1. (20 points) Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed positive random variables with \( P(X_1 > x) = e^{-x} \) for all \( x \geq 0 \). Show that
\[
\limsup_{n \to \infty} \frac{X_n - \ln n}{\ln \ln n} = 1
\]
almost surely.

2. (20 points) Let \( X_1, X_2, \ldots \) be independent and identically distributed random variables with \( E(X_1) = 0 \) and \( EX_1^2 = 1 \). Show that
\[
\frac{\sqrt{n} \sum_{m=1}^{n} X_m}{\sum_{m=1}^{n} X_m^2}
\]
converges weakly to a standard normal random variable.

3. (20 points) Let \( Y_1, Y_2, \ldots \) be nonnegative independent and identically distributed random variables with \( E(Y_1) = 1 \) and \( P(Y_1 = 1) < 1 \). Put \( \mathcal{F}_n = \sigma\{Y_1, \ldots, Y_n\} \). (i) Show that \( X_n = \prod_{m \leq n} Y_m \) is a martingale with respect to \( \mathcal{F}_n \). (ii) Show that \( X_n \) converges to zero almost surely as \( n \to \infty \).

4. (20 points) Let \( \xi_{i,n}, i, n \geq 0 \) be independent identically distributed non-negative integer valued random variables with a common expectation \( \mu \) and common variance \( \sigma^2 \in (0, \infty) \). Define a sequence \( Z_n, n \geq 0 \) by \( Z_0 = 1 \) and
\[
Z_{n+1} = \begin{cases} 
\xi_{1,n+1} + \cdots + \xi_{Z_n,n+1}, & Z_n > 0 \\
0, & Z_n = 0.
\end{cases}
\]
Put \( \mathcal{F}_n = \sigma(\xi_{i,m} : i \geq 1, 1 \leq m \leq n) \) and \( X_n = \frac{Z_n}{\mu^n} \). (a) Show that \( X_n \) is a martingale with respect to \( \mathcal{F}_n \). (b) Show that if \( \mu \leq 1 \), then \( Z_n = 0 \) for all \( n \) sufficiently large. (c) Show that
\[
EX_n^2 = 1 + \sigma^2 \sum_{k=2}^{n+1} \mu^{-k}
\]
by proving the identities
\[
E(X_n^2|\mathcal{F}_{n-1}) = X_{n-1}^2 + E((X_n - X_{n-1})^2|\mathcal{F}_{n-1})
\]
and
\[
E((X_n - X_{n-1})^2|\mathcal{F}_{n-1}) = \sigma^2 \mu^{-2n} Z_{n-1}.
\]
(d) Show that, if \( \mu > 1 \), \( X_n \) converges in \( L^2 \).

5. (20 points) Suppose that \( X_n \) is a nonnegative submartingale with respect to a filtration \( \mathcal{F}_n \). Show that for any \( a > 0 \) and any positive integer \( N \) we have the following Doob’s inequality
\[
P(\max_{1 \leq n \leq N} X_n \geq a) \leq \frac{1}{a} \int_{\{\max_{1 \leq n \leq N} X_n \geq a\}} X_N dP.
\]