Logic Comprehensive Exam (Math 570), January 27, 2021

Do all four problems. Explain your answers except when asked to “indicate” something. The four problems have equal weight. Throughout: \( m, n \) range over \( \mathbb{N} := \{0, 1, 2, 3, \ldots\} \); \( L \) is a language; given a set \( \Sigma \) of \( L \)-sentences, \( \text{Th}(\Sigma) \) is the set of \( L \)-sentences \( \sigma \) such that \( \Sigma \vdash \sigma \); for an \( L \)-structure \( A \), \( \text{Th}(A) \) is the set of \( L \)-sentences true in \( A \); computable has the same meaning as recursive, and computably generated the same as recursively enumerable (for those used to other terminology).

1. Let \( L \) have just the binary relation symbol \(<\). Let \( \sigma \) be the sentence \( \forall x \exists y (x < y) \).
   (i) Indicate a finite set \( \Sigma \) of \( L \)-sentences whose models are exactly the (nonempty) totally ordered sets \((A; <)\). Here “ordered” is taken in the strict sense where \( a < b \) implies \( a \neq b \).
   (ii) Show that \( \sigma \) is not \( \Sigma \)-equivalent to any existential \( L \)-sentence.
   (iii) Show that \( \sigma \) is not \( \Sigma \)-equivalent to any universal \( L \)-sentence.

2. Let \( L \) have just the unary relation symbol \( P \).
   (i) Indicate a set \( \Sigma \) of \( L \)-sentences whose models are exactly the \( L \)-structures \( A = (A; P) \) such that \( P \subseteq A \) is infinite.
   (ii) Determine the countable models of \( \Sigma \) up to isomorphism.
   (iii) Show that \( \Sigma \) is not complete.
   (iv) Indicate a family \( (\Sigma_i)_{i \in I} \) where each \( \Sigma_i \supseteq \Sigma \) is a complete set of \( L \)-sentences and every model of \( \Sigma \) is a model of \( \Sigma_i \) for exactly one \( i \in I \).
   (v) Show that \( \text{Th}(\Sigma) \) is decidable. (You can argue informally using “decidable” intuitively.)

3. Let \( \mathcal{N} = (\mathbb{N}; <, 0, S, +, \cdot) \) be the standard model of arithmetic. Let PA be the usual set of axioms of (first-order) Peano Arithmetic; recall that PA includes an induction scheme.
   (i) \( \mathcal{A} \equiv \mathcal{N} \) for all \( \mathcal{A} \models \text{PA} \). True or false?
   (ii) Is there a model \( \mathcal{A} \) of PA such that \( \text{Th}(\mathcal{A}) \) is decidable?
   (iii) Show that there is a countable model \( \mathcal{A} = (A; <, \ldots) \) of PA with an element \( a \in A \) such that \( n < a \) and \( a \in nA \) for all \( n \); here \( \mathbb{N} \) is identified with its image in \( A \) via the embedding \( n \mapsto (S^n 0)^A : \mathcal{N} \to \mathcal{A} \) and \( nA := \{n \cdot a : a \in A\} \).
   (iv) Let \( \mathcal{A} \) be as in (iii). Show that the subset \( \mathbb{N} \) of \( A \) is not definable in \( \mathcal{A} \).

4. Let \( f, g : \mathbb{N} \to \mathbb{N} \) be computable such that \( f \) is injective, \( f(\mathbb{N}) \) is computable, and \( f(n) \leq g(n) \) for all \( n \).
   (i) Show that \( g(\mathbb{N}) \) is computable. (You can argue informally using “computable” intuitively.)

Let \( A, B \subseteq \mathbb{N} \). (Continued on other side.)
(ii) Show that if $A, B$ are computably generated, then there are disjoint computably generated sets $A^* \subseteq A$ and $B^* \subseteq B$ such that $A^* \cup B^* = A \cup B$.

(iii) Suppose $A \cap B = \emptyset$ and the complements of $A$ and $B$ are computably generated. Use (ii) to show there is a computable set $S \subseteq \mathbb{N}$ such that $A \subseteq S$ and $S \cap B = \emptyset$. 