Comprehensive Exam | Math 500 | August 2019

All answers must contain proper justifications.

1. Let p, q be two prime integers. Prove that a group of order p^2q is not simple. (20 pts.)

2. Let S_n denote the symmetric group of n elements and let $n \geq 5$.

   a) Show that any 3-cycle is a commutator. (5 pts.)

   b) Let H be a subgroup of S_n and let H_1 be a normal subgroup of H such that H/H_1 is abelian. If H contains all 3-cycles then show that H_1 contains all 3-cycles. (5 pts.)

   c) Deduce that S_n is not solvable. (10 pts.)

3. Let V be a finite dimensional real vector space and $\phi : V \to V$ a linear transformation with invariant factors $q_1 = X^4 - 4X^3 + 5X^2 - 4X + 4 = (x-2)^2(x^2+1)$ and $q_2 = X^7 + 6X^6 + 14X^5 - 20X^4 + 25X^3 - 22X^2 + 12X - 8 = (X-2)^3(X^2+1)^2$ in $\mathbb{R}[X]$.

   a) Find the rational cononical form of ϕ with respect to some basis. (10 pts.)

   b) Suppose V is a complex vector space and $\psi : V \to V$ is a linear transformation with same invariant factors as above.

      i) Find the elementary divisors of ψ in $\mathbb{C}[X]$. (5 pts.)

      ii) Find the Jordan canonical form of ψ with respect to some basis. (5 pts.)

4. Consider the polynomial $f(X) = X^4 - 2$ on $\mathbb{Q}[X]$.

a) Show that $f(X)$ is irreducible in $\mathbb{Q}[X]$. (5 pts.)

b) Let L denote the splitting field of $f(X)$ and let G denote its galois group over $\mathbb{Q}$. Determine L and G. Also find a relation between the generators of G. (15 pts.)

5. Let p be a prime integer > 2.

a) Show that for any integer $n, n^p \equiv n \bmod p$. (5 pts.)

b) Let k be a field of characteristic p and let $f(X) = X^p - X - a \in k[X], a \in k$.

Show that

i) If $f(X)$ has a root in k, then $f(X)$ has all its roots in k. (5 pts.)

ii) If $f(X)$ does not have any root in k, then $f(X)$ is irreducible in $k[X]$. (5 pts.)

iii) In case ii) above, the galois group of $f(X)$ is cyclic of order p. (5 pts.)