Math 540 Comprehensive Examination, January 2020

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by $m$.

1. Let $I$ be a subset of $\mathbb{R}$. Suppose that for any $\alpha \in I$, $f_\alpha$ is measurable. Is $\sup\{f_\alpha(x) : \alpha \in I\}$ measurable? Justify your answer.

2. Let $f, f_k$ be measurable functions on $E$ with $\mu(E) < \infty$. Suppose that $\lim_{k \to \infty} f_k(x) = f(x)$ a.e. $x \in E$. Prove that there exists a collection of subsets $E_n$'s of $E$, such that $f_k$ converges to $f$ uniformly on each $E_n$, and

$$\mu(E \setminus \bigcup_{n=1}^\infty E_n) = 0.$$ 

3. Let $f$ be differentiable on $[a, b] \subset \mathbb{R}$. Recall that the total variation of $f$ on $[a, b]$ is given by

$$V(f) = \sup \left\{ \sum_{j=1}^n |f(x_j) - f(x_{j-1})| : a = x_0 < x_1 < \cdots < x_n = b \right\}.$$ 

Prove that if $f'$ is Riemann-integrable on $[a, b]$, then

$$V(f) = \int_a^b |f'(x)| \, dx.$$ 

4. (i) (3 pts.) Show that $g(x) = \frac{\sin x}{x}$ is not Lebesgue integrable on $([0, \infty), m)$.

(ii) (7 pts.) Employ the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt} \, dt \quad (x > 0)$$

to evaluate the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x} \, dx = \lim_{L \to \infty} \int_0^L \frac{\sin x}{x} \, dx.$$ 

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces and let $K : X \times Y \to \mathbb{C}$ be measurable with respect to the product $\sigma$-algebra $\mathcal{M} \otimes \mathcal{N}$. Assume there is a constant $0 < C < \infty$ such that

$$\forall x \in X, \quad \int_Y |K(x, y)| \, d\nu(y) \leq C$$

and

$$\forall y \in Y, \quad \int_X |K(x, y)| \, d\mu(x) \leq C.$$
Let $p \in [1, \infty)$ and for $f \in L^p(\mu)$ define
\[(Tf)(y) := \int_X f(x)K(x,y)\,d\mu(x).\]
Prove that $Tf \in L^p(\nu)$ and
\[\|Tf\|_{L^p(\nu)} \leq C\|f\|_{L^p(\mu)}.\]

6. Let $E$ be a Lebesgue measurable subset of $\mathbb{R}$. Define the function $f_E : [0, +\infty) \to \mathbb{R}$ by
\[f_E(x) = \mu(E \cap (-x,x)).\]
where $\mu$ denotes Lebesgue’s measure on $\mathbb{R}$. Prove:
(a) $f$ is a uniformly continuous function from $[0, +\infty)$ to $[0, \mu(E)]$.
(b) $\lim_{x \to \infty} f_E(x) = \mu(E)$.