Math 542 Comprehensive Examination, January 2020

Solve five of the following six. Each problem is worth 20 points. The open ball centered at $z$ of radius $r$ is denoted by $B_r(z)$.

1. Let $G \subseteq \mathbb{C}$ be open and connected and $f : G \rightarrow \mathbb{C}$ an analytic function. Suppose that there is a point in $G$ at which $f$ and all of its derivatives vanish. Prove that $f$ is identically zero on $G$.

2. Let $G = B_1(0) \setminus \{ x + iy \in B_1(0) : x \in (-1, 0], y = 0 \}$.
   a) Suppose $f : B_1(0) \rightarrow \mathbb{C}$ is analytic in $G$ and continuous in $B_1(0)$. Show that $f$ is analytic in $B_1(0)$.
   b) Show that there exists an analytic function $f : G \rightarrow \mathbb{C}$ that does not extend to an analytic function on $B_1(0)$.

3. Let $n \geq 2$ and set $P_n(z) = z^n + 3z + 1$. Show that $P_n(z)$ has exactly one zero inside the unit disk, and its remaining $(n - 1)$ zeros lie in the annulus $1 < |z| < 4^{1/(n-1)}$.

4. Using residue theorem evaluate the integral
   $$\int_{-\infty}^{\infty} \frac{\cos(x)}{(1 + x^2)^2} dx.$$  
   For full credit, the answer should be given as a real number.

5. Let $\Gamma = \{ e^{i\theta} : 0 \leq \theta \leq \pi \}$.
   a) Find a conformal bijection between the set $\mathbb{C} \setminus \Gamma$ and the punctured unit disc $B_1(0)^* = \{ z : 0 < |z| < 1 \}$.
   b) Prove that if $f$ is an entire function satisfying $f(\mathbb{C}) \cap \Gamma = \emptyset$, then $f$ is a constant.

6. Consider the series $f(z) = \sum_{n=0}^{\infty} \left( \frac{-z}{z + 4} \right)^n$.
   a) Find the largest open set $G \subseteq \mathbb{C}$ so that the series converges normally in $G$.
   b) The function $f$ has an analytic continuation $F$ to $\mathbb{C} \setminus \{ z_0 \}$ for some $z_0 \in \mathbb{C}$. Find $z_0$ and the residue of $F$ at $z_0$. 