Course Descriptions
Spring 2022

Department of Mathematics
University of Illinois

Mathematics 500 — Abstract Algebra I

Spring 2022
(11–12 MWF, in 141 Altgeld)

Instructor: Charles Rezk
Email: rezk@illinois.edu
Webpage: `http://faculty.math.illinois.edu/~rezk/`

Course requirements:
Homework: Weekly homework assignments. (50% of grade.)

Tests: Two midterms (10% each) and a final (30%), in class. The final exam will resemble a Comp Exam.

Texts: The primary text will be:

- Dummit & Foote, *Abstract Algebra,* (3rd edition). Wiley, ISBN 978-0-471-43334-7.

Course topics: The course will cover approximately chapters 1–8, and 10–14 of Dummit and Foote. It will be assumed that students are familiar with basic material from an undergraduate algebra class, such as in Math 417. We will cover the topics on the standard syllabus for 500, including:

- Free groups and presentations of groups.
- Group actions.
- The Sylow theorems.
- Basic ring theory.
- Basic module theory.
- Classification of modules over a PID.
- Fields and field extensions.
- Galois theory.

MATH 506 REPRESENTATION THEORY, SPRING 2022

PROFESSOR RINAT KEDEM

This course will cover the basics of representation theory of groups, algebras, and certain geometric objects.

Representation theory is a vast subject and comes in many flavors. This course will emphasize combinatorial representation theory, and aspects of representation theory useful in or motivated by mathematical physics.

We will start with the standard theory of representations of finite groups: Group homomorphisms from G into the general linear group $GL(V)$ where V is any finite dimensional vector space. The most important example is the symmetric group S_N for any finite N. This is best studied in conjunction with the representation theory of the general linear group itself, via the Schur-Weyl duality. This part of the course corresponds to the first six chapters of Fulton and Harris, together with a deep dive into the material in the appendices, particularly symmetric function theory.

In the rest of the course, we will cover select topics beyond finite groups: Lie algebra representations, quiver representations, and some affine algebras, as well as quantum deformations, as time allows.

References will be distributed throughout the semester, generally materials available online.

Expectation from the students: There will be suggested homework problems, and students are encouraged to work out the basic examples for themselves each week. It is highly recommended that solutions to selected problems designated to be handed in should be LaTex-ed. Students should be prepared for in-class discussion of the suggested problems. Group work is highly encouraged. At the end of the semester, each student will prepare and deliver a short in-class presentation about material related to the course.

Prerequisites: Knowledge of math 500 and prior review of linear algebra is assumed. The material in Math 501 is not assumed.

Math 511:
Professor Dodd

This is an introduction to classical algebraic geometry. As such, the course will develop the basic concepts of commutative algebra and algebraic geometry over an algebraically closed base field, with lots of examples. Topics include the Nullstellensatz, Noether normalization, projective space, basic theory of varieties, Bezout's theorem, and more.

Math 525, Spring 2022
Professor Lerman

Math 525 is an introduction to algebraic topology, a powerful tool for distinguishing and studying topological spaces by associating to them algebraic objects such as groups. The first part of the course will cover fundamental groups, fundamental groupoids and covering spaces. The second part will cover homology. The official syllabus of the course is available here:

https://math.illinois.edu/resources/department-resources/syllabus-math-525.

Text: Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. ISBN: 0521795400. The electronic version of the text is available for a free download:

http://pi.math.cornell.edu/~hatcher/AT/ATpage.html. I haven't decided yet how closely I will follow the text.

Homework: Homework problems are to be assigned once a week. They are due the following week.

Exams: There will be one midterm and a final.

Grade: The formula for the course grade is **roughly** as follows:
- final exam = 40 %
- midterm = 20%
- homework = 40%

SPRING 2022, Math 530: Algebraic Number Theory

Instructor: Alexandru Zaharescu

Office: 449 Altgeld Hall
Phone: 265-5439
E-mail: zaharesc@illinois.edu

Lectures: Tu -Th 9:30 - 10:50 am

Office hours: Tu - Th 10:50 - 12:30 pm

Prerequisite: MATH 500 or equivalent

Grading Policy: Comprehensive final exam: 40%; One midterm exam: 30%; Homework: 30%.

Recommended Textbook:

Daniel A. Marcus, Number Fields, Springer-Verlag, 1987, NY.

We will discuss material from the first five chapters, and then, as time permits, present other selected topics.

Math 532:
Professor Thorner

We will explore several advanced topics in analytic number theory with the goal of exposing students to a wide range of topics and ideas within the field. Possible topics include: Primes in short intervals, bounding the least prime in an arithmetic progression, the proportion of nontrivial zeros of the Riemann zeta function that satisfy the Riemann hypothesis, Selberg's central limit theorem for the Riemann zeta function, moments of the Riemann zeta function, the Chebotarev density theorem for Galois extensions of number fields, introduction to modular forms and elliptic curves and their L-functions, introduction to the Sato-Tate conjecture. No homework or exams will be given, but students will be expected to prepare and give presentations on these topics.

Math 541: Functional Analysis
Spring 2022
Lectures: in person, TBA
Instructor: Florin P. Boca (fboca@illinois.edu)

This course will provide an introduction to Functional Analysis. The main topics will include:

- Review of abstract measure theory, Riesz representation theorem, Lebesgue-Radon-Nykodim theorem.
- Basic topics on Banach spaces, linear and bounded maps on Banach spaces, open mapping theorem, closed graph theorem.
- Locally convex spaces.
- Hahn-Banach theorem, Banach-Alaoglu theorem, extreme points, Krein-Milman theorem. Applications.
- Compact operators, spectrum and the spectral theorem for compact operators on Hilbert spaces.
- Weak operator topologies.
- Representations of compact groups and the Peter-Weyl theorems.
- The Fourier transform on $\mathbb{R}^k$.
- Tempered distributions.
- Sobolev spaces.

Prerequisite: Math 540.

Textbook: There is no required textbook. The instructor will use his own notes.
Recommended textbooks:

- J. B. Conway, A Course in Abstract Analysis.
- J. B. Conway, A Course in Functional Analysis.
- W. Rudin, Functional Analysis.
- G. B. Folland, Real Analysis. Modern Techniques and their Applications.
- Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis.

Grading: The final grade will be based on six homework assignments (90%) and class participation (10%).

Math 561: Probability I

Instructor: Renming Song
Office: 338 Illini Hall
Phone (217) 244 6604
Email: rsong@illinois.edu <mailto:rsong@illinois.edu>
Homepage: http://www.math.uiuc.edu/~rsong Class Hours:
Classroom:
Office hours:

Test: R. Durrett: Probability: Theory and Examples (4th edition) Cambridge University Press, 2010

Course topics: This is the first half of the basic graduate course in probability theory.
The goal of this course is to understand the basic tools and language of modern probability theory. We will start with the basic concepts of probability theory: random variables, distributions, expectations, variances, independence and convergence of random variables.
Then we will cover the following topics: (1) the basic limit theorems (the law of large numbers, the central limit theorem and the large deviation principle); (2) martingales and their applications.
If time allows, we will give a brief introduction to Brownian motion. The prerequisite for Math 561 is Math 540.

Grading Policy: 40% of your grade will depend on homework assignment, 30% will depend on the midterm test and 30% on the final exam.

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Math 584
Professor Balog

PREREQUISITES:
Math 580 or consent of instructor, obtainable by familiarity with elementary combinatorics.

COURSE REQUIREMENTS:
There will be 4 homework assignments. Homework allows you to choose five out of six problems to write up. Problems are worth 5 points each, so the maximum score/homework is 25 points. For Math 584 students A is from 80%, grade drops by 5% (so 75% is A-). Make up possibilities include giving lecture.

TEXT:
Jiri Matousek: 33 Miniatures, Mathematical and Algorithmic Applications of Linear Algebra + selected research papers.

TOPICS:
The course is about the linear algebraic methods in combinatorics. Recent new breakthrough results should be included.

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C O U R S E   D E S C R I P T I O N

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Spring 2022

MATH 595

S M O O T H   A N D   E T A L E   E X T E N S I O N S

Prof. S. P. Dutta
TR: 9:30 am to 10:50 am

This course is intended to be a two-semester course covering several areas in commutative algebra and algebraic geometry on Smooth and Etale extensions. Our main focus will be on the following topics: Weierstrass Preparation Theorem; structure theorem for complete local rings; Zariski's Main Theorem; unramified, étale and smooth extensions and their corresponding structure theorems; Henselian Rings and Henselization; Artin's approximation theorem; Hochster's construction of big Cohen-Macaulay modules and finally Swan's exposition of Popescu's proof of Artin's conjecture on smooth extensions.

The following book covers several topics (not all) mentioned above.

Text: Birger Iversen - Generic local structure in commutative algebra--Lecture notes in Math 310, Springer Verlag, berlin Heidelberg, New York.

$2 \times 5 \times 31 \times 492467 \times 25794089$ (3)(2 4)(5 7)(1 6 8 10 9)

Graduate Course Description
Spring 2022
Math 595: Anatomy of Integers and Random Permutations

Instructor: Kevin Ford

Time/place: MWF 1–1:50; Altgeld 347

Prerequisites: Basic analytic number theory will be very helpful (equivalent of the first half of Math 531; elementary prime number estimates and multiplicative functions). Some knowledge of basic probability will be helpful but not necessary.

Recommended Texts: *Divisors*, by R. R. Hall and G. Tenenbaum, Cambridge Tracts in Math. **90**, paperback edition, 2008. (highly recommended for purchase).

Course Description.
Integers factor uniquely into a product of primes, and permutations factor uniquely into a product of cycles. Basic questions one can ask about these structures are

- How many prime factors does a typical integer $n \in [1, x]$ have? What is the distribution of those prime factors?
- How many cycles does a typical permutation of S_n have? How are the lengths of the cycles distributed?

Perhaps surprisingly, there is a close connection between these two problems, both distributions governed by the same probabilistic law. In the first part of the course we will examine carefully, on many scales, the distribution of the prime factors of typical integers and cycle decompositions of typical permutations. We will emphasize the connections between these two structures, stressing probabilistic techniques and ideas. In the second part, we will apply this knowledge to answer questions which about the distribution of divisors of integers, fixed sets of permutations (a subset of $\{1, \dots, n\}$ which is itself permuted by the permutation) and applications of these bounds. Some examples:

1. How likely is it that an integer has two divisors in a fixed dyadic interval $(y, 2y]$?
2. How likely is it that an integer has two divisors in *some* dyadic interval $(y, 2y]$?
3. How many *distinct entries* are there in and $N \times N$ multiplication table?
4. How likely is it that a random permutation $\pi \in S_n$ contains a fixed set of size equal to k (that is, contains cycles with lengths summing to k)?
5. What is the probability that a random polynomial is irreducible?

Problems 1,2,3 are classical problems of Erdős. Problem 4 generalizes the classical derangement problem ($k = 1$). Problem 5 is an application of the ideas about random permutations, and the answer is “almost all polynomials” under the proper probability measure.

Grades. The course grade will depend on homework assignments, which will be given periodically.

MATH 595: ROOT SYSTEMS AND GENERALIZED SCHUBERT CELLS

WILLIAM HABOUSH

This course will be devoted to root systems and their role in the structure of generalized Schubert cells. I will begin by discussing reduced root systems and their classification. I will introduce the notion of groups generated by reflections and their interaction with the geometry of root systems. Then I will give several purely descriptive lectures on the structure theory of semisimple groups with special attention to Borel sub-groups, maximal tori, Cartan subgroups, Weyl groups and the Bruhat decomposition. Then I will discuss homogeneous spaces and the descent theory necessary to prove the basic facts about homogeneous bundles. Then we will consider line bundles and divisors on the generalized flag variety and and the relations between the geometry of the generalized flag variety and the representation theory of the semisimple group. I hope to give an account of the Weyl character formula and perhaps the Demazure character formula. While most of the course will assume that the characteristic is null I will offer some perspectives on phenomena special to positive characteristic. Although I will try to explain basic algebraic geometric notions such as divisors, line bundles, and coherent cohomology, it would be good for registrants to have some basic knowledge of algebraic geometry.

SPRING 2022
MATH 595
MODERN ALGEBRAIC GEOMETRY II

Time: Tuesdays and Thursdays 11:00–12:20
Room: TBA
Instructor: Sheldon Katz
Text: Algebraic Geometry, R. Hartshorne, Graduate Texts in Mathematics 52, Springer NY 1977
Prerequisites: Modern Algebraic Geometry I (Math 512) or equivalent, or permission of the instructor

This full-semester course will begin by covering most of the content of Chapter III of Hartshorne's text, especially derived functors and cohomology. The course will include applications to curves (Chapter IV) and surfaces (Chapter V). The text will be frequently supplemented with additional materials designed to enhance geometric intuition, and the treatment in class will often diverge from the text.

The course will include weekly problem sessions. Homework will be assigned but will not be graded.

MATH 595-TOPICS IN GEOMETRIC PDE'S: INSANTONS AND MONOPOLES ON SMOOTH AND SINGULAR SPACES

The main leitmotif of the course will be the study of Geometric properties of smooth and singular manifolds and their Physical meaning in Quantum Field Theory, by means of studying suitable (typically non-linear) Geometric PDE's on such spaces.

The main focus will be on the structure of Instantons on 4-manifolds and monopoles on 3-manifolds (both on smooth and singular spaces). The singular spaces we consider are of wedge (or conical) type. We will discuss the moduli spaces of solutions (and geometric consequences like the existence of exotic smooth structures on $\mathbb{R}^4$ and Donaldson's polynomial invariants).

In a fundamental paper ("Topological Quantum Field Theory", Commun. Math. Phys. 117), E. Witten showed that Donaldson theory can be obtained from a (supersymmetric) Topological Quantum Field Theory (TQFT). Witten constructed a certain topologically twisted $\mathcal{N} = 2$ Super Yang-Mills (SYM) theory on a smooth four-manifold X with gauge group SU(2) and showed that in this theory, the correlators of some specific operators reproduce the Donaldson invariants of the four-manifold. Then Witten and Seiberg showed that the low energy effective theory of the $\mathcal{N} = 2$ SYM theory on $\mathbb{R}^4$ (the Seiberg-Witten theory) , as well as the definition of the Seiberg-Witten invariants led eventually to the work of Moore and Witten. The Main topics we will cover are:

- Basics of Riemannian and comparison geometry
- Foundamentals of Elliptic and Parabolic PDE's in the smooth and the *wedge* setting
- Gauge theory in math and physics
- Index theorems
- Donaldson invariants
- Supersymmetric Topological Quantum field theory with an eye to Donaldson-Witten and Seiberg Witten invariants
- Possible generalizations to the wedge case.

A familiarity with Riemannian Geometry, PDE's and Quantum Field Theory is recommended but not required, as we will review all the necessary tools. Familiarity with basic algebraic topology (fundamental group, covering spaces, homology and cohomology) and differential topology will be assumed.

MATH 595-TOPICS IN SUPERSTRING THEORY AND GEOMETRY

This course will be mostly about Superstring Theories and their topological incarnation. The term topological here is used as in the physical literature, that is to say that, in case the Lagrangian is written on a manifold M with the choice of a metric $g = g_{ij}dx^i dx^j$ (which is a background field in topological QFT's), Then, loosely speaking, the theory is called a topological field theory if the observables do not depend on the choice of metric g. Let us stress that it is part of the definition that h is a background field – in particular, we do not integrate over h in the 17 path integral. In topological string theories one makes the metric g dynamical. After briefly reviewing path integral quantization and basics of string theory, we will review Atiyah's fundamental paper "Topological quantum field theory," (Publ. Math. IHES 68 (1988) 175- 186). We will then start our journey through Topological String Theory by studying the notion of cohomological QFTs and the fundamental example of the Witten twist. We will then aim at understanding Mirror symmetry and possible generalizations of all this in the case of

The Main topics we will cover are:

- Basics of topological QFT (following Atiyah)
- Atiyah-Singer index theorems
- Topological Supersymmetric String Theory
- Kähler manifolds and Kodaira-Spencer theory
- Gromov-Witten invariants
- Mirror symmetry
- Large comples structure limits versus Cheger-Gromov limits
- Generalizations to the wedge case (in light of Cheeger-Colding theorem) of Witten's twists and Mirror symmetry
- Singular Gromov-Witten invariants

We will follow mostly the original articles but an excellent most comprehensive reference is the book K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. P. Thomas, C. Vafa, R. Vakil, and E. Zaslow, Mirror symmetry, vol. 1 of Clay Mathematics Monographs. American Mathematical Society, Providence, RI, 2003.

A familiarity with Riemannian Geometry, PDE's and Quantum Field Theory is recommended but not required, as we will review all the necessary tools.

Mathematics 595 — Higher category theory and quasicategories

(MWF 1pm-2pm, first half of semester, Spring 2022)

Instructor: Charles Rezk
Email: rezk@illinois.edu
Webpage: `http://faculty.math.illinois.edu/~rezk/`

Course description:
Higher category theory is the study of structures which are like categories, but are "higher-dimensional": while a category has objects (0 dimensions), and morphisms between objects (1 dimensions), higher dimensional analogues are allowed to have morphisms between morphisms (2 dimensions), and so on. Of special interest are "$(\infty, 1)$-categories", in which higher morphisms are always invertible. These have become essential for new research in homotopy theory and related areas.

The goal of this course is to descibe an approach to this called *quasicategories*. These were invented by Boardman and Vogt, and were developed further by André Joyal (in various papers and unpublished preprints) and Jacob Lurie (in his book *Higher topos theory* and subsequent works, where he calls them ∞-categories).

The goal of this course is to give a brief and *accessible* introduction to this theory. That is, we imagine that we are familiar with classical category theory, and we are confronted with the strange new notion of a quasicategory. We will attempt to develop basic concepts and results for quasicategories by analogy with what we know about classical categories.

This is, of course, not entirely straightforward. In order to proceed, we will need to develop the theory of "simplicial sets" and the theory of "model categories". These are not prerequisites: they will be introduced and developed through the course.

Prerequisites: Some familiarity with the basic notions of classical category theory is needed (e.g., functors, natural transformations, limits and colimits, etc.)

Familiarity with basic algebraic topology (e.g., fundamental group and singular homology, as in Math 525), or homological algebra, will be helpful, but not essential.

Texts: The main text are a revised version of notes that I have used the previous times I taught this topic: they are available from my homepage.