Math 500 Comprehensive Exam, January 2021

Complete all 4 problems for a maximum total of 100 points. This exam is "open book" and you may use any results from Math 500, but you must make accurate references to the theorems used in your solutions and explain your answers.

1. [30 points] Let G be a group of order 2057.

(a) Show that $G \simeq P \times Q$ where P is a group of order 17 and Q is a group of order 121. Determine all groups of order 2057 up to isomorphism.
(b) Show that $\mathrm{Aut}(G) \simeq \mathrm{Aut}(P) \times \mathrm{Aut}(Q)$.
(c) Show that if Q is cyclic, then so is $\mathrm{Aut}(Q)$. What is the order of $\mathrm{Aut}(Q)$ in this case?
(d) If Q is not cyclic, find an isomorphic description of $\mathrm{Aut}(Q)$ and compute its order.

2. [20 points]

(a) Let R be the ring of 3×3 matrices over $\mathbb{Q}$, and S denote the ring of 2×2 matrices over $\mathbb{Q}$. Is there a surjective ring homomorphism $\varphi\colon R \to S$? Justify your answer.
(b) Compute $\gcd(17 + i, 24 + 2i)$ in the ring $\mathbb{Z}[i]$.

3. [25 points] Suppose A is a 9×9 matrix over the field $\mathbb{F}_5$ with 5 elements such that the characteristic polynomial of A is $(x-1)^2(x-3)^4(x^3-1)$ and the minimal polynomial of A is $(x-1)(x-3)^3(x^3-1)$. Compute the following:

(a) The possible Jordan canonical form (or forms) of A over a suitable extension of $\mathbb{F}_5$;
(b) The possible rational canonical form (or forms) of A.

4. [25 points] Answer the following questions and provide justification.

(a) Let K be a field. Define the ring homomorphism $\varphi : \mathbb{Z} \to K$ by $\varphi(n) = n \cdot 1$. If φ is injective and $\iota : \mathbb{Z} \to \mathbb{Q}$ is the standard inclusion, prove that there exists an injective ring homomorphism $\bar{\varphi} : \mathbb{Q} \to K$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\varphi} & K \\ {\scriptstyle\iota}\downarrow & \nearrow{\scriptstyle\bar{\varphi}} & \\ \mathbb{Q} & & \end{array}$$

is commutative.
(b) Let $f(x) = x^4 + 4x^3 + 6x^2 + 4x \in \mathbb{Q}[x]$ and E be a splitting field of $f(x)$. Does $f(x)$ have four pairwise distinct roots in E?
(c) For E as in part (b), what is the order of the Galois group, $|Gal(E/\mathbb{Q})|$?
(d) For E as in part (b), is the extension $E/\mathbb{Q}$ a Galois extension?