Syllabus for Math 416

Course summary. Math 416 is a rigorous, abstract treatment of linear algebra. Topics to be covered include vector spaces, linear transformations, eigenvalues and eigenvectors, diagonalizability, and inner product spaces. The course concludes with a brief introduction to the theory of canonical forms for matrices and linear transformations.

Audience. Math 416 or Math 416 Honors is required of all math majors. The listed prerequisites are Math 241 or permission of the instructor, with Math 347 as a recommended prerequisite. In practice, Math 347 is "strongly recommended" by the math advisors as the content of Math 416 is theoretical and proof-oriented.

Math 416 does *not* have any other linear algebra course as a prerequisite. Instructors should not expect that students are familiar with matrix linear algebra, beyond the brief introduction to vectors and matrices (exclusively 2×2 and 3×3 matrices) in third-semester calculus (Math 241). For this reason, the course *must* start with a crash course on matrix linear algebra, including

- systems of linear equations, row reduction and echelon form
- review of Euclidean vectors, emphasizing linear dependence/independence and spanning sets
- matrices: arithmetic, invertibility, matrix inverses (2×2 and 3×3 only), nonsingular matrices, elementary matrices and their relationship to row reduction

In addition to introducing students to the basic ideas of linear algebra from a theoretical perspective, the course should give students practice in reading and writing proofs. The course will cover fewer applications than Math 415, but it should treat the material at a more conceptual level.

Students may take both Math 225 and Math 416 for credit. It is possible for students daunted by 416 to take 225 first to gain familiarity with matrix linear algebra.

Book. The textbook for the course is

Linear Algebra, fourth edition, by S. H. Friedberg, A. J. Insel and L. E. Spence.

The book begins by defining abstract vector spaces, and only discusses matrix operations and solvability of linear systems in Chapter 3. As mentioned above, Math 416 reorganizes the material to start off with concrete matrix linear algebra. There are a variety of instructional aids which can be used for this topic; a recommend one is the free online book *A First Course in Linear Algebra* by R. A. Beezer.

While the initial topic of concrete matrix linear algebra is crucial for students to succeed at Math 416, the course is still primarily one in theoretical linear algebra; it is important to move relatively quickly to a discussion of vector spaces and linear transformations from a more abstract perspective.

Syllabus. The essential ideas in the course are

(1) systems of linear equations, row reduction and echelon form
(2) vectors and matrices, matrix multiplication, invertibility and inverses
(3) vector spaces and linear transformations
(4) subspaces, linear combinations, spanning sets and bases
(5) representing linear transformations as matrices, change of basis
(6) kernel and image, row and column rank, Rank-Nullity theorem
(7) determinants
(8) eigenvalues and eigenvectors

(9) finding the eigenvalues of a transformation using the characteristic polynomial
(10) finding the eigenspace associated to an eigenvalue
(11) inner product spaces and their algebra and geometry, the Cauchy–Schwarz inequality
(12) orthogonal projections, Gram–Schmidt, least squares
(13) orthogonal and unitary matrices, spectral theory
(14) bilinear forms
(15) Jordan form

Schedule. Here is a typical schedule

- [**Material from Beezer, or Chapter 3 of FIS**] Systems of linear equations, elementary row operations, row echelon form, pivots, Gaussian elimination, matrix arithmetic, elementary matrices, rank of a matrix, invertibility, matrix inverses (6 hours)
- [**Chapter 1 of FIS, §§1.1–1.6**] Basic notions: vector spaces, subspaces, linear dependence and independence, bases and dimension (5 hours)
- [**Chapter 2 of FIS, §§2.1–2.5**] Linear transformations, kernel and range, matrix representations of linear transformations, isomorphisms and invertibility, change of basis (6 hours)
- [**Chapter 4 of FIS**] Determinants (4 hours)
- [**Chapter 5 of FIS, §§5.1, 5.2 and 5.4**] Diagonalization, eigenvalues and eigenvectors, invariant subspaces and the Cayley–Hamilton theorem (4 hours)
- [**Chapter 6 of FIS, §§6.1–6.7**] Inner product spaces, orthogonality, Gram–Schmidt, adjoints, normal and self-adjoint operators, orthogonal and unitary operators, orthogonal projections and the Spectral Theorem, bilinear forms (12 hours)
- [**Chapter 7 of FIS, §§7.1 and 7.2**] Advanced spectral theory: generalized eigenspaces and the Jordan canonical form (3 hours)

This schedule totals 40 hours, leaving 4 hours for exams, review and leeway. Some instructors may wish to emphasize other topics. For instance, one could omit section 5.4 on invariant subspaces and the Cayley–Hamilton theorem and instead cover the polar and/or singular value decomposition.