QUASICONFORMAL DISTORTION OF THE ASSOUAD SPECTRUM AND CLASSIFICATION OF POLYNOMIAL SPIRALS

EFSTATHIOS K. CHRONTSIOS GARITSIS AND JEREMY T. TYSON

Dedicated to Frederick W. Gehring (1925–2012) and Jussi Väisälä

Abstract. We investigate the distortion of Assouad dimension and the Assouad spectrum under Euclidean quasiconformal maps. Our results complement existing conclusions for Hausdorff and box-counting dimension due to Gehring–Väisälä and others. As an application, we classify polynomial spirals $S_a := \{x^{-a}e^{\mathbf{i}x} : x > 0\}$ up to quasiconformal equivalence, up to the level of the dilatation. Specifically, for $a > b > 0$ we show that there exists a quasiconformal map f of $\mathbb{C}$ with dilatation K_f and $f(S_a) = S_b$ if and only if $K_f \geq \frac{a}{b}$.

1. Introduction

Quasiconformal mappings can distort dimensions of sets. Gehring and Väisälä [13] gave dilatation-dependent bounds for Euclidean quasiconformal distortion of Hausdorff dimension. The precise formulation of their bounds involves the sharp exponent of higher Sobolev integrability for an n-dimensional quasiconformal map, whose precise value remains conjectural in dimension at least three. In dimension two, explicit, sharp results follow from Astala's theorem [2].

A dilatation-independent study of quasiconformal dimension distortion was initiated in the late 1990s, see e.g. [6], [21], [17], [23]. While the results of Gehring and Väisälä concern the distortion of dimensions of arbitrary subsets by a fixed quasiconformal mapping, these later results concern distortion of dimension for a fixed subset of $\mathbb{R}^n$ by arbitrary quasiconformal maps.

A more recent line of research (see, for instance, [4], [7], or [5]) addresses the question of simultaneous distortion of dimensions of large families of parallel subspaces, or generic elements in other parameterized families of subsets.

In all of the preceding discussion, the concept of dimension under consideration is the Hausdorff dimension. Hausdorff dimension is one of the most well-studied metric notions of dimension, and numerous tools exist for its computation and estimation. Some of the preceding theory extends to other notions of dimension, such as box-counting or packing dimension. For example, the distortion bounds in Gehring and Väisälä's original paper hold also for both box-counting dimension and packing dimension, as they rely only on higher Sobolev regularity. See Kaufman, [16], for a discussion of the distortion of Hausdorff and box-counting dimension by supercritical Sobolev maps.

In this paper we establish dilatation-dependent estimates for the distortion of Assouad dimension and the Assouad spectrum by quasiconformal maps. This paper can be seen as a companion to [22], which considered conformal Assouad dimension of sets and metric spaces. We improve and sharpen some dilatation-dependent estimates from [22], and we initiate a study of quasiconformal distortion of the recently defined Assouad spectrum.

As an application, we provide a precise classification of planar polynomial spirals up to quasiconformal equivalence which is sharp on the level of dilatation. This result indicates the relevance of the Assouad spectrum for classification problems in geometric mapping theory.

Date: December 5, 2021.

For a strictly decreasing function $\phi : (1,\infty) \to (0,1)$ with $\phi(x) \overset{x\to\infty}{\longrightarrow} 0$, define the *$\phi$-spiral*

$$S(\phi) := \{\phi(x)e^{\mathbf{i}x} \in \mathbb{C} : x > 0\}.$$

When $\phi(x) = e^{-cx}$, $c > 0$, we have the *logarithmic spiral.* Here we consider polynomial spirals. For $a > 0$ set $S_a := S(\phi_a)$ where

$$\phi_a(x) = x^{-a}.$$

The following result is an application of our main theorem.

Theorem 1.1. *For $a > b > 0$, there exists a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ with $f(S_a) = S_b$ if and only if $K_f \geq \frac{a}{b}$.*

One direction is trivial. The radial stretch map $f(z) = |z|^{1/K-1}z$ is K-quasiconformal, and maps S_a to S_b with $b = a/K$. The content of the theorem is the other implication, namely, if $K < \frac{a}{b}$, then no K-quasiconformal map of $\mathbb{C}$ satisfies $f(S_a) = S_b$. Distinguishing sets up to quasiconformal equivalence, particularly at the level of the dilatation, is in general a hard problem. One approach is to use dimension distortion bounds. Theorem 1.1 is the first application of the Assouad spectrum to such a question, and motivates the study of these notions of dimension in connection with geometric mapping theory.

We now discuss the history of quasiconformal dimension distortion estimates, to set the stage for our main result. Let us begin by recalling the celebrated estimates of Gehring and Väisälä from [13]. Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping between domains in Euclidean space $\mathbb{R}^n$, $n \geq 2$. (Here and throughout this paper we adopt the analytic definition for quasiconformality.) According to Gehring's well-known higher integrability theorem [12], f lies in a local Sobolev space $W^{1,p}_{loc}(\Omega : \mathbb{R}^n)$ for some $p > n$. In particular, the Jacobian of f satisfies a reverse Hölder inequality. For ease of exposition, we will state the reverse Hölder inequality in terms of the norm of the differential, $|Df|$, rather than the Jacobian of f.

Definition 1.2. For $n \geq 2$ and $K \geq 1$, define $p(n,K)$ to be the supremum of those values $p > n$ so that there exists a constant $C > 0$ so that for every K-quasiconformal map $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$, the estimate

$$\left(\fint_Q |Df|^p\right)^{1/p} \leq C\left(\fint_Q |Df|^n\right)^{1/n} \tag{1.1}$$

holds for every cube $Q \subset \Omega$ with $\operatorname{diam}(f(Q)) < \operatorname{dist}(f(Q), \partial\Omega')$

By [12, Theorem 1], $p(n,K) > n$ for each $n \geq 2$ and $K \geq 1$. The exact value of $p(n,K)$ remains conjectural for each $n \geq 3$, however, Astala [2] showed that

$$p(2,K) = \frac{2K}{K-1}. \tag{1.2}$$

See also [3, Theorem 13.2.3 and Corollary 13.2.4].

Returning to the results of [13], let E be a subset of Ω with $\dim_H(E) = \alpha \in (0,n)$. Here and henceforth we denote by $\dim_H(E)$ the Hausdorff dimension of a set E. Then

$$0 < \frac{(p(n,K)-n)\alpha}{p(n,K)-\alpha} \leq \dim_H f(E) \leq \frac{p(n,K)\alpha}{p(n,K)-n+\alpha} < n \tag{1.3}$$

for any K-quasiconformal map $f : \Omega \to \mathbb{R}^n$. In particular, quasiconformal maps in $\mathbb{R}^n$ preserve the dimension of sets of Hausdorff dimension 0 or n. The two-sided estimates in (1.3) are sometimes written in the 'symmetric' form

$$\left(1-\frac{n}{p(n,K)}\right)\left(\frac{1}{\dim_H E}-\frac{1}{n}\right)\le\frac{1}{\dim_H f(E)}-\frac{1}{n}\le\left(1-\frac{n}{p(n,K)}\right)^{-1}\left(\frac{1}{\dim_H E}-\frac{1}{n}\right), \tag{1.4}$$

illustrating the role of the local Hölder exponent $1-n/p$ for $W^{1,p}$ mappings in dimension n. In particular, when $n=2$ we have

$$\frac{1}{K}\left(\frac{1}{\dim_H E}-\frac{1}{2}\right)\le\frac{1}{\dim_H f(E)}-\frac{1}{2}\le K\left(\frac{1}{\dim_H E}-\frac{1}{2}\right), \tag{1.5}$$

for $E\subset\mathbb{R}^2$ and f a quasiconformal map of $\mathbb{R}^2$, as observed by Astala [2].

A panoply of metrically defined notions of dimension have been introduced to elucidate disparate features of sets and metric spaces. These include, for instance, box-counting (Minkowski) dimension and its countably stable regularization, packing dimension, as well as Assouad dimension. We denote by $\overline{\dim_B}(E)$ the upper box-counting dimension of a bounded set $E\subset\mathbb{R}^n$ and by $\dim_A(E)$ the Assouad dimension of an arbitrary set $E\subset\mathbb{R}^n$. We refer to section 2 for the full definitions of these and other notions of dimension, but for later purposes in this introduction we remind the reader that

$$\dim_A(E)=\inf\{s>0\,:\,E\text{ is }s\text{-homogeneous}\},$$

where a set E is said to be s-homogeneous (with s-homogeneity constant C) if the number $N(B(x,r)\cap E,r)$ of small sets of diameter at most r needed to cover a large ball $B(x,R)\cap E$ is bounded above by $C((R/r)^s)$, uniformly for $0<r\le R$ and $x\in E$. We recall that

$$\dim_H(E)\le\overline{\dim_B}(E)\le\dim_A(E)$$

for all bounded E, and

$$\dim_H(E)\le\dim_A(E)$$

for all E. Assouad dimension was introduced in connection with the existence (and non-existence) of bi-Lipschitz embeddings into Euclidean spaces. The past two decades have witnessed the increased role of Assouad dimension in quasisymmetric uniformization questions, with particular emphasis on quasisymmetric uniformization of metric 2-spheres. There is also substantial interest in Assouad dimension for its own sake, as a tool for the study of the metric geometry of Euclidean sets and sets in more general metric spaces. We refer the reader to [8] for a comprehensive study of Assouad dimension from the perspective of fractal geometry.

In [22], motivated by some (at that time unresolved) questions regarding dilatation-independent distortion of Hausdorff dimension by quasiconformal maps, the second author studied analogous questions for Assouad dimension. While the primary focus of [22] was on dilatation-independent results, the following dilatation-dependent analog for (1.3) was included. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a K-quasiconformal map ($n\ge2$) and let $E\subset\mathbb{R}^n$ satisfy $\dim_A(E)=\alpha\in(0,n)$. Then

$$0<\beta_-\le\dim_A f(E)\le\beta_+<n \tag{1.6}$$

where the constants $\beta_\pm$ depend only on n, K, α, and (in the case of β_+) an s-homogeneity constant for the set E for some exponent s, $\alpha<s<n$. The proof for this result was quite different from the classical proof by Gehring and Väisälä; the lower bound was derived using the fact that Euclidean quasiconformal mappings are power quasisymmetric, and the upper bound relied on the connection between Assouad dimension and porosity and the quasiconformal invariance of porosity. No explicit bounds for $\beta_\pm$ were given, and it was left as an open question whether the stated dependence of β_+ on the auxiliary homogeneity constant was necessary.

Our first main result (Theorem 1.3) addresses both of the above issues. We give precise estimates for the upper and lower bounds in (1.6) and we show that the upper bound can be chosen independent of any auxiliary homogeneity data.

Theorem 1.3. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal map between domains in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be a compact set satisfying $\dim_A(E) = \alpha$, $0 < \alpha < n$. Then*

$$\left(1 - \frac{n}{p(n,K)}\right)\left(\frac{1}{\dim_A E} - \frac{1}{n}\right) \leq \frac{1}{\dim_A f(E)} - \frac{1}{n} \leq \left(1 - \frac{n}{p(n,K)}\right)^{-1}\left(\frac{1}{\dim_A E} - \frac{1}{n}\right), \tag{1.7}$$

where $p(n,K) > n$ is as defined in (1.1). If $\Omega = \Omega' = \mathbb{R}^n$ then the conclusion holds for all subsets E with $\dim_A(E) = \alpha$ (not necessarily compact).

While the formulation of the upper and lower estimates in Theorem 1.3 is identical to that in the Gehring–Väisälä result (with Hausdorff dimension replaced by Assouad dimension), the proofs are quite different. In particular, the upper bound in (1.3) holds for any $W^{1,p}$ mapping (not necessarily quasiconformal, or even a homeomorphism), and the lower bound is obtained by applying the upper bound to the inverse mapping $g = f^{-1}$. (Note that g is again K-quasiconformal.) As previously mentioned, distortion of Hausdorff and box-counting (as well as packing) dimensions by Sobolev mappings was studied by Kaufman [16]. Our proof of Theorem 1.3 explicitly uses the fact that f is a quasiconformal homeomorphism, and not just its modulus of uniform continuity or membership in a suitable Sobolev space. Observe that Lipschitz mappings may increase Assouad dimension, see [18, Example A.6.2] or [8, Theorem 10.2.5] for examples.

Our second main theorem concerns the distortion of the Assouad spectrum by Euclidean quasiconformal maps. The Assouad spectrum, introduced by Fraser and Yu [11], is a one-parameter family of metrically defined dimensions which interpolates between the upper box-counting dimension and the (quasi-)Assouad dimension. Specifically, the Assouad spectrum of a set $E \subset \mathbb{R}^n$ is a collection of values

$$\{\dim_A^\theta(E) : 0 < \theta < 1\},$$

where $\dim_A^\theta(E)$ captures the growth rate of the covering number $N(B(x,R) \cap E, r)$ for scales $0 < r \leq R < 1$ related by $R = r^\theta$. The map $\theta \mapsto \dim_A^\theta(E)$ is continuous (even locally Lipschitz) when $0 < \theta < 1$, and

$$\dim_A^\theta(E) \to \overline{\dim_B}(E) \quad \text{as } \theta \to 0, \qquad \dim_A^\theta(E) \to \dim_{qA}(E) \quad \text{as } \theta \to 1\,.$$

Here $\dim_{qA}(E)$ denotes the *quasi-Assouad dimension* of E, a variant of Assouad dimension introduced by Lü and Xi [LX]. We always have $\dim_{qA}(E) \leq \dim_A(E)$, and equality holds in many situations (see [8, Section 3.3] for details).

In fact, we use a slightly modified version of the Assouad spectrum where the relationship $R = r^\theta$ between the two scales is relaxed to an inequality $R \geq r^\theta$. This modification leads to the notion of *upper Assouad spectrum*, denoted $\overline{\dim_A^\theta}(E)$ in the literature: see [10] or [8, Section 3.3.2] for more information. The key relationship between the two values (see Theorem 3.3.6 in [8]) is that

$$\overline{\dim_A^\theta}(E) = \sup_{0<\theta'<\theta} \dim_A^{\theta'}(E). \tag{1.8}$$

In this paper, we propose the term *regularized Assouad spectrum* in lieu of upper Assouad spectrum, and use the notation $\dim_{A,reg}^\theta(E)$ in place of $\overline{\dim_A^\theta}(E)$.

For $t > 0$ define

$$\theta(t) = \frac{1}{1+t}.$$

Note that the next two statements still hold formally when one of the denominators is equal to 0, with the convention that $1/0 = \infty$. For example, if $\dim_{A,reg}^{\theta(t/K)}(E) = 0$, then Theorem 1.4 implies that $\dim_{A,reg}^{\theta(t)}(f(E))$ needs to be equal to 0 as well.

Theorem 1.4. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal map as in Theorem 1.3, For $t > 0$ and a compact set E contained in Ω, we have*

(1.9)

$$\left(1 - \frac{n}{p(n,K)}\right)\left(\frac{1}{\dim_{A,reg}^{\theta(t/K)}(E)} - \frac{1}{n}\right) \le \frac{1}{\dim_{A,reg}^{\theta(t)}(f(E))} - \frac{1}{n} \le \left(1 - \frac{n}{p(n,K)}\right)^{-1}\left(\frac{1}{\dim_{A,reg}^{\theta(Kt)}(E)} - \frac{1}{n}\right).$$

Using the known sharp value for $p(2,K)$ we obtain the following corollary.

Corollary 1.5. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal map between domains in $\mathbb{C}$. For $t > 0$ and a compact set $E \subset \Omega$, we have*

$$\frac{1}{K}\left(\frac{1}{\dim_{A,reg}^{\theta(t/K)}(E)} - \frac{1}{2}\right) \le \frac{1}{\dim_{A,reg}^{\theta(t)}(f(E))} - \frac{1}{2} \le K\left(\frac{1}{\dim_{A,reg}^{\theta(Kt)}(E)} - \frac{1}{2}\right).$$

Note that the conclusion of Theorem 1.4 remains restricted to compact sets, even in the case when the quasiconformal mapping in question is globally defined. The reason for this distinction between Theorems 1.3 and 1.4 is that the Assouad dimension is Möbius invariant, which allows us to reduce to the compact case via inversion in a small sphere in the complementary region.

This paper is organized as follows. Section 2 reviews the precise definitions for, and basic properties of, the Assouad dimension and the (regularized) Assouad spectrum. Section 3 indicates how to derive the classification theorem for polynomial spirals, Theorem 1.1, from Corollary 1.5. In Section 4 we prove Theorems 1.3 and 1.4. Section 5 contains open questions and further remarks motivated by this study.

Acknowledgements. The second author acknowledges support from the National Science Foundation under grant DMS-1600650, 'Mappings and Measures in Sub-Riemannian and Metric Spaces' and from the Simons Foundation under grant #852888, 'Geometric mapping theory and geometric measure theory in sub-Riemannian and metric spaces'. In addition, this material is based upon work conducted while the second author is serving at the National Science Foundation. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

2. Background

2.1. **Quasiconformal mappings.** A homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$, $n \ge 2$, is said to be *K-quasiconformal*, $K \ge 1$, if f lies in the local Sobolev space $W^{1,n}_{loc}$ and the inequality

$$|Df|^n \le K \det Df \tag{2.1}$$

holds a.e. in Ω. Here Df denotes the (a.e. defined) differential matrix and $|\mathbf{A}| = \max\{|\mathbf{A}(\mathbf{v})| : |\mathbf{v}| = 1\}$ denotes the operator norm of a matrix $\mathbf{A}$. The smallest value $K \ge 1$ for which (2.1) holds a.e. in Ω is known as the *outer dilatation* of f and is denoted $K_O(f)$. Alternatively, set $\ell(\mathbf{A}) := \min\{|\mathbf{A}(\mathbf{v})| : |\mathbf{v}| = 1\}$ and replace (2.1) with the inequality

$$\det Df \le K \ell(Df)^n\,. \tag{2.2}$$

The smallest choice of K for which (2.2) holds a.e. is the *inner dilatation* of f and is denoted $K_I(f)$. These two dilatation measures are related by the mutual inequalities $K_O(f) \le K_I(f)^{n-1}$ and $K_I(f) \le K_O(f)^{n-1}$; thus $K_O(f) = K_I(f) =: K(f)$ when $n = 2$.

In this paper, we will not have any need to consider the inner dilatation, apart from the final remark in section 5. We thus omit the adjective 'outer' henceforth in describing the dilatation $K_O(f)$, and we write $K_f = K_O(f)$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the (closed) Euclidean ball with center x and radius r, and by $Q(x,r)$ the axes-parallel closed cube centered at x with side length $2r$. In other words, if $x = (x_1, \ldots, x_n)$, then $Q(x,r) = [x_1 - r, x_1 + r] \times \cdots \times [x_n - r, x_n + r]$. Alternatively, $Q(x,r)$ is the metric ball in the ℓ^∞ norm on $\mathbb{R}^n$, with center x and radius r.

We record the following elementary inclusions, valid for any $x \in \mathbb{R}^n$ and $r > 0$:

$$Q(x, \tfrac{1}{\sqrt{n}} r) \subset B(x,r) \subset Q(x,r). \tag{2.3}$$

For $\lambda > 0$ and a ball $B = B(x,r)$ (resp. cube $Q = Q(x,r)$), we denote by λB (resp. λQ) the set obtained by dilating with scale factor λ, i.e., $\lambda B = B(x, \lambda r)$ and $\lambda Q = Q(x, \lambda r)$.

We recall the notion of Whitney decomposition of a domain $\Omega \subsetneq \mathbb{R}^n$. For each $k \ge 1$, and for such a domain Ω, we can write Ω as an essentially disjoint union of closed cubes $\mathcal{W} = \{Q_i\}_{i \in I}$, where each cube Q_i satisfies

$$\frac{1}{4k} \operatorname{dist}(Q_i, \partial\Omega) \le \operatorname{diam}(Q_i) \le \frac{1}{k} \operatorname{dist}(Q_i, \partial\Omega). \tag{2.4}$$

See, e.g., [19].

The following property of quasiconformal mappings is popularly known as the 'egg yolk principle'. See, for instance, [14, Theorem 11.14].

Proposition 2.1. *Fix $n \ge 2$ and $K \ge 1$. Then there exists an increasing homeomorphism $\eta = \eta_{K,n}$ of $[0, \infty)$ so that for any K-quasiconformal homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$ and any cube $Q \subset \Omega$ with $\operatorname{diam}(Q) \le \operatorname{dist}(Q, \partial\Omega)$, the restriction $f|_Q$ is η-quasisymmetric. More precisely, if $x, y, z \in Q$, then*

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \eta\left(\frac{|x-y|}{|x-z|}\right).$$

The following corollary is standard, but for the benefit of the reader we provide a short proof.

Corollary 2.2. *For each $n \ge 2$, $K \ge 1$, and $c > 1$, there exists $k > 1$ so that if $f : \Omega \to \Omega'$ is a K-quasiconformal mapping between domains in $\mathbb{R}^n$ and $Q \subset \Omega$ is a cube with $\operatorname{diam}(Q) \le \frac{1}{k} \operatorname{dist}(Q, \partial\Omega)$, then there exists a cube $Q' \subset \Omega'$ so that $\operatorname{diam}(Q') \le \frac{1}{c} \operatorname{dist}(Q', \partial\Omega')$ and*

$$f(Q) \subset Q' \subset f(kQ). \tag{2.5}$$

Proof. We will prove the corresponding statement for balls instead of cubes; the analogous result for cubes follows immediately upon appeal to (2.3).

With η as in Proposition 2.1, choose $k \ge 1$ so that $\eta(1/k) < 1/c$. Let $B = B(x,R)$ and assume that $\operatorname{diam}(B) \le \frac{1}{k} \operatorname{dist}(B, \partial\Omega)$. Set $B' = B(f(x), R')$ where

$$R' := \max\{|f(x) - f(y)| \,:\, |x-y| = R\}\,. \tag{2.6}$$

We claim that $cB' \subset \Omega'$ and $f(B) \subset B' \subset f(kB)$. The inclusion $f(B) \subset B'$ is immediate from the definition of R. Fix $y, z \in \Omega$ so that $|x-y| = R$ and $|f(x) - f(y)| = R'$, while $|x-z| = kR$ and

$$|f(x) - f(z)| = \min\{|f(x) - f(w)| \,:\, |x-w| = kR\}\,. \tag{2.7}$$

Appealing to Proposition 2.1, we obtain

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \eta\left(\frac{|x-y|}{|x-z|}\right) = \eta(\frac{1}{k}) < \frac{1}{c}.$$

Since $kB \subset \Omega$ we conclude from (2.7) that $cR' = c|f(x)-f(y)| \le |f(x)-f(z)| < \mathrm{dist}(f(x), \partial\Omega')$. Hence $cB' \subset \Omega'$. Moreover, $B' = B(f(x), |f(x)-f(y)|) \subset B(f(x), |f(x)-f(z)|) \subset f(kB)$. □

2.2. **Assouad dimension and the Assouad spectrum.** Let F be a bounded subset of $\mathbb{R}^n$. For $r > 0$, denote by $N(F, r)$ the smallest number of sets of diameter at most r needed to cover F. The *(upper) box-counting dimension* of F is

$$\overline{\dim_B}(F) = \limsup_{r\to 0} \frac{\log N(F,r)}{\log(1/r)}.$$

We drop the adjective 'upper' throughout this paper as we will make no reference to the lower box-counting dimension. An equivalent formulation is

$$\overline{\dim_B}(F) = \inf\{\alpha > 0 \,:\, \exists\, C > 0 \text{ s.t. } N(F,r) \le Cr^{-\alpha} \text{ for all } 0 < r \le \mathrm{diam}(F)\}.$$

Since any bounded set can be covered by finitely many balls of radius 1, another equivalent formulation is

$$\overline{\dim_B}(F) = \inf\left\{\alpha > 0 \,:\, \begin{array}{c} \exists\, C > 0 \text{ s.t. } N(B(x,1)\cap F, r) \le C(1/r)^\alpha \\ \text{for all } 0 < r \le 1 \text{ and all } x \in F \end{array}\right\},$$

which makes sense also for unbounded F. The latter formulation also shows the connection between box-counting and Assouad dimension. For an arbitrary (not necessarily bounded) set $F \subset \mathbb{R}^n$, the *Assouad dimension* of F is

$$\dim_A(F) = \inf\left\{\alpha > 0 \,:\, \begin{array}{c} \exists\, C > 0 \text{ s.t. } N(B(x,R)\cap F, r) \le C(R/r)^\alpha \\ \text{for all } 0 < r \le R \text{ and all } x \in F \end{array}\right\}.$$

Assouad dimension first appeared (under a different name) in a 1983 paper of Assouad on metric embedding problems [1]. Luukkainen [18] gave a detailed presentation of the state-of-the-art in the theory of Assouad dimension as of the late 1990s. We also recommend the recent book by Fraser [8]. It is perhaps important to specify that in the aforementioned book the author defines $N(F, r)$ to be the smallest number of open sets of diameter at most r needed to cover F, and uses this covering number in the definition of all dimensions and spectra. It is an elementary exercise to prove that this yields the same exact notions of dimension and spectrum.

It is clear from the definitions that $\overline{\dim_B}(F) \le \dim_A(F)$ for bounded sets F. Both box-counting dimension and Assouad dimension are monotonic and finitely stable, but neither quantity is countably stable. Indeed, both notions of dimension are invariant under passing to the closure, thus the dimension of any dense subset of $\mathbb{R}^n$ is equal to n. Such behavior is in sharp contrast to that exhibited by Hausdorff dimension, which is zero for any countable set. An illustrative example is $F = \{0\} \cup \{m^{-1} \,:\, m \in \mathbb{N}\} \subset \mathbb{R}$, for which $\dim_H(F) = 0$, $\overline{\dim_B}(F) = \frac{1}{2}$, and $\dim_A(F) = 1$.

Fraser and Yu [11] introduced the Assouad spectrum as an interpolation between upper box-counting dimension and Assouad dimension. As discussed in the introduction, we employ a slight modification of the definition, as discussed in [10] and [8, Section 3.3.2]. For $0 < \theta < 1$ and a set $F \subset \mathbb{R}^n$, define

$$\dim^\theta_{A,reg}(F) = \inf\left\{\alpha > 0 \,:\, \begin{array}{c} \exists\, C > 0 \text{ s.t. } N(B(x,R)\cap F, r) \le C(R/r)^\alpha \\ \text{for all } 0 < r \le R^{1/\theta} < R < 1 \text{ and all } x \in F \end{array}\right\}. \tag{2.8}$$

Thus $\dim^\theta_A(F)$ is defined by the same process as $\dim_A(F)$, but with the restriction that the two scales r and R involved in the definition of the latter are related by the inequality $R \ge r^\theta$. The set of values $\{\dim_{A,reg}^{\theta}(F) : 0 < \theta < 1\}$ is called the *regularized Assouad spectrum* of F. We collect various properties of the regularized Assouad spectrum in the following proposition. Proofs of these results for the (original) Assouad spectrum can be found in [8, Sections 3.3 and 3.4], and the corresponding results for the regularized Assouad spectrum follow easily from the regularization identity (1.8).

Proposition 2.3. *The regularized Assouad spectrum enjoys the following features.*

(1) *For fixed* θ*, the set function* $F \mapsto \dim_{A,reg}^{\theta}(F)$ *is*
  (a) *monotonic, i.e.,* $E \subset F$ *implies* $\dim_{A,reg}^{\theta}(E) \le \dim_{A,reg}^{\theta}(F)$*,*
  (b) *finitely stable, i.e.,* $\dim_{A,reg}^{\theta}(E \cup F) = \max\{\dim_{A,reg}^{\theta}(E), \dim_{A,reg}^{\theta}(F)\}$*,*
  (c) *invariant under taking closures, and*
  (d) *invariant under bi-Lipschitz transformation.*

(2) *For a fixed set* F*, the function* $\theta \mapsto \dim_{A,reg}^{\theta}(F)$ *is nondecreasing, continuous on* $(0,1)$*, and Lipschitz on compact subsets of* $(0,1)$*.*

(3) *For fixed* F*,* $\lim_{\theta\to 0^+} \dim_{A,reg}^{\theta}(F)$ *exists and equals* $\overline{\dim_B}(F)$*. Moreover,* $\lim_{\theta\to 1^-} \dim_{A,reg}^{\theta}(F)$ *coincides with the so-called quasi-Assouad dimension of* F*, denoted* $\dim_{qA}(F)$*. For any* θ*, one has* $\dim_{A,reg}^{\theta}(F) \le \dim_{qA}(F) \le \dim_A(F)$*. If* $\dim_{A,reg}^{\theta}(F) = \dim_{qA}(F)$ *for some* $0 < \theta < 1$*, then* $\dim_{A,reg}^{\theta'}(F) = \dim_{qA}(F)$ *for all* $\theta \le \theta' < 1$*.*

(4) *Set*

$$\rho = \rho(F) := \inf\{\theta \in (0,1) : \dim_{A,reg}^{\theta}(F) = \dim_{qA}(F)\}, \tag{2.9}$$

or $\rho = 1$ *if no such* θ *exists. Then*

$$\dim_{A,reg}^{\theta}(F) \ge \left(\tfrac{1-\rho}{1-\theta}\right) \dim_{qA}(F)$$

for all $0 < \theta < \rho$*.*

(5) *For* F *bounded and* $0 < \theta < 1$*,*

$$\dim_{A,reg}^{\theta}(F) \le \frac{\overline{\dim_B}(F)}{1-\theta}\,. \tag{2.10}$$

Hence the phase transition $\rho(F)$ *defined in* (2.9) *satisfies*

$$\rho(F) \ge 1 - \frac{\overline{\dim_B}(F)}{\dim_{qA}(F)}\,. \tag{2.11}$$

Note that the Assouad spectrum function $\theta \mapsto \dim_A^{\theta}(F)$ is **not** always monotonically increasing. For an example, see [8, Section 3.4.4].

In the proof of our main theorems in section 4, it will be convenient to take advantage of several alternative descriptions for the regularized Assouad spectrum. We collect several such descriptions in the following proposition. In part (ii) of the proposition we make use of the standard dyadic decomposition. Specifically, for a fixed axes-parallel cube $Q_0 \subset \mathbb{R}^n$, we subdivide Q_0 into 2^n essentially disjoint subcubes, each with side length equal to half of the side length of Q_0, and then we subdivide each of those cubes in the same fashion, and so on. Let $\mathcal{W}(Q_0)$ denote the collection of all such cubes obtained at any level of the construction, and let $\mathcal{W}_m(Q_0)$ denote the collection of all cubes obtained after m steps.

Proposition 2.4. *Let $F \subset \mathbb{R}^n$ be a bounded subset and let $0 < \theta < 1$.*

(i) *Fix a value $C_1 \geq 1$. Then the regularized Assouad spectrum $\dim_{A,reg}^{\theta}(F)$ is equal to the infimum of all $\alpha > 0$ for which there exists $C > 0$ so that*
$$N(F \cap B(x,R), r) \leq C(\tfrac{R}{r})^{\alpha}$$
for all $x \in F$ and all $0 < r/C_1 \leq R^{1/\theta} < R < 1$.

(ii) *Fix an axes-parallel cube Q_0 with $F \subset Q_0$. For $m \in \mathbb{N}$, let $N_d(F,m)$ denote the number of dyadic cubes in $\mathcal{W}_m(Q_0)$ needed to cover F. Then $\dim_{A,reg}^{\theta}(F)$ is equal to the infimum of all $\alpha > 0$ for which there exists $C > 0$ so that*
$$N_d(B(x,R) \cap F, m) \leq C 2^{m\alpha}$$
for all $x \in F$ and all $0 < 2^{-m}R \leq R^{1/\theta} < R < 1$.

Proof. (i) Denote by
$$A_\theta := \left\{ \alpha > 0 \ : \ \begin{array}{c} \exists C > 0 \text{ s.t. } N(B(x,R) \cap F, r) \leq C(R/r)^{\alpha} \\ \text{for all } 0 < r \leq R^{1/\theta} < R < 1 \text{ and all } x \in F \end{array} \right\}$$
and by
$$B_\theta := \left\{ \alpha > 0 \ : \ \begin{array}{c} \exists C > 0 \text{ s.t. } N(B(x,R) \cap F, r) \leq C(R/r)^{\alpha} \\ \text{for all } 0 < r/C_1 \leq R^{1/\theta} < R < 1 \text{ and all } x \in F \end{array} \right\}.$$
We know by the definition of the regularized Assouad spectrum that $\dim_{A,reg}^{\theta}(F) = \inf A_\theta$, so it is enough to show that $A_\theta = B_\theta$.

Let $\alpha \in A_\theta$, $x \in F$ and $r, R > 0$ with $0 < r/C_1 \leq R^{1/\theta} < R < 1$. Then
$$N(B(x,R) \cap F, r) \leq N(B(x,R) \cap F, r/C_1) \leq C \left(\frac{R}{r/C_1} \right)^{\alpha} = C C_1^{\alpha} (R/r)^{\alpha}$$
which implies that $\alpha \in B_\theta$. Hence, $A_\theta \subset B_\theta$.

Since $C_1 \geq 1$, for any $0 < r \leq R^{1/\theta} < R < 1$ we have that $r/C_1 < r$, which makes the inclusion $B_\theta \subset A_\theta$ trivial. Hence $A_\theta = B_\theta$.

(ii) Denote by
$$D_\theta := \left\{ \alpha > 0 \ : \ \begin{array}{c} \exists C > 0 \text{ s.t. } N_d(B(x,R) \cap F, m) \leq C 2^{m\alpha} \\ \text{for all } m \in \mathbb{N}, R > 0 \text{ with } 0 < 2^{-m}R \leq R^{1/\theta} < R < 1 \text{ and all } x \in F \end{array} \right\}.$$
Similarly, it is enough to show that $A_\theta = D_\theta$. For this we will need the inequalities
$$N(F, 2^{-m}R\sqrt{n}) \leq N_d(F,m) \leq 3^n N(F, 2^{-m}R). \tag{2.12}$$
Since cubes of side length $2^{-m}R$ are sets of diameter at most $2^{-m}R\sqrt{n}$, the left inequality is trivial. For the right inequality it is enough to notice that every set of diameter at most $2^{-m}R$ cannot possibly intersect more than 3^n axes-paralleled cubes of side length $2^{-m}R$.

Let $\alpha \in A_\theta$, $x \in F$ and $m \in \mathbb{N}, R > 0$ with $0 < 2^{-m}R \leq R^{1/\theta} < R < 1$. Then
$$N(B(x,R) \cap F, 2^{-m}R) \leq C \left(\frac{R}{2^{-m}R} \right)^{\alpha}$$
which by (2.12) implies that
$$N_d(B(x,R) \cap F, m) \leq C 3^n 2^{m\alpha}$$
and, thus, $\alpha \in D_\theta$. Hence $A_\theta \subset D_\theta$.

Let $\alpha \in D_\theta$, $x \in F$ and $r, R > 0$ with $0 < r \le R^{1/\theta} < R < 1$. Let $m \in \mathbb{N}$ be the smallest number for which $2^{-m} R\sqrt{n} \le r \le 2^{-m+1} R\sqrt{n}$. Then

$$N(B(x,R) \cap F, r) \le N(B(x,R) \cap F, 2^{-m} R\sqrt{n})$$

which by (2.12) implies that

$$N(B(x,R) \cap F, r) \le N_d(B(x,R) \cap F, m) \le C 2^{m\alpha} = C \left(\frac{R}{2^{-m+1} R\sqrt{n}} \right)^\alpha (2\sqrt{n})^\alpha.$$

Hence, $N(B(x,R) \cap F, r) \le C(2\sqrt{n})^\alpha (R/r)^\alpha$ which means that $\alpha \in A_\theta$. As a result, $D_\theta \subset A_\theta$ and the proof is complete. □

3. Quasiconformal classification of polynomial spirals

Recall from the introduction that S_a denotes the polynomial spiral

$$\{x^{-a} e^{\mathbf{i}x} \in \mathbb{C} : x > 0\}.$$

Theorem 1.1 asserts that S_a can be mapped to S_b by a quasiconformal map f of $\mathbb{C}$ if and only if $K_f \ge \frac{a}{b}$. We will prove this result as an application of Corollary 1.5. Let us first observe why other notions of dimension are insufficient for this purpose. Clearly, since $\dim_H(S_a) = 1$ for every $a > 0$, Astala's result (1.5) cannot be used to quasiconformally distinguish any pair of polynomial spirals.

For any $W^{1,p}$ mapping f (not necessarily quasiconformal) of domains in $\mathbb{R}^n$ and for any set E with $\dim_H E = \alpha \in (0, n)$, the estimate

$$\dim_H f(E) \le \frac{p\alpha}{p - n + \alpha}$$

holds true. This is an easy covering argument using the Morrey–Sobolev inequality and the characterization of Hausdorff dimension using coverings by dyadic cubes. Kaufman [16] proved the analogous statement for the box-counting dimension $\dim_B$. Using again the sharp exponent of Sobolev integrability for planar quasiconformal maps one concludes that if $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and $E \subset \mathbb{C}$ is bounded, then

$$\frac{1}{K} \left(\frac{1}{\dim_B(E)} - \frac{1}{2} \right) \le \frac{1}{\dim_B(f(E))} - \frac{1}{2} \le K \left(\frac{1}{\dim_B(E)} - \frac{1}{2} \right).$$

One may try to use this estimate to answer the question about quasiconformal equivalence of polynomial spirals. Fraser [9] computed the box-counting dimensions of such spirals: for $a > 0$,

$$\dim_B(S_a) = \max \left\{ \frac{2}{1+a}, 1 \right\}.$$

It follows that $\frac{1}{\dim_B(S_a)} - \frac{1}{2} = \min\{\frac{a}{2}, \frac{1}{2}\}$. Thus if $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal with $f(S_a) = S_b$, $a > b > 0$, then

$$\frac{1}{\dim_B(S_a)} - \frac{1}{2} \le K \left(\frac{1}{\dim_B(S_b)} - \frac{1}{2} \right)$$

and so

$$K \ge \frac{\min\{a, 1\}}{\min\{b, 1\}}.$$

This proves Theorem 1.1, but only in the case $0 < b < a \le 1$. If $a > 1$ then $\dim_B(S_a) = 1$ and the preceding lower bound for K does not match the upper bound given by the radial stretch map. Moreover, if $b > 1$ then $\dim_B(S_a) = \dim_B(S_b) = 1$ and we obtain no nontrivial information about the dilatation K of the mapping.

To resolve the remaining case, we consider the Assouad spectrum. Fraser [9] also computed these quantities for the polynomial spirals. For $a > 0$ and $0 < \theta < 1$,

$$\dim_A^\theta(S_a) = \begin{cases} \min\left\{\frac{2}{(1+a)(1-\theta)}, 2\right\}, & \text{if } 0 < a \le 1, \\ \min\left\{1 + \frac{\theta}{a(1-\theta)}, 2\right\}, & \text{if } a \ge 1. \end{cases}$$

Note that since these expressions are monotonically increasing as functions of θ, they also agree with the regularized Assouad spectrum $\dim_{A,reg}^\theta(S_a)$. The critical parameter $\rho(S_a)$, as in (2.9), is equal to $a/(1+a)$. Note that equality in (2.11) holds only in the case $0 < a \le 1$. We have $\dim_A^\theta(S_a) < 2$ if $\theta < \rho(S_a)$ and $\dim_A^\theta(S_a) = 2$ if $\theta \ge \rho(S_a)$. It follows that the quasi-Assouad dimension (and hence also the Assouad dimension) of S_a is equal to 2 for all $a > 0$. Hence Assouad dimension cannot be used to distinguish any two polynomial spirals up to quasiconformal equivalence.

Proof of Theorem 1.1. As discussed above, it suffices to show that if $K < \frac{a}{b}$, $a > b > 0$, then there does not exist a K-quasiconformal map f of $\mathbb{C}$ with $f(S_a) = f_b$.

Suppose that such a map exists. Set $t = 1/b$. Then $\theta(t) = 1/(1+t) = b/(1+b)$ so $\dim_A^{\theta(t)}(S_b) = 2$. On the other hand, $\theta(t/K) = K/(K+t) < a/(1+a)$ so $\dim_A^{\theta(t/K)}(S_a) < 2$. This leads to a contradiction with the conclusion of Corollary 1.5. Note that

$$\frac{K}{K+\frac{1}{b}} < \frac{a}{1+a} \quad \Leftrightarrow \quad K < \frac{a}{b}.$$

□

Remark 3.1. In the preceding proof, distinguishing different polynomial spirals S_a up to K-quasiconformal equivalence relies on an understanding of the behavior of the Assouad spectrum $\dim_A^\theta(S_a)$ as a function of θ, and more precisely, determining the threshold parameter $\rho(S_a)$ where the value of $\dim_A^\theta(S_a)$ reaches the dimension of the ambient space $\mathbb{R}^2$. The precise form of the upper bound for the Assouad spectrum of $f(E)$ in terms of that of E does not play a role. It would be interesting to identify a situation in which the precise bounds in (1.9) feature in the quasiconformal classification problem.

4. Quasiconformal distortion of Assouad dimension and the Assouad spectrum

In this section, we prove Theorems 1.3 and 1.4. In subsection 4.1 we prove Theorem 1.4 on the quasiconformal distortion of the Assouad spectrum. The proof of Theorem 1.3, on the distortion of Assouad dimension, proceeds along similar lines. We present this proof in subsection 4.2 in an abbreviated form, focusing on those aspects of the argument in subsection 4.1 which must be modified.

4.1. **Proof of Theorem 1.4.**

Recall that our goal here is to prove the dimension distortion estimates (1.9) for any K-quasiconformal map $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$ and for compact sets $E \subset \Omega$. We begin by performing some preliminary reductions.

If $\Omega = \mathbb{R}^n$ then also $\Omega' = \mathbb{R}^n$. Using the bi-Lipschitz invariance of the Assouad spectrum (Proposition 2.3(1)(d)), and pre- and post-composing with suitable homotheties, we may assume without loss of generality that $E \subset Q_0$ and $f(E) \subset Q_0$, where

$$Q_0 = Q(0, \tfrac{1}{2}) = [-\frac{1}{2}, \frac{1}{2}]^n.$$

If $\Omega \subsetneq \mathbb{R}^n$ then also $\Omega' \subsetneq \mathbb{R}^n$. In this case we consider a suitable Whitney decomposition of Ω. Let η be a local quasisymmetric distortion function as in Proposition 2.1 and let $c =$

$\eta(1)$. Choose $k \geq 1$ as in Corollary 2.2 and select a Whitney decomposition $\mathcal{W} = \{Q_i\}_{i\in I}$ for Ω satisfying (2.4) for each $Q \in \mathcal{W}$. Since E is compact, it has nonempty intersection with only finitely many cubes in the Whitney decomposition of Ω. In view of the finite stability of the Assouad spectrum (Proposition 2.3(1)(b)) we may without loss of generality assume that E is contained in one such cube $Q \in \mathcal{W}$. A further appeal to Corollary 2.2 yields a cube Q' so that $f(Q) \subset Q'$ and $\operatorname{diam} f(Q) \leq \operatorname{diam} Q' \leq \frac{1}{c}\operatorname{dist}(Q', \partial\Omega') \leq \frac{1}{c}\operatorname{dist}(f(Q), \partial\Omega')$. Using again the bi-Lipschitz invariance of the Assouad spectrum, we may assume without loss of generality that $Q = Q' = Q_0$.

In conclusion, and regardless of which of the above two cases holds, we may assume with no loss of generality that both E and $f(E)$ are contained in Q_0, that $f(Q_0) \subset Q_0$, and that

$$\operatorname{diam} Q_0 \leq \delta \min\{\operatorname{dist}(Q_0, \partial\Omega), \operatorname{dist}(Q_0, \partial\Omega')\}, \tag{4.1}$$

where we set $\delta = \min\{\frac{1}{k}, \frac{1}{c}\}$ and we interpret the right hand side as $+\infty$ if $\Omega = \Omega' = \mathbb{R}^n$.

We now begin the proof in earnest. It suffices to prove one of the two inequalities in (1.9), as the other inequality follows by considering the inverse map f^{-1}. We prove the left hand inequality, which we rewrite in the form

$$\dim_{A,reg}^{\theta(t)}(f(E)) \leq \beta_0 := \frac{p(n,K)\alpha_0}{p(n,K) - n + \alpha_0}, \qquad \alpha_0 = \dim_{A,reg}^{\theta(t/K)}(E).$$

This follows if we prove

$$\dim_{A,reg}^{\theta(t)}(f(E)) \leq \beta := \frac{p\alpha}{p - n + \alpha}$$

for all $p < p(n,K)$ and $\alpha > \alpha_0$.

Fix p and α as above, and let $\beta = \frac{p\alpha}{p-n+\alpha}$. Let $y \in f(E)$ and $0 < R' \leq 1$. We will find a constant $C_1' > 0$ and we will cover $f(E) \cap B(y, R')$ by sets of diameter at most r_m' for all m so that $r_m' \leq C_1'(R')^{1/\theta(t)}$, where (r_m') is a decreasing sequence with $r_m' \searrow 0$ and $\frac{r_{m+1}'}{r_m'} \geq c$ for some $c > 0$ independent of m. This covering leads to an estimate for $N(B(y,R') \cap f(E), r_m')$ from above by $C(R'/r_m')^\beta$ for some fixed constant C; in view of Proposition 2.4(i) we conclude that $\dim_{A,reg}^{\theta(t)}(f(E)) \leq \beta$, and letting $\alpha \searrow \alpha_0$ and $p \nearrow p(K)$ finishes the proof.

For the given choice of y and R', let $B = B(y, R')$. We apply Corollary 2.2 to the inverse map $g = f^{-1}$ and choose a ball $B(x, R)$ with $x = g(y)$. If we cover $B(x,R) \cap E$ with sets from a covering $\mathcal{U}$, then the set $B(y,R') \cap f(E) \subset f(B(x,R)) \cap f(E)$ will be covered by the images of the elements in $\mathcal{U}$.

We consider cubes obtained via dyadic decomposition of $Q(x,R)$. The side lengths of such cubes assume values $2^{-m}R$ for values $m \geq -1$. Let $m_0 = m_0(R)$ be the unique integer so that

$$2^{-m_0}R \leq R^{1/\theta(t/K)} = R^{1+t/K} \leq 2^{-m_0+1}R,$$

and let

$$\begin{aligned} r_m &:= 2^{-m}R, \\ r_m' &:= 2^{-m\alpha/\beta}R' \end{aligned}$$

for $m \geq m_0$. Since $\alpha > \alpha_0$, for any such choice of m we need at most

$$C\left(\frac{R}{r_m}\right)^\alpha = C2^{m\alpha} \tag{4.2}$$

dyadic cubes of side length $2^{-m}R$ to cover $E \cap B(x,R)$, see Proposition 2.4(ii).

Recalling the choice of R in (2.6) and using standard local Hölder continuity estimates for f and f^{-1} (see, for instance, [15, Theorem 7.7.1]), we conclude that

$$R \geq c(R')^K$$

for some $c > 0$. It follows that there exists C_1' depending only on c, t, and K, so that if we denote by $m_0' = m_0'(R')$ the unique integer so that

$$r'_{m_0} = 2^{-m_0'\alpha/\beta} R' \leq C_1'(R')^{1/\theta(t)} = C_1'(R')^{1+t} \leq 2^{(-m_0'+1)\alpha/\beta} R' = r'_{m_0-1}, \tag{4.3}$$

then

$$m_0' \geq m_0.$$

To see this, note that since $\alpha \leq \beta$ we have

$$2^{(-m_0+1)\alpha/\beta} R' \geq 2^{-m_0+1} R' \geq R^{t/K} R' \geq c^{t/K} (R')^{1+t}$$

so (4.3) holds with $C_1' = c^{t/K}$, which means that we have $r'_m \leq C_1'(R')^{1/\theta(t)}$ for all $m \geq m_0'(R')$.

Fix an integer $m \geq m_0'$. Following the terminology in [16], we call a dyadic cube Q **minor** if $\operatorname{diam} f(Q) \leq r'_m$ and **major** otherwise. Since f is uniformly continuous, we can subdivide any dyadic cube Q into dyadic minor subcubes of varying sizes, where all of the dyadic minor subcubes in question have the property that their dyadic parent is major. We will estimate the number of dyadic minor cubes which may arise in this fashion.

Lemma 4.1. *For any cube $Q(x, R) \subset Q_0$, the total number of major cubes of side length at most $2^{-m}R$ contained in $Q(x, R)$ is bounded above by $C2^{m\alpha}$, where the constant C depends only on K and n.*

Proof. For fixed $j \geq 0$, let $M(j)$ be the number of major cubes in $Q(x, R)$ of side length $2^{-j}R$. Denote by Q_i^j a typical such cube. The Morrey–Sobolev inequality on Q_j^i takes the form

$$\operatorname{diam} f(Q_i^j) \leq C_2(\operatorname{diam}(Q_i^j))^{1-n/p} \left(\int_{Q_i^j} |Df|^p \right)^{1/p}.$$

Since Q_i^j is major, this implies that

$$2^{-\frac{m\alpha p}{\beta}} (R')^p \leq C_3\, 2^{-j(p-n)} R^{p-n} \int_{Q_i^j} |Df|^p$$

from which it follows that

$$\begin{aligned} 2^{-\frac{m\alpha p}{\beta}} (R')^p M(j) &\leq C_3\, 2^{-j(p-n)} R^{p-n} \int_{\bigcup_i Q_i^j} |Df|^p \\ &\leq C_3\, 2^{-j(p-n)} R^{p-n} \int_{Q(x,R)} |Df|^p . \end{aligned}$$

Summing over all $j \geq m$ yields

$$2^{-\frac{m\alpha p}{\beta}} (R')^p \sum_{j=m}^{\infty} M(j) \leq C_4\, 2^{-m(p-n)} R^{p-n} \int_{Q(x,R)} |Df|^p.$$

Hence

$$\sum_{j=m}^{\infty} M(j) \leq C_4\, 2^{\frac{m\alpha p}{\beta} - m(p-n)} (R')^{-p} R^{p-n} \int_{Q(x,R)} |Df|^p . \tag{4.4}$$

Since any such cube Q is contained in Q_0, (4.1) implies that $\operatorname{diam} f(Q) \le \operatorname{dist}(f(Q), \partial\Omega')$; since $p < p(n,K)$ we conclude that the reverse Hölder inequality (1.1) is satisfied. Thus

$$\frac{1}{|Q(x,R)|^{1/p}} \left(\int_{Q(x,R)} |Df|^p \right)^{1/p} \le \frac{1}{|Q(x,R)|^{1/n}} \left(\int_{Q(x,R)} |Df|^n \right)^{1/n}$$

and using (2.5) we can bound the integral on the right hand side of (4.4) (up to a global constant) by

$$\frac{R^n}{R^p} |f(Q(x,R))|^{p/n} \le R^{n-p} |Q(y,R')|^{p/n} \le C(n,p) R^{n-p} (R')^p \,.$$

Hence, by the definition of β, we obtain

$$\sum_{j=m}^{\infty} M(j) \le C_5 \, 2^{m\alpha} \,.$$

This completes the proof of the lemma. □

We now count the number of minor cubes which may arise under successive dyadic decomposition of $Q(x,R)$ as described above. Any such minor cube arises as a dyadic child of a major cube, and there are 2^n such children. Hence, the number of such minor cubes which we will obtain is at most

$$\sum_{j=m}^{\infty} 2^n M(j) \le C_5 \, 2^{m\alpha+n} \le C_6 2^{m\alpha} \,. \tag{4.5}$$

The subcollection of these minor cubes consisting of those cubes whose image under f meets $f(E)$ forms a suitable cover of $f(E)$ by sets of diameter at most r'_m. By previous comments, the cardinality of this set is at most

$$C_6 2^{m\alpha} = C_6 \left(\frac{R'}{r'_m} \right)^{\beta} ,$$

where we recall that $\beta = p\alpha/(p-n+\alpha)$. It follows that

$$N(B(y,R') \cap f(E), r'_m) \le C_6 \left(\frac{R'}{r'_m} \right)^{\beta}$$

which implies

$$\dim_{A,reg}^{\theta(t)}(f(E)) \le \beta = \frac{p\alpha}{p-n+\alpha}$$

and the conclusion follows by letting $\alpha \searrow \dim_{A,reg}^{\theta(t/K)}(E)$ and $p \nearrow p(K)$.

This concludes the proof of Theorem 1.4.

4.2. **Proof of Theorem 1.3.** We first observe that similar estimates hold for quasiconformal distortion of the quasi-Assouad dimension $\dim_{qA}(E)$. Specifically, if $f : \Omega \to \Omega'$ is a K-quasiconformal mapping between domains in $\mathbb{R}^n$, and E is a compact subset of Ω, then

$$\left(1 - \frac{n}{p(n,K)}\right) \left(\frac{1}{\dim_{qA}(E)} - \frac{1}{n} \right) \le \frac{1}{\dim_{qA}(f(E))} - \frac{1}{n} \le \left(1 - \frac{n}{p(n,K)}\right)^{-1} \left(\frac{1}{\dim_{qA}(E)} - \frac{1}{n} \right) .$$

This is an immediate consequence of Theorem 1.4, obtaining by letting $\theta \nearrow 1$ and using the continuity of $\theta \mapsto \dim_{A,reg}^{\theta}(E)$ as $\theta \to 1^-$.

The quasi-Assouad dimension $\dim_{qA}(E)$ and the Assouad dimension $\dim_A(E)$ do not agree in general, so Theorem 1.3 requires an additional argument. Nevertheless, the proof of Theorem 1.3 is substantially similar to that of Theorem 1.4 given in the previous subsection.

Proof of Theorem 1.3. As in the proof of Theorem 1.4, we begin with a series of reductions. Suppose that $\Omega = \Omega' = \mathbb{R}^n$ and that E is unbounded. Since the Assouad spectrum of a set is unchanged upon passing to the closure, we may assume without loss of generality that E is closed. If $E = \mathbb{R}^n$ then also $f(E) = \mathbb{R}^n$ and the result is trivial. If $E \subsetneq \mathbb{R}^n$, then choose an open ball $B(x_0, \delta)$ in the complement of E. Conformal inversion in the boundary of this ball preserves the Assouad dimension of sets, and maps E to a compact set. Moreover, precomposition by a conformal map does not alter the dilatation of the original quasiconformal map f. Thus it suffices to assume that E is compact.

We now perform additional reductions as in the beginning of the proof of Theorem 1.4, and assume without loss of generality that both E and $f(E)$ are contained in Q_0 and that (4.1) holds.

Similar to the proof of Theorem 1.4, it suffices to prove one of the two inequalities, as the other inequality follows by considering the inverse map f^{-1}. We prove the left hand inequality, which we rewrite in the form

$$\dim_A(f(E)) \le \beta_0 := \frac{p(n,K)\alpha_0}{p(n,K) - n + \alpha_0}, \qquad \alpha_0 = \dim_A(E).$$

This follows if we prove

$$\dim_A(f(E)) \le \beta := \frac{p\alpha}{p - n + \alpha}$$

for all $p < p(n,K)$ and $\alpha > \alpha_0$.

Fix such p and α, and let $y \in f(E)$ and $R' > 0$. As before, for the ball $B(y, R')$ we select a ball $B(x, R)$, $x = f^{-1}(y)$ as described in Corollary 2.2. Again, we will cover $f(E) \cap B(y, R')$ by sets of diameter at most $r'_m = 2^{-m\alpha/\beta} R'$ for all $m \ge 0$; this suffices for the desired conclusion. Note that here we allow all nonnegative m in the set of scales, and do not impose an R-dependent lower bound on the allowable scales. For $m \ge 0$ we define $r_m := 2^{-m} R$ as before. Since $\alpha > \alpha_0$, for each such m we need at most

$$C \left(\frac{R}{r_m} \right)^{\alpha} = C 2^{m\alpha} \tag{4.6}$$

dyadic cubes of side length $2^{-m} R$ to cover $E \cap B(x, R)$. The remainder of the proof follows by Lemma 4.1 in the same fashion as the proof of Theorem 1.4. □

Fraser and Yu [11] (see also Lemma 3.4.13 in [8]) studied the distortion of the Assouad spectrum by bi-Hölder homeomorphisms. Since quasiconformal maps are locally bi-Hölder, it is instructive to consider the relationship between Theorem 1.4 and the results of [11].

Recall that a homeomorphism $f : X \to Y$ between metric spaces is said to be (α, β)-*bi-Hölder*, for $0 < \alpha \le 1 \le \beta < \infty$, if there exists a constant $C > 0$ so that

$$C^{-1} d(x, x')^{\beta} \le d(f(x), f(x')) \le C d(x, x')^{\alpha} \qquad \forall\, x, x' \in X.$$

Since $\alpha \le \beta$ it is clear that X must be a bounded space in order for a bi-Hölder homeomorphism from X to exist. According to [8, Lemma 3.4.13], if $X \subset \mathbb{R}^n$ is bounded, $f : X \to \mathbb{R}^n$ is a (α, β)-bi-Hölder homeomorphism, and $0 < \theta < \alpha/\beta$, then

$$\frac{1 - \beta\theta/\alpha}{\beta(1-\theta)} \dim_{A,reg}^{\beta\theta/\alpha}(X) \le \dim_{A,reg}^{\theta}(f(X)) \le \frac{1 - \alpha\theta/\beta}{\alpha(1-\theta)} \dim_{A,reg}^{\alpha\theta/\beta}(X). \tag{4.7}$$

(The result is stated in [8] for the (unregularized) Assouad spectrum $\dim_A^{\theta}$, but easily transfers to the regularized version.)

Every K-quasiconformal map in $\mathbb{C}$ is locally $(\frac{1}{K}, K)$-bi-Hölder continuous, see e.g. [3, Corollary 3.10.3]. It follows from (4.7) that if $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, $E \subset \mathbb{C}$ is bounded, and $0 < \theta < 1/K^2$, then

$$\dim_{A,reg}^{\theta}(f(E)) \le K \frac{1 - \frac{\theta}{K^2}}{1-\theta} \dim_{A,reg}^{\theta/K^2}(E). \tag{4.8}$$

On the other hand, Corollary 1.5 implies that if $t > 0$ then

$$\dim_{A,reg}^{\theta(t)}(f(E)) \le K \frac{\dim_{A,reg}^{\theta(t/K)}(E)}{1 + \frac{K-1}{2} \dim_{A,reg}^{\theta(t/K)}(E)}. \tag{4.9}$$

We conclude this section with the following observation, which indicates the range of Assouad spectrum parameters for which (4.9) improves upon (4.8).

Proposition 4.2. *Let* $d := \dim_{A,reg}^{\theta(t/K)}(E)$ *and assume that* $0 < d \le 1$. *If* $\theta(t) \le \min\{1/K^2, d/2\}$, *then*

$$\frac{d}{1 + \frac{K-1}{2} d} \le \frac{1 - \frac{\theta(t)}{K^2}}{1 - \theta(t)} \dim_{A,reg}^{\theta(t)/K^2}(E).$$

The inequality $1/(1+t) \le d/2$ is an implicit bound for t, since d is also a function of t. However, if $\overline{\dim}_B(E) > 0$ then it suffices to assume $\theta(t) \le \min\{1/K^2, \overline{\dim}_B(E)/2\}$.

Proof of Proposition 4.2. Let $\theta_1 = \theta(t)/K^2$ and $\theta_2 = \theta(t/K)$. Then $d = \dim_{A,reg}^{\theta_2}(E)$ and the conclusion reads

$$\frac{d}{1 + \frac{K-1}{2} d} \le \frac{1 - \theta_1}{1 - \theta(t)} \dim_{A,reg}^{\theta_1}(E). \tag{4.10}$$

Since $\theta_1 \le \theta_2$, Theorem 3.3.1 of [8] implies that

$$\dim_{A,reg}^{\theta_1}(E) \ge \frac{1-\theta_2}{1-\theta_1} \dim_{A,reg}^{\theta_2}(E)$$

and so (4.10) is implied by

$$\frac{d}{1 + \frac{K-1}{2} d} \le \frac{1 - \theta_2}{1 - \theta(t)} d. \tag{4.11}$$

Since $1 - \theta_2 = \frac{t}{K+t}$ and $1 - \theta(t) = \frac{1}{1+t}$, (4.11) reads

$$\frac{1}{1 + \frac{K-1}{2} d} \le \frac{t(1+t)}{K+t}. \tag{4.12}$$

The function $h(t) := t(1+t)/(K+t)$ is increasing for $t > 0$. The assumption $\theta(t) \le d/2$ is equivalent to $t \ge (2/d) - 1$. Hence

$$\frac{t(1+t)}{K+t} \ge \frac{(\frac{2}{d} - 1)(\frac{2}{d})}{K - 1 + \frac{2}{d}} = \frac{\frac{2}{d} - 1}{1 + \frac{K-1}{2} d}$$

and now the hypothesis $d \le 1$ ensures that (4.12) is satisfied. The assumption $\theta(t) \le 1/K^2$ is included in order to apply (4.8). ∎

5. Concluding remarks and open questions

Remark 5.1. The upper bound for Assouad dimension in Theorem 1.3 has the same form as in the analogous theorem for Hausdorff dimension. The upper bound depends only on the dimension of the source set E, on n, and on the optimal exponent of higher Sobolev regularity. In particular, if $f:\mathbb{R}^n\to\mathbb{R}^n$ is quasiconformal and Lipschitz, then $\dim_A f(E)\le\dim_A(E)$ for all sets $E\subset\mathbb{R}^n$. As previously noted, Lipschitz mappings can in general raise the Assouad dimension of sets.

Question 5.2. What can be said about upper bounds for distortion of Assouad dimension or the Assouad spectrum under (not necessarily quasiconformal, or even injective) mappings $f\in W^{1,p}(\mathbb{R}^n:\mathbb{R}^N)$, $p>n$?

The analogous question for Hausdorff and box-counting dimensions was considered by Kaufman [16] for individual sets $E\subset\mathbb{R}^n$, and by Balogh, Monti and the second author in [4] for generic elements in parameterized families of subsets of $\mathbb{R}^n$.

Another natural question which arises is the following.

Question 5.3. Give quantitative upper and lower bounds for distortion of Assouad dimension under quasisymmetric maps of $\mathbb{R}$.

It is known that if $f:\mathbb{R}\to\mathbb{R}$ is η-quasisymmetric and $E\subset\mathbb{R}$ has $\dim_A(E)=\alpha\in(0,1)$, then $\dim_A(f(E))\le\beta=\beta(\alpha,\eta)<1$. This follows from the quantitative equivalence of porosity with having non-full Assouad dimension (valid in any $\mathbb{R}^n$) [18, Theorem 5.2] and the quantitative invariance of porosity under quasisymmetric maps [24]. The exact formula for the upper bound $\beta(\alpha,\eta)$ stemming from this argument is complicated, and unlikely to be sharp. It is well-known that the analogous statement for Hausdorff dimension is false. Indeed, there exist quasisymmetric maps $f:\mathbb{R}\to\mathbb{R}$ and subsets $E\subset\mathbb{R}$ for which both E and $\mathbb{R}\setminus f(E)$ have Hausdorff dimension as small as we please. See, for example, [20].

Remark 5.4. Another consequence of Theorem 1.3 is that Assouad dimension of compact sets is invariant under conformal mappings. In dimensions three and higher, this statement provides no new information, due to Liouville's theorem and the Möbius invariance of Assouad dimension [18, Theorem A.10]. However, it is a new result for planar conformal maps.

Example 5.5. The restriction to compact sets in the previous remark is necessary. Let Ω be the set of $z\in\mathbb{C}$ so that $\mathrm{Re}(z)>0$ and $|\mathrm{Im}(z)|<\pi$. Let $f(z)=\exp(-z)$. Let $E=\mathbb{N}\subset\Omega$. Then $\dim_A(E)=1$, but $\dim_A(f(E))=0$.

Dilatation-independent results for these notions of dimension may also be of interest. The *global quasiconformal dimension* of a set $E\subset\mathbb{R}^n$ is the infimum of dimensions of images $f(E)$, where the infimum is taken over all quasiconformal self-maps of $\mathbb{R}^n$. For values $\alpha\in[1,n)$ there exist Ahlfors α-regular sets $E\subset\mathbb{R}^n$ which are minimal for quasiconformal dimension distortion. Note that all notions of dimension considered in this paper (Hausdorff, box-counting, Assouad, and the Assouad spectrum) agree for an Ahlfors regular set. Thus for any $0<\theta<1$ and any $1\le\alpha<n$ there exists a set $E\subset\mathbb{R}^n$ with $\dim_A^\theta(E)=\alpha$ and which is minimal for global quasiconformal Assouad spectrum dimension with parameter θ. It is known that sets of Hausdorff (respectively, Assouad) dimension strictly less than one have global quasiconformal Hausdorff (respectively, Assouad) dimension zero; these results can be found in [17] and (respectively) [22].

Conjecture 5.6. Let $0\le\theta<1$ and let $E\subset\mathbb{R}^n$ satisfy $\dim_A^\theta(E)<1$. Then the global quasiconformal Assouad spectrum dimension of E with parameter θ is equal to zero. Here we interpret $\dim_A^0$ to be the upper box-counting dimension $\overline{\dim}_B$.

To prove Conjecture 5.6 it suffices to establish the case $\theta = 0$, i.e., to prove the result for the upper box-counting dimension. This follows from a known estimate for Assouad spectrum in terms of box-counting dimension; see Proposition 2.3(6). Assume that Conjecture 5.6 has been established for $\theta = 0$. Let E and $0 < \theta < 1$ be such that $\dim_A^\theta(E) < 1$. Then $\overline{\dim}_B(E) < 1$ and hence there exist quasiconformal maps f of $\mathbb{R}^n$ for which $\overline{\dim}_B(f(E))$ is arbitrarily small. Inequality (2.10) then implies that $\dim_A^\theta(f(E))$ can also be made arbitrarily small by varying over all quasiconformal self-maps f of $\mathbb{R}^n$.

Remark 5.7. A direct computation using the radial stretch map $g(x) = |x|^{1/K-1}x$ in $\mathbb{R}^n$ shows that

$$p(n,K) \le \frac{nK}{K-1}, \tag{5.1}$$

and a well-known conjecture states that equality holds in (5.1) for any $n \ge 2$. As previously mentioned, such equality is currently only known to hold when $n = 2$. Iwaniec and Martin [15] have shown that for any $n \ge 3$ there exists a constant $\lambda = \lambda(n) \ge 1$ so that

$$p(n,K) \ge \frac{n\lambda K}{\lambda K - 1}. \tag{5.2}$$

The value of $\lambda(n)$ obtained in [15] is the smallest possible constant $\lambda \ge 1$ which makes the inequality

$$\left| \int |Df|^{p-n} (\det Df) \right| \le \lambda \left| 1 - \frac{n}{p} \right| \int |Df|^p \tag{5.3}$$

valid for arbitrary distributional mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ with L^p differential $Df : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, and a stronger conjecture (also due to Iwaniec and Martin) is that (5.3) holds with $\lambda = 1$ for all such mappings f. A more precise version of the dimension estimates in (1.4) follows, see [15, Theorem 17.4.1]. Specifically, if $f : \Omega \to \Omega'$ is quasiconformal between domains in $\mathbb{R}^n$ and E is a closed subset of Ω, then

$$\frac{1}{K_O(f)\lambda(n)} \left(\frac{1}{\dim_H(E)} - \frac{1}{n} \right) \le \frac{1}{\dim_H(f(E))} - \frac{1}{n} \le \lambda(n) K_I(f) \left(\frac{1}{\dim_H(E)} - \frac{1}{n} \right). \tag{5.4}$$

Conjecture 5.8. The estimates in (1.7) for quasiconformal distortion of the Assouad spectrum can be sharpened to match those in (5.4). Specifically, the coefficients $1 - \frac{n}{p(n,K)}$ and $(1 - \frac{n}{p(n,K)})^{-1}$ in (1.7) can be replaced by $(K_O(f)\lambda(n))^{-1}$ and $\lambda(n)K_I(f)$, respectively.

References

[1] Assouad, P. Plongements lipschitziens dans $\mathbf{R}^n$. *Bull. Soc. Math. France 111*, 4 (1983), 429–448.

[2] Astala, K. Area distortion of quasiconformal mappings. *Acta Math. 173*, 1 (1994), 37–60.

[3] Astala, K., Iwaniec, T., and Martin, G. *Elliptic partial differential equations and quasiconformal mappings in the plane*, vol. 48 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2009.

[4] Balogh, Z. M., Monti, R., and Tyson, J. T. Frequency of Sobolev and quasiconformal dimension distortion. *J. Math. Pures Appl. (9) 99*, 2 (2013), 125–149.

[5] Balogh, Z. M., Tyson, J. T., and Wildrick, K. Quasiconformal mappings that highly distort the dimensions of many parallel lines. *Ann. Acad. Sci. Fenn. Ser. A I Math. 42* (2017), 61–72.

[6] Bishop, C. J. Quasiconformal mappings which increase dimension. *Ann. Acad. Sci. Fenn. Math. 24*, 2 (1999), 397–407.

[7] Bishop, C. J., Hakobyan, H., and Williams, M. Quasisymmetric dimension distortion of Ahlfors regular subsets of a metric space. *Geom. Funct. Anal. 26*, 2 (2016), 379–421.

[8] Fraser, J. *Assouad dimension and fractal geometry*, vol. 222 of *Cambridge Trats in Mathematics*. Cambridge University Press, 2020.

[9] Fraser, J. M. On Hölder solutions to the spiral winding problem. *Nonlinearity 34*, 5 (2021), 3251–3270.

[10] Fraser, J. M., Hare, K. E., Hare, K. G., Troscheit, S., and Yu, H. The Assouad spectrum and the quasi-Assouad dimension: a tale of two spectra. *Ann. Acad. Sci. Fenn. A I Math. 44* (2019), 379–387.

[11] Fraser, J. M., and Yu, H. New dimension spectra: finer information on scaling and homogeneity. *Adv. Math. 329* (2018), 273–328.

[12] Gehring, F. W. The L^p-integrability of the partial derivatives of a quasiconformal mapping. *Acta Math. 130* (1973), 265–277.

[13] Gehring, F. W., and Väisälä, J. Hausdorff dimension and quasiconformal mappings. *J. London Math. Soc. (2) 6* (1973), 504–512.

[14] Heinonen, J. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[15] Iwaniec, T., and Martin, G. *Geometric function theory and non-linear analysis.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2001.

[16] Kaufman, R. P. Sobolev spaces, dimension, and random series. *Proc. Amer. Math. Soc. 128*, 2 (2000), 427–431.

[17] Kovalev, L. V. Conformal dimension does not assume values between zero and one. *Duke Math. J. 134*, 1 (2006), 1–13.

[18] Luukkainen, J. Assouad dimension: antifractal metrization, porous sets, and homogeneous measures. *J. Korean Math. Soc. 35*, 1 (1998), 23–76.

[19] Stein, E. M. *Singular integrals and differentiability properties of functions*, vol. 30 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1970.

[20] Tukia, P. A quasiconformal group not isomorphic to a Möbius group. *Ann. Acad. Sci. Fenn. Ser. A I Math. 6* (1981), 149–160.

[21] Tyson, J. T. Sets of minimal Hausdorff dimension for quasiconformal maps. *Proc. Amer. Math. Soc. 128*, 11 (2000), 3361–3367.

[22] Tyson, J. T. Lowering the Assouad dimension by quasisymmetric mappings. *Illinois J. Math. 45* (2001), 641–656.

[23] Tyson, J. T., and Wu, J.-M. Quasiconformal dimensions of self-similar fractals. *Rev. Mat. Iberoam. 22*, 1 (2006), 205–258.

[24] Väisälä, J. Porous sets and quasisymmetric maps. *Trans. Amer. Math. Soc. 299* (1987), 525–533.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

E-mail address: echronts@gmail.com

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801 and National Science Foundation, 2415 Eisenhower Avenue, Alexandria, VA 22314

E-mail address: tyson@illinois.edu