2\mathbb{Z}_+$ which satisfy

\[
0 < b < \min\{b_0, b_1, \rho/10\},
\]

and

\[
4b^2/3 \leq 2\pi/m \leq 3b^2/2.
\]

Let $\lambda: \mathbb{R}^4 \to \mathbb{R}^4$ be the scaling map

\[x \mapsto bx.\]

In the construction of the Cantor set, we iterate two geometric model configurations $T$ and $\tilde{T}$ each of which is a solid 4-tube. In what follows, $m$ is the number of 4-tubes inside the initial configuration, and $b$ is the scaling constant. Note that $b$ is not comparable to the reciprocal of $m$. We will return to this particular point later.

We now discuss the iteration process formally assuming that the 4-tubes have been fixed. We set for each $j = 1, \ldots, m$,

\[
X_{1,j} = \begin{cases} (\rho^j \circ \Phi \circ \lambda)T, & j \text{ even,} \\ (\rho^j \circ \Phi \circ \lambda)\tilde{T}, & j \text{ odd.} \end{cases}
\]

and

\[
\tilde{X}_{1,j} = \begin{cases} (\rho^j \circ \Psi \circ \lambda)T, & j \text{ even,} \\ (\rho^j \circ \Psi \circ \lambda)\tilde{T}, & j \text{ odd.} \end{cases}
\]

Let also

\[X_1 = \bigcup_{j=1}^{m} X_{1,j} \quad \text{and} \quad \tilde{X}_1 = \bigcup_{j=1}^{m} \tilde{X}_{1,j}.
\]

By suitable choices of $c_0, c_1, b_0,$ and $b_1$, sets $X_1$ and $\tilde{X}_1$ are unions of mutually disjoint 4-tubes. Moreover in $X_1$ and also in $\tilde{X}_1$, half of the 4-tubes are similar to the model tube $T$ and half are similar to the model tube $\tilde{T}$. This choice allows us to establish the quasi-self-similarity (instead of the self-similarity) of the Cantor set.

For the iteration, we need the following proposition.

**Proposition 20.3.** There exist constants $b_0, b_1, c_0,$ and $c_1$ in $(0, 1)$, and bilipschitz equivalent 4-tubes $T$ and $\tilde{T}$ for which the following holds: for each $j = 1, \ldots, m$,

1. $X_{1,j} \subset \text{int}T$ and $\tilde{X}_{1,j} \subset \text{int}\tilde{T}$;
2. for $i \neq j$, $X_{1,i} \cap X_{1,j} = \emptyset$ and $\tilde{X}_{1,i} \cap \tilde{X}_{1,j} = \emptyset$;
3. $X_{1,j}$ and $X_{1,j+1}$ (similarly $\tilde{X}_{1,j}$ and $\tilde{X}_{1,j+1}$) are disjoint tubes, Hopf linked in $\mathbb{R}^4$; here $m + 1 \equiv 1(\text{mod} m)$.

Assuming for now the validity of the proposition, we fix a bilipschitz homeomorphism (of pairs) $H: (T, X_1) \to (\tilde{T}, \tilde{X}_1)$ having the property that for each $j = 1, \ldots, m$, the restriction $H|_{X_{1,j}}$ satisfies

\[H|_{X_{1,j}} = \rho^j \circ \Psi \circ \Phi^{-1} \circ \rho^{-j}.\]

Note that, in particular, each restriction $H|_{X_{1,j}}: X_{1,j} \to \tilde{X}_{1,j}$ is an isometry.

For each $j = 1, \ldots, m$, let $\varphi_j: T \to X_{1,j}$ be the homeomorphism

\[
\varphi_j = \begin{cases} \rho^j \circ \Phi \circ \lambda, & j \text{ even,} \\ \rho^j \circ \Phi \circ \lambda \circ H, & j \text{ odd.} \end{cases}
\]
Then \( X_1 = \bigcup_{j=1}^{m} \varphi_j T \). We set, for each \( k \geq 1 \),
\[
X_{k+1} = \bigcup_{|\alpha|=k} \bigcup_{j=1}^{m} (\varphi_{\alpha_1} \circ \cdots \circ \varphi_{\alpha_k} \circ \varphi_j) T,
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, m\}^k \), and set \( X_0 = T \) for completeness. 
Note again that, for each \( k \in \mathbb{Z}_+ \), half of the connected components of \( X_k \) are similar to \( T \) and half are similar to \( T \).

The intersection \( X = \bigcap_{k=0}^{\infty} X_k \) is a wild Cantor set. The quasi-self-similarity of \( X \) follows from the fact that each map \( \varphi_{\alpha_1} \circ \cdots \circ \varphi_{\alpha_k} \) is \((b^k, L)\)-quasi-similar, where \( L \) is a bilipschitz constant for the mapping \( H \). The wildness follows now directly from [8, Section 2].

For the proof of Theorem 20.2, it remains to verify Proposition 20.3.

**20.2. Proof of Proposition 20.3.** Let \( b \in (0, 1/10) \) be the constant in \([1]\) to be determined, and \( m \) be an even integer satisfying \([2]\).

Since a solid 4-tube \( B^2 \times S^1 \times S^1 \) may be considered as the regular neighborhood of its core \( \{0\} \times S^1 \times S^1 \) in \( \mathbb{R}^4 \), we first construct the toroidal cores. 

Heuristically, starting with a torus \( S^1(b) \times S^1(1) \) in \( \mathbb{R}^4 \), we place \( m \) tori comparable in size to \( S^1(b) \times S^1(b^2) \) in such a way that their smaller generating circles \( S^1(b^2) \) are linked in \( \mathbb{R}^3 \), and go around the larger generating circle \( S^1(1) \) of the initial torus. In the next step, corresponding to each one of these \( m \) tori, we place \( m \) smaller tori comparable in size to \( S^1(b^3) \times S^1(b^2) \) so that their \( S^1(b^3) \)-circles are linked in a Euclidean 3-space and go around the \( S^1(b) \)-circle of their predecessor. The construction may be continued by induction. We note here that the scaling constant is roughly \( b \), and we need \( m \approx 1/b^2 \) small circles to go around the previous generating circle. The condition \([2]\) stems from this observation.

**Four circles.** We fix four (families of) circles which are meridians and longitudes of the cores of two (families of) tubes.

For the first two families of tori, we set
\[
\gamma \equiv \gamma(b) = \{(0, x_2, x_3, 0) \in \mathbb{R}^4 : x_2^2 + (x_3 - 1)^2 = b^2 \}
\]
and
\[
l \equiv l(b) = \{(0, 0, x_3, x_4) \in \mathbb{R}^4 : x_3^2 + x_4^2 = (1 - b)^2 \}.
\]

For the next two families, we set
\[
\tilde{\gamma} \equiv \tilde{\gamma}(b) = S^1(b) \times \{(1, 0)\}
\]
and
\[
\tilde{l} \equiv l(b) = \{(b, 0)\} \times S^1.
\]
The circles \( \gamma \) and \( \tilde{\gamma} \) are orthogonal and they have a common center \( e_3 = (0, 0, 1, 0) \), and circles \( l \) and \( \tilde{l} \) are invariant under the rotation \( \varphi \).

**Toroidal cores.** Let \( \kappa \) be the round torus obtained by revolving \( \gamma \) in \( \mathbb{R}^4 \) with respect to the hyperplane \( P = \mathbb{R}^2 \times \{(0, 0)\} \) in \( \mathbb{R}^4 \), that is,
\[
k \equiv \kappa(b) = \{(0, b \cos \phi, (1 + b \sin \phi) \cos \theta, (1 + b \sin \phi) \sin \theta) : \phi, \theta \in [0, 2\pi] \}.
\]
Then $\gamma$ is a meridian and $l$ is a longitude of $\kappa$; they will be designated as the marked meridian and the marked longitude of $\kappa$, respectively.

Let $\bar{\kappa} = S^1(b) \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ be the flat torus in $\mathbb{R}^4$, that is,$$
\bar{\kappa} \equiv \bar{\kappa}(b) = \{(b \cos \psi, b \sin \psi, \cos \theta, \sin \theta) : \psi, \theta \in [0, 2\pi]\}.$$

Then $\tilde{\gamma}$ is a meridian and $\tilde{l}$ is a longitude of $\tilde{\kappa}$. In fact, $\bar{\kappa}$ is the surface of revolution of $\tilde{\gamma}$ with respect to $P$. We call $\tilde{\gamma}$ and $\tilde{l}$ the marked meridian and the marked longitude of $\bar{\kappa}$, respectively.

We summarize the properties of toroidal cores $\kappa$ and $\bar{\kappa}$ with respect to the properties of $\Phi$ and $\Psi$ as follows. Since $\Phi(P) = \mathbb{R} \times \{(0, 1)\} \times \mathbb{R}$, we have that

(1) the embedded 2-tori $(\Phi \circ \lambda)\kappa$ and $(\Phi \circ \lambda)\bar{\kappa}$ are surfaces of revolution, with respect to the hyperplane $\mathbb{R} \times \{(0, 1)\} \times \mathbb{R}$, of the marked meridians
\[
(\Phi \circ \lambda)\gamma = \{(0, 0, x_3, x_4) : (x_3 - (1 - b))^2 + x_4^2 = b^4\},
\]
and
\[
(\Phi \circ \lambda)\tilde{\gamma} = \{(x_1, 0, 1 - b, x_4) : x_1^2 + x_4^2 = b^4\},
\]
respectively;

(2) the marked meridians $(\Phi \circ \lambda)\gamma$ and $(\Phi \circ \lambda)\tilde{\gamma}$ have a common center $(0, 0, 1 - b, 0)$, which lies on the longitude $l$ of the torus $\kappa$, and the axle of tori $(\Phi \circ \lambda)\kappa$ and $(\Phi \circ \lambda)\bar{\kappa}$, by which we mean the circle of revolution of the center $(0, 0, 1 - b, 0)$ with respect to the hyperplane $\Phi(P)$, is the marked meridian $\gamma$ of $\kappa$; and

(3)\[
\max\{\text{dist} (y, \kappa) : y \in (\Phi \circ \lambda)\kappa\} \leq b^2,
\]
and
\[
\max\{\text{dist} (y, \kappa) : y \in (\Phi \circ \lambda)\bar{\kappa}\} \leq b^2.
\]

Similarly, regarding the embedding $\Psi \circ \lambda$, we have that

(1) tori $(\Psi \circ \lambda)\kappa$ and $(\Psi \circ \lambda)\bar{\kappa}$ are the surfaces of revolution, with respect to the hyperplane $\{(0, 0)\} \times \mathbb{R}^2 = \Psi(P)$, of the meridians
\[
(\Psi \circ \lambda)\gamma = \{(x_1, 0, 1, x_4) : (x_1 - b)^2 + x_4^2 = b^4\},
\]
and
\[
(\Psi \circ \lambda)\tilde{\gamma} = \{(b, 0, x_3, x_4) : (x_3 - 1)^2 + x_4^2 = b^4\},
\]
respectively;

(2) these two meridians $(\Psi \circ \lambda)\gamma$ and $(\Psi \circ \lambda)\tilde{\gamma}$ have a common center $(b, 0, 1, 0)$, which lies on the longitude $\tilde{l}$ of $\bar{\kappa}$, and the axle of $(\Psi \circ \lambda)\kappa$ and $(\Psi \circ \lambda)\bar{\kappa}$, i.e., the circle of revolution of the center $(b, 0, 1, 0)$ with respect to $\Psi(P)$, is the marked meridian $\gamma$ of $\bar{\kappa}$; and

(3) we have
\[
\max\{\text{dist} (y, \kappa) : y \in (\Psi \circ \lambda)\kappa\} \leq b^2,
\]
and
\[
\max\{\text{dist} (y, \kappa) : y \in (\Psi \circ \lambda)\bar{\kappa}\} \leq b^2.
\]
Lemma 20.4. There exist absolute constants $b_0 > 0$ and $c_0 > 0$ with the property that if $0 < b < b_0$, then

$$\text{dist} (\tau_i, \tau_j) \geq c_0 b^2$$

for $i, j \in \{1, \ldots, m\}, i \neq j$. Similarly, there exist absolute constants $b_1 > 0$ and $c_1 > 0$ so that if $0 < b < b_1$ then

$$\text{dist} (\tilde{\tau}_i, \tilde{\tau}_j) \geq c_1 b^2$$

for all $i, j \in \{1, \ldots, m\}$ and $i \neq j$.

Proof. Note that each set of distance estimates claimed in the lemma is rotational invariant. For the first claim, it suffices to check that

(3) $\text{dist} (((\Phi \circ \lambda) \kappa, (\varrho \circ \Phi \circ \lambda) \kappa) > c_0 b^2$,

for some absolute constants $b_0$ and $c_0$ in $(0, 1)$ and for $0 < b < b_0$.

Recall that $\varrho$ is a rotation by an angle $2\pi/m$ and $4b^2/3 < 2\pi/m < 3b^2/2$; denote, in the following, $\beta = 2\pi/m$. Then

$$(\Phi \circ \lambda) \kappa = p([0, 2\pi)^2)$$

and

$$(\varrho \circ \Phi \circ \lambda) \kappa = q([0, 2\pi)^2),$$

where $p: [0, 2\pi)^2 \to \mathbb{R}^4$ is the mapping

$$p(\phi, \theta) = (0, (b + b^2 \sin \phi) \sin \theta, 1 - (b + b^2 \sin \phi) \cos \theta, b^2 \cos \phi)$$

Cyclically linked cores. For each $j = 1, \ldots, m$, let $\sigma_j \subset \mathbb{R}^4$ be the circle

$$\sigma_j = \begin{cases} (\varrho^j \circ \Phi \circ \lambda) \gamma_j, & j \text{ even,} \\ (\varrho^j \circ \Phi \circ \lambda) \tilde{\gamma}_j, & j \text{ odd.} \end{cases}$$

The circles $\sigma_1, \ldots, \sigma_m$ form a necklace chain in $\mathbb{R} \times \{0\} \times \mathbb{R}^2$. More precisely,

1. circles $\sigma_1, \ldots, \sigma_m$ are pairwise disjoint, and their centers $z_1, \ldots, z_m$ are equally spaced on the longitude $l$ of $\kappa$,

2. $\varrho^2(\sigma_j) = \sigma_{j+2}$ for each $j = 1, \ldots, m - 2$, $\varrho^2(\sigma_{m-1}) = \sigma_1$, and $\varrho^2(\sigma_m) = \sigma_2$, and

3. circles $\sigma_1$ and $\sigma_j$ are (Hopf) linked in $\mathbb{R} \times \{0\} \times \mathbb{R}^2$ if and only if $i - j \equiv \pm 1 (\mod m)$.

Similarly circles

$$\tilde{\sigma}_j = \begin{cases} (\varrho^j \circ \Psi \circ \lambda) \gamma_j, & j \text{ even,} \\ (\varrho^j \circ \Psi \circ \lambda) \tilde{\gamma}_j, & j \text{ odd,} \end{cases}$$

for $j = 1, \ldots, m$, also form a necklace chain in $\mathbb{R} \times \{0\} \times \mathbb{R}^2$.

Let $\tau_1, \ldots, \tau_m$ and $\tilde{\tau}_1, \ldots, \tilde{\tau}_m$, where

$$\tau_j = \begin{cases} (\varrho^j \circ \Phi \circ \lambda) \kappa, & j \text{ even,} \\ (\varrho^j \circ \Phi \circ \lambda) \tilde{\kappa}, & j \text{ odd,} \end{cases} \quad \text{and} \quad \tilde{\tau}_j = \begin{cases} (\varrho^j \circ \Psi \circ \lambda) \kappa, & j \text{ even,} \\ (\varrho^j \circ \Psi \circ \lambda) \tilde{\kappa}, & j \text{ odd,} \end{cases}$$

be two sets of cyclically linked 2-tori. For small enough $b$, tori $\tau_1, \ldots, \tau_m$ are mutually disjoint; the same holds for tori $\tilde{\tau}_1, \ldots, \tilde{\tau}_m$. The distance between these tori may be estimated quantitatively as follows.
for $\phi, \theta \in [0, 2\pi]$, and $q \colon [0, 2\pi)^2 \to \mathbb{R}^4$ is the mapping

$$q(\psi, \theta) = (b^2 \cos \psi, b \sin \theta, (1 - b \cos \theta) \cos \beta - b^2 \sin \psi \sin \beta, (1 - b \cos \theta) \sin \beta + b^2 \sin \psi \cos \beta)$$

for $\psi, \theta \in [0, 2\pi]$.

Since $\sum_{i=1}^4 |x_i| \leq 2(\sum_{i=1}^4 |x_i|^2)^{1/2}$ for $x \in \mathbb{R}^4$, a direct computation, using the fact that $\sin \beta = \beta + O(\beta^3)$ and $\cos \beta = 1 + O(\beta^2)$ as $\beta \to 0$, yields the following estimates

$$2|p(\phi, \theta) - q(\psi, \theta')| \geq b^2 |\cos \psi| + |b \sin \theta' - (b + b^2 \sin \phi) \sin \theta| + |(1 - b \cos \theta') \cos \beta - b^2 \sin \psi \sin \beta - (1 - (b + b^2 \sin \phi) \cos \theta)| + |(1 - b \cos \theta') \sin \beta + b^2 \sin \psi \cos \beta - b^2 \cos \phi|$$

$$b^2 |\cos \psi| + |b \sin \theta' - (b + b^2 \sin \phi) \sin \theta| + |b \cos \theta' - (b + b^2 \sin \phi) \cos \theta| + |\beta + b^2 (\sin \psi - \cos \phi)| + O(b^3)$$

$$b^2 |\cos \psi| + b(1 + (1 - 2(1 + b \sin \phi) \cos(\theta - \theta') + b \sin \phi)^2)^{1/2} + |\beta + b^2 (\sin \psi - \cos \phi)| + O(b^3)$$

$$b^2 |\cos \psi| + b^2 |\sin \phi| + |\beta + b^2 (\sin \psi - \cos \phi)| + O(b^3).$$

If $|\beta + b^2 (\sin \psi - \cos \phi)| \geq b^2/4$, then (3) holds trivially. Otherwise, $|\beta + b^2 (\sin \psi - \cos \phi)| < b^2/4$, which yields

$$13/12 < \cos \phi - \sin \psi < 7/4.$$ 

Thus $\cos \phi > 0$ and $\sin \psi < 0$. Hence, at least one of the two inequalities $0 < \cos \phi < 7/8$ and $0 < -\sin \psi < 7/8$ holds. As a consequence, either $|\sin \phi| > 3/8$ or $|\cos \psi| > 3/8$. In either case (3) holds true.

The estimate of the distances between tori $\tilde{T}_1, \ldots, \tilde{T}_m$ is similar. This completes the proof of the lemma. \(\Box\)

**The 4-tubes.** We are now ready to choose the 4-tubes $T$ and $\tilde{T}$, and to verify the claims of Proposition 20.3. Let $b_0, b_1, c_0$, and $c_1$ be the constants in Lemma 20.3 and let $b, \rho$ and $m$ be constants satisfying (1) and (2). We now define the model 4-tubes by

$$T \equiv T(\rho, b) = \{x \in \mathbb{R}^4 : \text{dist}(x, \kappa) \leq \rho b\}$$

and

$$\tilde{T} \equiv \tilde{T}(\rho, b) = \{x \in \mathbb{R}^4 : \text{dist}(x, \tilde{\kappa}) \leq \rho b\}.$$ 

Note that $T$ and $\tilde{T}$ are not isometric, since $\kappa$ and $\tilde{\kappa}$ are not isometric. So the 4-tubes in $X_{1,1}, \ldots, X_{1,m}, \tilde{X}_{1,1}, \ldots, \tilde{X}_{1,m}$ are similar to either $T$ or $\tilde{T}$. Since these tubes have cores $\tau_1, \ldots, \tau_m, \tilde{\tau}_1, \ldots, \tilde{\tau}_m$, respectively. From (1) and Lemma 20.4, it follows that the 4-tubes $X_{1,1}, \ldots, X_{1,m}$ are pairwise disjoint; the same holds for 4-tubes $\tilde{X}_{1,1}, \ldots, \tilde{X}_{1,m}$. So statement (ii) in the claim of proposition holds.
Since $X_{1,1}, \ldots, X_{1,m}$ are cyclically (Hopf) linked in $\mathbb{R}^4$ by the construction and the same holds for 4-tubes $\tilde{X}_{1,1}, \ldots, \tilde{X}_{1,m}$. Thus statement (iii) also holds.

Since the core tori $\tau$ and $\tilde{\tau}$ are bilipschitz equivalent, the 4-tubes $T$ and $\tilde{T}$ are bilipschitz equivalent as claimed in the proposition.

It remains to verify statement (i) in the proposition.

**Lemma 20.5.** Under the conditions in (1), $X_1 \subset \text{int} X_0$ and $\tilde{X}_1 \subset \text{int} \tilde{X}_0$.

**Proof.** Recall that the core $\tau_m$ of $X_{1,m}$ is the torus $(\Phi \circ \lambda) \kappa$, and the axle of $\tau_m$ is the meridian $\gamma$ of $\kappa$. Recall also that $\max\{\text{dist} (y, \kappa) : y \in (\Phi \circ \lambda) \kappa\} \leq b^2$.

Therefore, for $x \in X_{1,m}$, we have

$$\text{dist} (x, \kappa) \leq \text{dist} (x, \tau_m) + \max\{\text{dist} (y, \kappa) : y \in \tau_m\} \leq \rho b^2 + b^2 < \rho b/5.$$  

Thus, $X_{1,m} \subset \text{int} X_0$ and, by rotation, $X_{1,j} \subset \text{int} X_0$ for all even $j$; the proof of inclusion for odd $j$ is similar. Hence $X_1 \subset \text{int} X_0$. Similarly $\tilde{X}_1 \subset \text{int} \tilde{X}_0$.  

This completes the proof of Proposition 20.3 and the proof of Theorem 20.2.

**Remark 20.6.** The wild Cantor set $X$ constructed for Theorem 20.2 satisfies a condition sharper than the quasi-self-similarity. In fact, the Cantor set

$$X = \bigcap_{k=0}^{\infty} \bigcup_{|\alpha| = k} \varphi_\alpha T,$$

where, for each $\alpha \in \{1, \ldots, m\}^k$, $\varphi_\alpha$ is a $(b^k, L)$-quasi-similarity, and $0 < b < 1$ and $L \geq 1$ are constants.

**Remark 20.7.** In the construction above, $T$ and $\tilde{T}$ are both homeomorphic to $B^2 \times (S^1)^2$ but are not geometrically similar to each other. Therefore the wild Cantor set $X$ constructed above is only quasi-self-similar.

We do not know whether the 4-tubes $T$ and $\tilde{T}$ may be chosen to be the same set. Such a choice, if possible, would yield a self-similar (instead of quasi-self-similar) wild Cantor set.

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