Math 595 - Lie Groupoids and Lie Algebroids - Fall 2021

(section LG)

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Office Hours: TBD
Class meets: 02:00-03:20 TR, 345 Altgeld Hall
Prerequisites: Math 518 or equivalent.

This course is an introduction to the theory of Lie groupoids and their infinitesimal counterparts, called Lie algebroids. This is a far reaching extension of the usual Lie theory, which finds application in many areas of Mathematics.
Groups typically arise as the symmetries of some given object. The concept of a groupoid allows for more general symmetries, acting on a collection of objects rather than just a single one. Groupoid elements may be pictured as arrows from a source object to a target object, and two such arrows can be composed if and only if the second arrow starts where the first arrow ends. Just as Lie groups (as introduced by Lie around 1900) describe smooth symmetries of an object, Lie groupoids (as introduced by Ehresmann in the late 1950's) describe smooth symmetries of a smooth family of objects. That is, the collection of arrows is a manifold G, the set of objects is a manifold M, and all the structure maps of the groupoid are smooth. Ehresmann's original work was motivated by applications to differential equations. Since then, Lie groupoids have appeared in many other branches of mathematics and physics. These include:

- algebraic geometry: Grothendieck introduced stacks in the late 1960's via fibered categories over a site. Fibered categories can be viewed as a type of sheaf of groupoids. More recently, this has led to the concept of a gerbe.
- foliation theory: Haefliger introduced transversal structures to foliations in the 1970's, using the concept of a holonomy groupoid. This approach allows for a systematic study of transversal structures, and has been central to the subsequent development of the subject.
- noncommutative geometry and index theory: Lie groupoids made their appearance in noncommutative geometry through the monumental work of Connes in the 1980's. He introduced the tangent groupoid of a space as a central ingredient in his approach to the

Atiyah-Singer index theorem. This approach led to a number of refinements of the index theorem, such as the Connes-Skandalis index theory for foliations.
- Poisson geometry: motivated by quantization problems, Karasev and Weinstein introduced the symplectic groupoid of a Poisson manifold in the late 1980's, as a way to ``untwist" the complicated behavior of the symplectic foliation underlying the Poisson manifold.

In this course I will be following my Lecture Notes:

- M. Crainic and R.L. Fernandes, Lectures on Integrability of Lie Brackets also available as *Geometry & Topology Monographs* **17** (2011) 1-107.

Students taking this course are assumed to know differential geometry at the level of Math 518 - Differentiable Manifolds. A knowledge of ordinary Lie Theorey at the level of Math 522 is recommended but not strictly necessary.

Course Contents:

- **Lie groupoids**
- **Lie algebroids**
- **Lie functor and integrability**
- **Differentiable stacks**
- **Special Topics.** To be chosen from the interests of the students.

Textbooks:

I will provide to participants some lecture notes as the course progresses, but the following two references should also be very helpful:

- A. Cannas da Silva and A. Weinstein, ***Geometric models for noncommutative algebras***, Berkeley Mathematics Lecture Notes, 10. American Mathematical Society, Providence, RI, 1999.
- M. Crainic and R.L. Fernandes, Lectures on Integrability of Lie Brackets, available as *Geometry & Topology Monographs* **17** (2011) 1-107.
- D. Metzler, Topological and Smooth Stacks, Preprint arXiv:math/0306176.
- A. Vistoli, Grothendieck topologies, fibered categories and descent theory, Fundamental algebraic geometry, 1104, Math. Surveys Monogr., 123, AMS Providence, RI (2005).

Grading Policy

- **Expository Paper:** Students will be encouraged to write (in LaTeX) and present a paper. This is not mandatory. Following the tradition of topics courses, there will be no homework and no written exams.