AN OBSERVATION OF RANKIN ON HANKEL DETERMINANTS

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Abstract

While studying the location of the zeros of the Eisenstein series $E_k(z)$, Rankin considered the determinants Δ_n of an associated Hankel matrix. He observed that the first few possess remarkable factorizations, and expressed the hope that a general theorem explaining these factorizations could be found. In this note we provide such a theorem by giving an explicit formula for Δ_n using work of Kaneko and Zagier on Atkin polynomials.

1. Introduction

The zeros of Eisenstein series were studied by R. Rankin in [5], where he showed that for $k = 28, 30, 32, 34$ and 38 the zeros of E_k lie on the unit circle. Soon after R. Rankin's result, F.K.C. Rankin and Swinnerton-Dyer [4] proved that the zeros of E_k lie on the unit circle for all even $k \geq 4$. In this note we confirm an observation made in [5] about the determinants $\Delta_n = |H_n|$ of the Hankel matrix given by

$$H_n = \begin{pmatrix} g_0 & g_1 & \cdots & g_n \\ g_1 & g_2 & \cdots & g_{n+1} \\ g_2 & g_3 & \cdots & g_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ g_n & g_{n+1} & \cdots & g_{2n} \end{pmatrix}, \tag{1}$$

where the g_v are defined as follows. Let

$$E_k(z) := \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k}$$

and let $\Delta(z)$ be the unique normalized weight 12 cusp form. Let j be the modular invariant defined by $j(z) := \frac{E_4^3(z)}{\Delta(z)}$ and with Fourier expansion

$$j(z) = q^{-1} \sum_{n=0}^{\infty} a_n q^n,$$

where $q = e^{2\pi iz}$. For a function F that is meromorphic on a fundamental domain $\mathcal{F}$, write $R(F)$ for the sum of the residues of F at points of $\mathcal{F}$ and

$$j^\nu(z) = q^{-\nu} \sum_{n=0}^{\infty} a_n^{(\nu)} q^n.$$

Then g_ν is defined by

$$g_\nu := 2\pi i R(j^\nu E_2) = a_\nu^{(\nu)} - 24 \sum_{m=1}^{\nu} a_{\nu-m}^{(\nu)} \sigma(m), \tag{2}$$

where $\sigma(n) = \sum_{d|n} d$ and $E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n$. The first few values are

$$g_0 = 1, \quad g_1 = 720, \quad g_2 = 911520 \quad g_3 = 1301011200, \quad g_4 = 1958042030400.$$

This is sequence A030185 in [6]. Rankin gave the values up to Δ_{13} in a table by their prime factorization; the last of these is of size approximately $2.79 \cdot 10^{483}$ (the computations in Rankin's paper were made by Mr. Stephen Muir of the Atlas Computer Laboratory in Chilton, Didcot). Rankin went on to say that "... they possess remarkable factorizations; each of them is a highly composite number expressible as a product of powers of small primes. These results are given in §4 in the hope that they may stimulate someone to prove a general theorem about these determinants."

Here we prove such a theorem.

Theorem 1. *For $n \geq 1$, let H_n be as in* (1) *and $\Delta_n = |H_n|$. Then,*

$$\Delta_n = 2^{n^2+4n} \cdot 3^{n^2+2n} \cdot 5^n \cdot 7^n \cdot 13^n \cdot \prod_{r=2}^{n} \Big(\frac{(12r-13)(12r-7)(12r-5)(12r+1)}{(2r-1)^2(r-1)r} \Big)^{n-r+1}.$$

Note that the largest prime that can appear in the factorization of Δ_n is at most $12n + 1$.

2. Proof of Theorem 1

Recall that if V is the space of polynomials in one variable over a field K, and $\phi : V \to K$ is a linear functional, then one can consider the scalar product on V defined by $(f, g) = \phi(fg)$. One can also consider the family (which for generic ϕ exists and is unique) of monic polynomials which are mutually orthogonal with respect to the scalar product.

Atkin [2, page 3] defined a sequence of polynomials $A_n(j) \in \mathbb{Q}[j]$, one for each degree n, as the orthogonal polynomials with respect to a scalar product. The particular scalar product used by Atkin is defined in several equivalent ways in [2, Proposition 3], one of them being

$$(f, g) := \text{ constant term of } fgE_2 \text{ as a Laurent series in } q.$$

Then from the definition (2) we see that

$$g_\nu = (j^\nu, 1).$$

Proof of Theorem 1. A recursion for the Atkin polynomials A_n is given by ([2, equation (18)]):

$$A_{n+1}(j) = (j - (\lambda_{2n} + \lambda_{2n+1}))A_n(j) - \lambda_{2n-1}\lambda_{2n}A_{n-1}(j), \tag{3}$$

where the numbers λ_n are defined by the continued fraction expansion

$$\sum_{k=0}^{\infty} g_k x^k = \cfrac{g_0}{1 - \cfrac{\lambda_1 x}{1 - \cfrac{\lambda_2 x}{1 - \cdots}}}. \tag{4}$$

By [3, Theorem 29], we can use the recurrence in (3) to give a formula for Δ_n in terms of the λ_n:

$$\Delta_n = \det_{0 \le i, r \le n} (g_{i+r}) = \prod_{r=1}^{n} (\lambda_{2r-1}\lambda_{2r})^{n-r+1}. \tag{5}$$

Equation (19) of [2] gives an explicit formula for the λ_n:

$$\lambda_1 = 720, \qquad \lambda_n = 12\Big(6 + \frac{(-1)^n}{n-1}\Big)\Big(6 + \frac{(-1)^n}{n}\Big) \quad \text{for } n > 1. \tag{6}$$

For $r > 1$ this gives

$$\begin{aligned} \lambda_{2r-1}\lambda_{2r} &= 12\Big(6 + \frac{(-1)^{2r-1}}{(2r-1)-1}\Big)\Big(6 + \frac{(-1)^{2r-1}}{2r-1}\Big)12\Big(6 + \frac{(-1)^{2r}}{(2r)-1}\Big)\Big(6 + \frac{(-1)^{2r}}{2r}\Big) \\ &= \frac{36(12r-13)(12r-7)(12r-5)(12r+1)}{(2r-1)^2(r-1)r}. \end{aligned}$$

Plugging this formula into equation (5) and simplifying yields the result. □

Let $\nu_p(m)$ be the highest power of p that divides a non-zero integer m. From Theorem 1 one can obtain $\nu_p(\Delta_n)$ for any prime p. In the case $p = 2$ it has a simple expression.

Corollary 1. *We have*

$$\nu_2(\Delta_n) = 4n - s_2(n) + 2\sum_{r=1}^{n} s_2(r),$$

where $s_2(r)$ is the sum of the digits of r in base 2.

Proof. From Theorem 1 we see that

$$\nu_2(\Delta_n) = n^2 + 4n - \nu_2\Big(\prod_{r=2}^{n} \big((r-1)r\big)^{n-r+1}\Big),$$

so we only need to show that $\nu_2\Big(\prod_{r=2}^{n}\big((r-1)r\big)^{n-r+1}\Big) = n^2 - 2\sum_{r=1}^{n} s_2(r) + s_2(n)$. Using the fact that $\nu_2(r) = 1 + s_2(r-1) - s_2(r)$ we obtain

$$\begin{aligned}
&\nu_2\Big(\prod_{r=2}^{n} \big((r-1)r\big)^{n-r+1}\Big)\\
&= \sum_{r=2}^{n}(n-r+1)(\nu_2(r) + \nu_2(r-1))\\
&= 2\sum_{r=2}^{n}(n-r+1) - \sum_{r=2}^{n}(n-r+1)s_2(r) + \sum_{r=1}^{n-2}(n-r-1)s_2(r)\\
&= n^2 - s_2(n) - 2s_2(n-1) + \sum_{r=1}^{n-2} s_2(r)(n-r-1-(n-r+1))\\
&= n^2 - 2\sum_{r=1}^{n} s_2(r) + s_2(n).
\end{aligned}$$

□

We point out that the sequence $(\sum_{r=1}^{n} s_2(r))_n$ is sequence A000788 in [6].

As seen in [2], there are many ways to approach the Atkin polynomials. In this spirit, we briefly explain another way in which one could obtain a closed formula for Δ_n. From Section 4 of [1] we have

$$\Delta_n = ||A_0(j)||^2 \cdot ||A_1(j)||^2 \cdots ||A_n(j)||^2 = \prod_{i=0}^{n}(A_i, A_i). \tag{7}$$

For $n \geq 1$, ([2, Proposition 6])

$$(A_n, A_n) = -12^{6n+1}\frac{(-1/12)_n(5/12)_n(7/12)_n(13/12)_n}{(2n-1)!(2n)!}, \tag{8}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ and $(A_0, A_0) = (1,1) = 1$. Thus,

$$\Delta_n = \prod_{i=1}^{n} -12^{6i+1}\frac{(-1/12)_i(5/12)_i(7/12)_i(13/12)_i}{(2i-1)!(2i)!}. \tag{9}$$

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