UIUC Math 580 Comprehensive Exam, January 2020

Each problem below is worth 10 points. The exam will be scored out of 100. A good performance in both Part 1 and Part 2 is required to pass this exam.

Answers without a proof or with an incorrect proof may not receive no points.

Part 1

1. Give a combinatorial (counting) argument that proves
\[ \sum_{k=1}^{n} k \cdot k! = (n + 1)! - 1. \]

2. Let \( c(m, n) \) be the number of surjective functions \( f : [n] \to [m] \) where \([j] := \{1, 2, 3, \ldots, j\}\).
By the inclusion-exclusion principle, or otherwise, determine an explicit formula for \( c(m, n) \).

3. Let \( F_k \) be the adjusted Fibonacci number, i.e.,
\( (F_1, F_2, F_3, F_4, F_5, \ldots) = (1, 1, 2, 3, 5, \ldots), \)
and \( F_k = F_{k-1} + F_{k-2} \) for \( k \geq 2 \). Evaluate (with proof)
\[ \sum_{k=0}^{n} F_{k+1} \binom{n}{k}. \]

4. By a combinatorial argument involving integer partitions/Ferrers diagrams, prove the following identity of ordinary generating series
\[ \prod_{k=1}^{\infty} 1 + x^k = \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{i=1}^{j}(1 - x^{2i})}. \]

5. Let \( K_n \) be the complete graph on \( n \) vertices. A collection \( C \) of (undirected) paths covers \( K_n \) if each vertex is in exactly one of the paths. Prove that the number of ways to cover \( K_n \), by paths with at least one vertex, is
\[ \left[ \frac{x^n}{n!} \right] \exp \left( \frac{x(2 - x)}{2(1 - x)} \right). \]

Part 2

1. Prove that a graph \( G = (V, E) \) with at least 4 vertices is 3-connected if and only if for arbitrary three vertices \( x, y, z \in V \) and any edge \( e \) disjoint from \( z \) there exists an \( x, y \)-path that passes through \( e \) and avoids \( z \).
2. In a plane 3-regular connected graph $G$, every vertex belongs to two faces of length 4 and to one face of length 36. How many edges does $G$ have?

3. Let $k \geq 4$ and $n - k \geq 3$. How few edges may have an $n$-vertex graph $G$ with chromatic number $k$ and minimum degree at least 2?

4. State the Szemerédi Regularity Lemma. This means that you need to define all notions used, such as $\epsilon$-regular partitions, etc.

5. For each statement below, determine if it is true or false. If it is true, give a proof. If it is false, provide a counterexample.
   a) Every 4-regular 4-connected simple graph has a perfect matching;
   b) Every 3-regular 2-connected graph has a perfect matching;
   c) Every 3-regular 3-connected graph has three disjoint perfect matchings;
   d) Every 3-regular bipartite graph has three disjoint perfect matchings.