Math 542 Comprehensive Examination, January 2020

Solve five of the following six. Each problem is worth 20 points. The open ball centered at z of radius r is denoted by $B_r(z)$.

1. Let $G \subseteq \mathbb{C}$ be open and connected and $f : G \longrightarrow \mathbb{C}$ an analytic function. Suppose that there is a point in G at which f and all of its derivatives vanish. Prove that f is identically zero on G.

2. Let

$$G = B_1(0) \setminus \{x + iy \in B_1(0) : x \in (-1, 0], \mathbf{y} = \mathbf{0}\}.$$

a) Suppose $f : B_1(0) \longrightarrow \mathbb{C}$ is analytic in G and continuous in $B_1(0)$. Show that f is analytic in $B_1(0)$.

b) Show that there exists an analytic function $f : G \longrightarrow \mathbb{C}$ that does not extend to an analytic function on $B_1(0)$.

3. Let $n \geq 2$ and set $P_n(z) = z^n + 3z + 1$. Show that $P_n(z)$ has exactly one zero inside the unit disk, and its remaining $(n - 1)$ zeros lie in the annulus $1 < |z| < 4^{1/(n-1)}$.

4. Using residue theorem evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(1 + x^2)^2} dx.$$

For full credit, the answer should be given as a real number.

5. Let $\Gamma = \{e^{i\theta} : 0 \leq \theta \leq \pi\}$.

a) Find a conformal bijection between the set $\mathbb{C} \backslash \Gamma$ and the punctured unit disc

$$B_1(0)^* = \{z : 0 < |z| < 1\}.$$

b) Prove that if f is an entire function satisfying $f(\mathbb{C}) \cap \Gamma = \emptyset$, then f is a constant.

6. Consider the series

$$f(z) = \sum_{n=0}^{\infty} \big(\frac{-z}{z+4}\big)^n.$$

a) Find the largest open set $G \subset \mathbb{C}$ so that the series converges normally in G.

b) The function f has an analytic continuation F to $\mathbb{C} \setminus \{z_0\}$ for some $z_0 \in \mathbb{C}$. Find z_0 and the residue of F at z_0.