Math 540 Comprehensive Examination, August 2019

Solve five of the following six. Each problem is worth 20 points.

The Lebesgue measure is denoted by m, and $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

1. Let $E \subset \mathbb{R}$ be Lebesgue measurable. Define

$$f(x) = \mathrm{dist}(x, E) = \inf\{|x - e| : e \in E\}.$$

Prove that

$$\lim_{r\to 0} \frac{f(x+r)}{r} = 0 \quad \text{for } m \text{ a.e. } x \in \mathbb{R}.$$

2. Consider the sequence of functions

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \frac{n}{1 + n^4 x^2}, \quad x \in \mathbb{R}.$$

(i) Does this sequence converge to zero almost uniformly on $\mathbb{R}$?
(ii) Determine all $p \in [1, \infty]$ for which the sequence converge to zero in $L^p(\mathbb{R}, m)$.

3. Let $f \in L^\infty(\mu)$ where μ is a finite measure. Suppose that $\|f\|_\infty > 0$. Does

$$\lim_{p\to\infty} \frac{\|f\|_{p+1}^{p+1}}{\|f\|_p^p}$$

exist? If it does, find the limit and verify your answer. Otherwise give a counterexample.

4. Evaluate the following limit for $a = 0$ and for $a > 0$

$$\lim_{n\to\infty} \int_a^\infty \frac{n^2 x}{1 + x^2} e^{-n^2 x^2} dx\,.$$

5. (i) Show that $g(x) = \frac{\sin x}{x}$ is not Lebesgue integrable on $([0, \infty), m)$.
(ii) Employ the identity

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt \quad (x > 0)$$

to evaluate the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\, dx \;=\; \lim_{L\to\infty} \int_0^L \frac{\sin x}{x}\, dx.$$

6. For any functions $f, g \in C(\mathbb{T})$ prove that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(nt)\ \mathrm{d}t \;=\; \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ \mathrm{d}t\right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\ \mathrm{d}t\right).$$

Here the limit is taken over $n \in \mathbb{N}$.