Math 542 **Comprehensive Examination** **August 2019**

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems. Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3, and 4. Each problem is worth 10 points. Justify all your answers. Good Luck!

Notation:
We denote the set of all complex numbers by $\mathbb{C}$. If D is a domain in $\mathbb{C}$, we denote the closure of D by $\overline{D}$.

1. Let $D = \{z \in \mathbb{C} : |z| < 1\} \setminus [\frac{1}{2}, 1]$, that is, D is the unit disk with the line segment from $\frac{1}{2}$ to 1 removed. Find a conformal mapping of D onto the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. (Hint: You can find the mapping as a composition of linear-fractional transformations and powers of z. Carefully sketch all your intermediate domains if any.)

2. Evaluate
$$\int_0^\infty \frac{\sqrt{x}\, dx}{x^2 + x + 1}.$$

3. Let f be an analytic function in $\mathbb{C} \setminus \{0\}$ that satisfies the estimate $|f(z)| \leq |z|^{5/2}$. Prove $f = 0$ identically. (Hint: Use the Laurent series and the Cauchy estimates.)

4. Let f be an analytic function in a neighborhood of $z_0 \in \mathbb{C}$. Suppose f has a zero of order $m \geq 1$ at z_0. Prove that there is $\delta_0 > 0$ so that for every $0 < \delta < \delta_0$, there exists $\epsilon > 0$ such that for every $w \in \mathbb{C}$ with $0 < |w| < \epsilon$, the equation $f(z) = w$ has exactly m distinct solutions z in the disk $|z - z_0| < \delta$. (Hint: Use the Rouché theorem.)

5. Present the following part of the proof of the Runge theorem. Let $D \subset \mathbb{C}$ be a simply connected bounded domain. Prove that for every $z_0 \in \mathbb{C} \setminus \overline{D}$, the function $f(z) = \frac{1}{z - z_0}$ can be approximated by polynomials uniformly on D.

6. Prove the identity
$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$