CONGRUENCES FOR LEVEL 1 CUSP FORMS OF HALF-INTEGRAL WEIGHT

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ABSTRACT. Suppose that $\ell \geq 5$ is prime. For a positive integer N with $4 \mid N$, previous works studied properties of half-integral weight modular forms on $\Gamma_0(N)$ which are supported on finitely many square classes modulo ℓ, in some cases proving that these forms are congruent to the image of a single variable theta series under some number of iterations of the Ramanujan Θ-operator. Here, we study the analogous problem for modular forms of half-integral weight on $\mathrm{SL}_2(\mathbb{Z})$. Let η be the Dedekind eta function. For a wide range of weights, we prove that every half-integral weight modular form on $\mathrm{SL}_2(\mathbb{Z})$ which is supported on finitely many square classes modulo ℓ can be written modulo ℓ in terms of η^ℓ and an iterated derivative of η.

1. Introduction

Suppose that $\ell \geq 5$ is prime. Many papers [OS98] [Bru99] [BO03] [AB05] [AB07] [ACR09] [AR10] study half-integral weight modular forms with few non-vanishing coefficients modulo ℓ, and give applications for divisibility properties of the algebraic parts of the central critical values of modular L-functions and the orders of Tate-Shafarevich groups of elliptic curves.

These results are modulo ℓ analogues of a theorem of Vignèras in characteristic 0. If λ is a non-negative integer and N is a positive integer with $4 \mid N$, let $M_{\lambda+\frac{1}{2}}(\Gamma_1(N))$ be the space of modular forms of weight $\lambda + \frac{1}{2}$ (in the sense of [Shi73]) on $\Gamma_1(N)$. Vignèras proved that a form $F(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_1(N))$ whose coefficients are supported on finitely many square classes of integers is a linear combination of single-variable theta series. The precise result is below (Bruinier [Bru98] gave a different proof of this theorem).

Theorem 1.1. [Vig77] *Suppose that $\lambda \geq 0$ is an integer, that N is a positive integer with $4 \mid N$, and that $F(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_1(N))$. If there exist finitely many square-free integers t_1,t_2,...,t_m for which*

$$F(z) = \sum_{i=1}^{m} \sum_{n=0}^{\infty} a(t_i n^2) q^{t_i n^2}, \quad q = e^{2\pi i z}$$

then $\lambda = 0$ or 1 and $F(z)$ is a linear combination of theta series.

A recent result of Bellaïche, Green and Soundararajan [BGS18] implies for any half-integral weight modular form that the number of coefficients $\leq X$ which do not vanish modulo ℓ is $\gg \frac{\sqrt{X}}{\log\log X}$. It is natural to suspect that the only half-integral weight forms for which the number of non-vanishing coefficients is close to this upper bound are those which are supported on finitely many square classes modulo ℓ. Forms of half-integral weight on $\mathrm{SL}_2(\mathbb{Z})$ whose coefficients are sparse modulo ℓ play an important role in the recent work of Ahlgren, Beckwith and Raum [ABR20] on scarcity of congruences for the partition function.

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Ahlgren, Choi and Rouse proved a modulo ℓ analogue of Theorem 1.1 for forms $f(z)$ in the Kohnen plus-space $S^{+}_{\lambda+\frac{1}{2}}(\Gamma_0(4))$. Their main theorem was the following.

Theorem 1.2. [ACR09] *Suppose that $\ell \geq 5$ is prime, that K is a number field, and that v is a prime of K above ℓ. Let $\mathcal{O}_v$ denote the ring of v-integral elements of K. Suppose that $f \in S^{+}_{\lambda+\frac{1}{2}}(\Gamma_0(4)) \cap \mathcal{O}_v[[q]]$ satisfies*

$$f \equiv \sum_{i=1}^{m}\sum_{n=1}^{\infty} a(t_i n^2) q^{t_i n^2} \not\equiv 0 \pmod{v},$$

where each t_i is a square-free positive integer. If $\lambda + \frac{1}{2} < \ell(\ell + 1 + \frac{1}{2})$, then λ is even and

$$f \equiv a(1) \sum_{n=1}^{\infty} n^{\lambda} q^{n^2} \pmod{v}.$$

In this paper, we study the analogous question for half-integral weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$. Before we state our main result, we introduce some notation. If $\lambda \geq 0$ is an integer, N is a positive integer, and ν is a multiplier system on $\Gamma_0(N)$ in weight $\lambda + \frac{1}{2}$, we denote by $S_{\lambda+\frac{1}{2}}(N, \nu)$ the space of cusp forms of weight $\lambda + \frac{1}{2}$ and multiplier ν on $\Gamma_0(N)$ (details will be given in the next section). Let ν_η be the multiplier for the Dedekind eta function defined in (2.1). With this notation, we prove the following theorem.

Theorem 1.3. *Suppose that $\ell \geq 5$ is prime, that K is a number field, and that v is a prime above ℓ. Let $\mathcal{O}_v$ denote the ring of v-integral elements in K. Suppose that λ is a non-negative integer satisfying $\lambda + \frac{1}{2} < \frac{\ell^2}{2}$. Suppose that r is a positive integer with $(r, 6) = 1$ and that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta^r) \cap \mathcal{O}_v[[q^{\frac{1}{24}}]]$ satisfies*

$$f \equiv \sum_{i=1}^{m}\sum_{n=1}^{\infty} a(t_i n^2) q^{\frac{t_i n^2}{24}} \not\equiv 0 \pmod{v},$$

where each t_i is a square-free positive integer. Then one of the following is true.

(1) $f \equiv a(1) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) n^{\lambda} q^{\frac{n^2}{24}} \pmod{v}$.
In this case, $r \equiv 1 \pmod{24}$ and λ is even.

(2) $f \equiv a(\ell) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{v}$.
In this case, $r \equiv \ell \pmod{24}$ and $\lambda \equiv \frac{\ell-1}{2} \pmod{\ell - 1}$.

(3) $f \equiv a(1) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) n^{\lambda} q^{\frac{n^2}{24}} + a(\ell) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{v}$*, where* $a(1) \not\equiv 0 \pmod{v}$ *and* $a(\ell) \not\equiv 0 \pmod{v}$*. In this case,* $r \equiv \ell \equiv 1 \pmod{24}$ *and* $\lambda \equiv \frac{\ell-1}{2} \pmod{\ell - 1}$.

Remark. For an example of case (1) of Theorem 1.3, let $\ell \geq 5$ be prime and λ be a nonnegative integer. Lemma 3.1 below implies that there exists a form $f \in S_{(\frac{\lambda}{2})(\ell+1)}(1, \nu_\eta)$ such that $f \equiv \Theta^{\frac{\lambda}{2}}(\eta) \pmod{\ell}$, where Θ is the Ramanujan Θ-operator defined in (2.11). We have

$$f \equiv \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) n^{\lambda} q^{\frac{n^2}{24}} \pmod{\ell}.$$

For an example of case (2) of Theorem 1.3, set $f = \eta^\ell$. Since $\eta^\ell(z) \equiv \eta(\ell z) \pmod{\ell}$, we have

$$f \equiv \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{\ell}.$$

For an example of case (3), suppose that ℓ is a prime such that $\ell \equiv 1 \pmod{24}$. Lemma 3.1 implies that there exists a form $g \in S_{(\frac{\ell-1}{4})(\ell+1)+\frac{1}{2}}(1, \nu_\eta)$ such that $g \equiv \Theta^{\frac{\ell-1}{4}}(\eta) \pmod{\ell}$. Set $f = 24^{\frac{\ell-1}{2}} g + \eta^\ell E_{\ell-1}^{\frac{\ell-1}{4}}$. We have

$$f \equiv \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) n^{\frac{\ell-1}{2}} q^{\frac{n^2}{24}} + \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{\ell}.$$

For this example, note that $\frac{\ell-1}{4}(\ell+1) \equiv \frac{\ell-1}{2} \pmod{\ell-1}$.

Remark. The upper bound on λ is sharp. For an example which illustrates this, set $f = \eta^{\ell^2}$. Then

$$f \equiv \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell^2 n^2}{24}} \pmod{v}.$$

Note that we have $\lambda + \frac{1}{2} = \frac{\ell^2}{2}$ in this case.

The paper is organized as follows. In Section 2, we give some background results on modular forms of integral and half-integral weight. In Section 3, we prove some preliminary results. In Section 4, we make a preliminary reduction for the proof of Theorem 1.3, and in Section 5, we prove the theorem.

2. Background

Suppose that $k \in \frac{1}{2}\mathbb{Z}$, that N is a positive integer, and that χ is a Dirichlet character modulo N. For a function $f(z)$ on the upper half plane and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}),$$

we have the weight k slash operator

$$f(z)\big|_k \gamma := \det(\gamma)^{\frac{k}{2}} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Let K be a number field, $\ell \geq 5$ a prime, and v a prime of K above ℓ. Let $\mathcal{O}_v$ be the ring of v-integral elements of K. If ν is a multiplier system on $\Gamma_0(N)$, we denote by $M_k(N,\nu)$, $S_k(N,\nu)$ and $M_k^!(N,\nu)$ the spaces of modular forms, cusp forms, and weakly holomorphic modular forms of weight k and multiplier ν on $\Gamma_0(N)$ whose Fourier coefficients are in $\mathcal{O}_v$. When k is an integer and the multiplier ν is trivial, we write $M_k(N)$, $S_k(N)$ and $M_k^!(N)$. Forms in these spaces satisfy the transformation law

$$f\big|_k \gamma = \nu(\gamma) f \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

and the appropriate conditions at the cusps of $\Gamma_0(N)$.

Throughout, let $q := e(z) = e^{2\pi i z}$. We define the eta function by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

and the theta function by

$$\theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

The eta function has a multiplier ν_η satisfying

$$\eta(\gamma z) = \nu_\eta(\gamma)(cz+d)^{\frac{1}{2}} \eta(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z});$$

throughout, we choose the principal branch of the square root. For $c > 0$, we have the formula [Kno70, §4.1]

$$\nu_\eta(\gamma) = \begin{cases} \left(\frac{d}{c}\right) e\left(\frac{1}{24}((a+d)c - bd(c^2-1) - 3c)\right), & \text{if } c \text{ is odd}, \\ \left(\frac{c}{d}\right) e\left(\frac{1}{24}((a+d)c - bd(c^2-1) + 3d - 3 - 3cd)\right) & \text{if } c \text{ is even}. \end{cases} \tag{2.1}$$

For the multiplier of the theta function we have

$$\nu_\theta(\gamma) := (cz+d)^{-\frac{1}{2}} \frac{\theta(\gamma z)}{\theta(z)} = \left(\frac{c}{d}\right) \epsilon_d^{-1}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4, \\ i, & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

In the next several paragraphs, we follow the exposition in [ABR20]. If $f \in M_k(N, \chi\nu_\eta^r)$, then $\eta^{-r} f \in M^!_{k-\frac{r}{2}}(N, \chi)$. This implies that f has a Fourier expansion of the form

$$f = \sum_{n \equiv r(24)} a(n) q^{\frac{n}{24}}. \tag{2.2}$$

These facts together imply the following lemma.

Lemma 2.1. *Suppose that $0 < r < 24$ is an integer with $(r, 6) = 1$ and that $f \in S_k(N, \chi\nu_\eta^r)$. Then we have*

$$\eta^{-r} f \in M_{k-\frac{r}{2}}(N, \chi).$$

We also have

$$M_k(N, \chi\nu_\eta^r) \neq \{0\} \quad \text{ only if } \quad 2k - r \equiv 1 - \chi(-1) \pmod 4,$$

so if $(r, 6) = 1$, then $k \notin \mathbb{Z}$. It follows from a computation using (2.1) and the formulas

$$e\left(\frac{1-d}{8}\right) = \left(\frac{2}{d}\right) \epsilon_d \quad \text{and} \quad \epsilon_{d_1 d_2} = \epsilon_{d_1} \epsilon_{d_2} (-1)^{\frac{d_1-1}{2} \frac{d_2-1}{2}}$$

for odd d, d_1, and d_2 that we have

$$f(z) \in S_k(N, \chi\nu_\eta^r) \implies f(24z) \in S_k(576N, \chi \left(\tfrac{12}{\bullet}\right) \nu_\theta^r). \tag{2.3}$$

This fact, together with the usual Shimura lift on $S_k(576, \left(\frac{12}{\bullet}\right) \nu_\theta^r)$, allows us to define a Shimura lift Sh_t on $S_k(1, \nu_\eta^r)$ for each squarefree integer t with $(t, 6) = 1$. Its action on Fourier expansions is

$$\mathrm{Sh}_t\left(\sum a(n)q^{\frac{n}{24}}\right) = \sum A_t(n)q^n, \tag{2.4}$$

where the coefficients $A_t(n)$ are given by

$$A_t(n) = \sum_{d|n} \left(\frac{-1}{d}\right)^{k-\frac{1}{2}} \left(\frac{12t}{d}\right) d^{k-\frac{3}{2}} a\left(\frac{tn^2}{d^2}\right). \tag{2.5}$$

The work of Shimura [Shi73] and Niwa [Niw75] shows that, for each squarefree integer t with $(t, 6) = 1$, we have

$$f \in S_k(1, \nu_\eta^r) \implies \mathrm{Sh}_t(f) \in S_{2k-1}(288).$$

We make use of Hecke operators on the spaces we will consider. If k is an integer, we denote the Hecke operator on $S_k(N)$ by $T(p, k, 1)$. If $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $4 \mid N$ and r is a positive integer with $(r, 6) = 1$, we denote the Hecke operator on $S_k(N, \chi \left(\frac{12}{\bullet}\right) \nu_\theta^r)$ by $T(p^2, k, \chi)$ (these forms are half-integral weight forms in the sense of [Shi73]).

We next recall the U and V operators. For a positive integer m, we define them on Fourier expansions by

$$\left(\sum_{n=1}^{\infty} a(n)q^{\frac{n}{24}}\right) \big| U_m := \sum_{n=1}^{\infty} a(mn)q^{\frac{n}{24}},$$

$$\left(\sum_{n=1}^{\infty} a(n)q^{\frac{n}{24}}\right) \big| V_m := \sum_{n=1}^{\infty} a(n)q^{\frac{mn}{24}}.$$

The following facts appear as Lemma 2.1 in [ABR20].

Lemma 2.2. *Suppose that r is a positive integer with $(r, 6) = 1$, that $f \in M_k(N, \chi\nu_\eta^r)$, and that m is a positive integer. Then*

$$f\big|U_m = m^{\frac{k}{2}-1} \sum_{t\ (m)} f\big|_k \begin{pmatrix} 1 & 24t \\ 0 & m \end{pmatrix} \quad \text{and} \quad f\big|V_m = m^{-\frac{k}{2}} f\big|_k \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}.$$

A computation using Lemma 2.2 and (2.1) implies that if $(r, 6) = 1$ and $p \geq 5$ is prime, then

$$U_p : M_k(N, \chi\nu_\eta^r) \to M_k\left(N\frac{p}{(N,p)}, \chi\chi_p\nu_\eta^{pr}\right), \tag{2.6}$$

$$V_p : M_k(N, \chi\nu_\eta^r) \to M_k\left(Np, \chi\chi_p\nu_\eta^{pr}\right). \tag{2.7}$$

Denote by $\chi_p = \left(\frac{\bullet}{p}\right)$ the quadratic character of modulus p and by χ_p^{triv} the trivial character with modulus p. If $(r, 6) = 1$ and if $\psi = \chi_p$ or χ_p^{triv}, we define the twist of $f \in M_k(N, \chi\nu_\eta^r)$ with Fourier expansion $f = \sum_{n=1}^{\infty} a(n)q^{\frac{n}{24}}$ by

$$f \otimes \psi := \sum_{n=1}^{\infty} \psi(n)a(n)q^{\frac{n}{24}}. \tag{2.8}$$

If $p \geq 5$ is prime, the fact that $p^2 \equiv 1 \pmod{24}$ together with (2.6) and (2.7) implies that we have

$$f \in M_k(N, \chi\nu_\eta^r) \implies f \otimes \chi_p^{\text{triv}} \in M_k(Np^2, \chi\nu_\eta^r). \tag{2.9}$$

We also have

$$f \otimes \chi_p = \frac{1}{\epsilon_p \sqrt{p}} \sum_{t\ (p)} \chi_p(t) f\big|_k \begin{pmatrix} 1 & \frac{24t}{p} \\ 0 & 1 \end{pmatrix}.$$

A similar computation as for (2.6) and (2.7) shows that

$$f \otimes \chi_p \in M_k(Np^2, \chi\nu_\eta^r). \tag{2.10}$$

We next review some facts about the algebra of modular forms modulo ℓ and filtrations. Suppose that $\ell \geq 5$ is prime, that K is a number field, and that v is a prime ideal of K above ℓ. Let $\mathcal{O}_v$ be the ring of v-integral elements of K, and let the residue field be $\mathbb{F}_v$. For Fourier expansions $\sum a(n) q^{\frac{n}{24}} \in \mathcal{O}_v[[q^{\frac{1}{24}}]]$, we define $\overline{f} := \sum \overline{a(n)} q^{\frac{n}{24}} \in \mathbb{F}_v[[q^{\frac{1}{24}}]]$, and we define

$$\overline{M_k(N,\nu)} := \{\overline{f} : \quad f \in M_k(N,\nu)\},$$
$$\overline{S_k(N,\nu)} := \{\overline{f} : \quad f \in S_k(N,\nu)\}.$$

If k is an integer and $f \in M_k(N)$, then we define the filtration of f to be

$$\omega(f) = \omega(\overline{f}) := \inf\{k' : \text{ there exists } g \in M_{k'}(N) \text{ with } \overline{f} = \overline{g}\}.$$

If $f \in M_k(N,\nu)$, we define

$$\Theta := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}. \tag{2.11}$$

We make use of the following facts from [Gro90, §4].

Proposition 2.3. *Suppose that N is a positive integer, that k is an integer, that $\ell \geq 5$ is prime with $\ell \nmid N$, and that $f \in M_k(N)$. Then*

1. *There exists a form $g \in M_{k+\ell+1}(N)$ with $\overline{g} = \overline{\Theta(f)}$.*
2. *$\omega(\Theta(f)) \leq \omega(f) + \ell + 1$, with equality if and only if $\ell \nmid \omega(f)$.*
3. *$\omega(f^i) = i\omega(f)$ for all $i \geq 1$.*
4. *If $g \in M_{k'}(N)$ has $\overline{f} = \overline{g} \neq 0$, then $k \equiv k' \pmod{\ell - 1}$.*

Finally, if $k \geq 2$ is an even integer, we denote the weight k Eisenstein series by E_k.

3. Preliminary Results

We record here some preliminary results which we require for the proof of Theorem 1.3.

Lemma 3.1. *Let $\ell \geq 5$ be a prime and K is a number field. Suppose that r is a positive integer with $(r,6) = 1$ and that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta^r)$. Then there is a form $F \in S_{\lambda+\ell+1+\frac{1}{2}}(1, \nu_\eta^r)$ such that $F \equiv \Theta(f) \pmod{\ell}$.*

Proof. Choose an integer j such that $\ell j + r \equiv 0 \pmod{24}$. Define

$$g := \eta^{\ell j} f \in S_{\lambda + \frac{\ell j}{2} + \frac{1}{2}}(1).$$

There exists a form $G \in S_{\lambda+\frac{\ell j}{2}+\ell+1+\frac{1}{2}}(1)$ such that

$$G \equiv \Theta(g) \equiv \eta^{\ell j}\Theta(f) \pmod{\ell}.$$

Set $F := \frac{G}{\eta^{\ell j}}$, which vanishes at infinity. We have

$$F \equiv \Theta(f) \pmod{\ell}.$$

□

Lemma 3.2. *Suppose that $\ell \geq 5$ is prime, that K is a number field which is Galois over $\mathbb{Q}$, and that v is a prime of K above ℓ. Suppose that λ is a non-negative integer, that r is a positive integer with $(r, 6) = 1$, and that $g \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta^r)$ satisfies*

$$g \equiv \sum_{n=1}^{\infty} a(n) q^{\frac{\ell n}{24}} \not\equiv 0 \pmod{v}.$$

Then there exists $\lambda' \geq 0$ with $\lambda' + \frac{1}{2} \leq \frac{1}{\ell}(\lambda + \frac{1}{2})$, and a form $f \in S_{\lambda'+\frac{1}{2}}(1, \nu_\eta^{r\ell})$ such that

$$f \equiv g \big| U_\ell \equiv \sum_{n=1}^{\infty} a(n) q^{\frac{n}{24}} \pmod{v}.$$

Proof. Choose an integer $j > 0$ such that $\ell j + r \equiv 0 \pmod{24}$, and define

$$h := \eta^{\ell j} g \in S_{\lambda+\frac{\ell j}{2}+\frac{1}{2}}(1).$$

Suppose that $x \in \mathcal{O}_v$ and that $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ is a Frobenius automorphism for the prime v. Then we have $x^\sigma \in \mathcal{O}_v$ and

$$x^\sigma \equiv x^\ell \pmod{v}.$$

Note that σ preserves the space $S_{\lambda+\frac{\ell j}{2}+\frac{1}{2}}(1)$. Since U_ℓ acts as $T(\ell, \lambda + \frac{\ell j}{2} + \frac{1}{2}, 1)$ modulo v, we see that $\overline{h|U_\ell} \in \overline{S_{\lambda+\frac{\ell j}{2}+\frac{1}{2}}(1)}$. We have

$$\overline{h^\sigma} = (\overline{h|U_\ell})^\ell.$$

By (4) of Proposition 2.3, we know that there exists an integer $\beta \geq 0$ such that

$$k := \omega(\overline{h|U_\ell}) = \frac{1}{\ell}\omega(\overline{h^\sigma}) = \frac{1}{\ell}\left(\lambda - \beta(\ell - 1) + \frac{\ell j}{2} + \frac{1}{2}\right).$$

Therefore, there exists $H \in S_k(1)$ such that $\overline{H} = \overline{h|U_\ell} = \overline{\eta^j g|U_\ell}$. Define $f := \frac{H}{\eta^j}$. Then, we see that $f \in S_{k-\frac{j}{2}}(1, \nu_\eta^{r\ell})$, and we have $\overline{f} = \overline{g|U_\ell}$. The lemma follows since $k - \frac{j}{2} \leq \frac{1}{\ell}(\lambda + \frac{1}{2})$. □

4. Preliminary Reduction

Before proving Theorem 1.3, we reduce the number of square classes on which our forms may be supported and the number of multipliers which we must consider.

Proposition 4.1. *Suppose that $\ell \geq 5$ is prime, that K is a number field, and that v is a prime above ℓ. Suppose that λ is a non-negative integer, that r is a positive integer with $(r, 6) = 1$, and that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta^r)$. Further, suppose that*

$$f \equiv \sum_{i=1}^{m} \sum_{n=1}^{\infty} a(t_i n^2) q^{\frac{t_i n^2}{24}} \not\equiv 0 \pmod{v}, \tag{4.1}$$

where each t_i is a square-free positive integer. Then

$$f \equiv \sum_{n=1}^{\infty} a(n^2) q^{\frac{n^2}{24}} + \sum_{n=1}^{\infty} a(\ell n^2) q^{\frac{\ell n^2}{24}} \pmod{v}. \tag{4.2}$$

Proof. Fix an $i \in \{1, ..., m\}$. We may assume that there exists an integer n_i for which $a(t_i n_i^2) \not\equiv 0 \pmod{v}$. Recalling our notation (2.8) and the facts (2.9) and (2.10), we follow the argument in the proof of Lemma 4.1 of [AB05] to find primes $p_1, ..., p_n \geq 5$, each relatively prime to $n_i t_i \ell$ and a form

$$G_i \in S_{\lambda+\frac{1}{2}}(p_1^2 \cdots p_n^2, \nu_\eta^r)$$

satisfying

$$G_i \equiv \sum_{(n, \prod p_j)=1} a(t_i n^2) q^{\frac{t_i n^2}{24}} \not\equiv 0 \pmod{v}.$$

Note that

$$G_i^{24} \in S_{24\lambda+12}(p_1^2, ..., p_s^2).$$

Since

$$G_i^{24} \equiv \sum_{n=1}^{\infty} b(t_i n) q^{t_i n} \pmod{v}$$

for some coefficients $b(t_i n)$, we can apply the following result to conclude that $t_i = 1$ or ℓ.

Theorem 4.2. [ACR09, Thm 3.1] *Suppose that K is a number field and that v is a prime above ℓ with ring of v-integral elements $\mathcal{O}_v$. Suppose that k is positive integer and that*

$$f = \sum_{n=1}^{\infty} a(n) q^n \in S_{2k}(\Gamma_0(N)) \cap \mathcal{O}_v[[q]].$$

If $t > 1$ satisfies $(t, \ell N) = 1$ and

$$f \equiv \sum_{n=1}^{\infty} a(tn) q^{tn} \pmod{v},$$

then $f \equiv 0 \pmod{v}$.

□

The next result reduces the number of multipliers which we must consider.

Lemma 4.3. *Suppose that r is a positive integer with $(r, 6) = 1$ and that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta^r)$ satisfies (4.2). Then we have*

$$r \equiv 1 \pmod{24} \quad \text{or} \quad r \equiv \ell \pmod{24}. \tag{4.3}$$

Proof. Since f satisfies (4.2), it follows that either $a(n^2) \neq 0$ or $a(\ell n^2) \neq 0$ for some positive integer n. It follows from (2.2) and the fact that $r^2 \equiv 1 \pmod{24}$ whenever $(r, 6) = 1$ that we have (4.3). □

5. Proof of Theorem 1.3

The proof of Theorem 1.3 will proceed in several steps. We first consider the case when $r \equiv 1 \pmod{24}$ and λ is even.

Theorem 5.1. *Suppose that $\ell \geq 5$ is prime, that K is a number field, and that v is a prime above ℓ. Suppose that λ is a non-negative integer and that $f \in S_{\lambda+\frac{1}{2}}(1,\nu_\eta)$ has the form (4.2). If λ is even and $\lambda < 2\ell^2+\ell-1$, then*

$$\sum_{\ell \nmid n} a(n^2) q^{\frac{n^2}{24}} \equiv a(1) \sum_{\ell \nmid n} \left(\frac{12}{n}\right) n^\lambda q^{\frac{n^2}{24}} \pmod{v}.$$

Proof. Define $\overline{\lambda} := \lambda \pmod{\ell-1}$. By Lemma 3.1, we have forms $g(z) \in S_{\lambda+\ell+1+\frac{1}{2}}(1,\nu_\eta)$ and $h(z) \in S_{(\ell+1)\frac{\overline{\lambda}+2}{2}+\frac{1}{2}}(1,\nu_\eta)$ such that

$$g = \sum_{n=1}^{\infty} c(n) q^{\frac{n}{24}} \equiv \Theta(f) \equiv \sum_{n=1}^{\infty} \frac{n^2}{24} a(n^2) q^{\frac{n^2}{24}} \pmod{v}$$

and

$$h = \sum_{n=1}^{\infty} b(n) q^{\frac{n}{24}} \equiv 24^{\frac{\overline{\lambda}}{2}} a(1) \Theta^{\frac{\overline{\lambda}+2}{2}}(\eta) \equiv a(1) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) \frac{n^{\overline{\lambda}+2}}{24} q^{\frac{n^2}{24}} \pmod{v}.$$

It suffices to show that $g \equiv h \pmod{v}$. To this end, we make use of this theorem, which follows from an argument of Bruinier and Ono [BO03, Thm 3.1] (see [ACR09, Thm 2.1]).

Theorem 5.2. *Suppose that N is a positive integer with $4 \mid N$. Suppose that $\ell \geq 5$ is prime, that K is a number field, and that v is a prime of K above ℓ. Suppose that λ is a non-negative integer and that r is a positive integer with $(r,6)=1$. Suppose that*

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n \in S_{\lambda+\frac{1}{2}}(N, \chi \left(\tfrac{12}{\bullet}\right) \nu_\theta^r),$$

that $\ell \nmid N$, and that $p \nmid N\ell$ is prime. If there exists $\epsilon_p \in \{\pm 1\}$ such that

$$f(z) \equiv \sum_{\left(\frac{n}{p}\right) \in \{0,\epsilon_p\}} a(n) q^n \pmod{v},$$

then we have

$$(p-1) f(z) \big| T\left(p^2, \lambda+\frac{1}{2}, \chi\right) \equiv \epsilon_p \chi(p) \left(\tfrac{(-1)^\lambda}{p}\right) (p^\lambda + p^{\lambda-1})(p-1) f(z) \pmod{v}.$$

By (2.3), we can apply Theorem 5.2 to $g(24z)$ to conclude that, for odd primes $p \geq 5$ with $p \not\equiv 0, 1 \pmod{\ell}$, we have

$$g(24z) \big| T\left(p^2, \lambda+\ell+1+\frac{1}{2}, 1\right) \equiv \left(\frac{12}{p}\right) (p^{\overline{\lambda}+2} + p^{\overline{\lambda}+1}) g(24z) \pmod{v}. \qquad (5.1)$$

Suppose that n is a positive integer satisfying $(n,6)=1$ which is divisible only by primes $p \not\equiv 0, 1 \pmod{\ell}$. If p is such a prime, write $n = p^a n_0$ if $p^a \mid\mid n$. The definition of the Hecke operator on $S_{\lambda+\ell+1+\frac{1}{2}}(576, (\frac{12}{\bullet})\nu_\theta^r)$ implies that we have

$$c(n^2p^2) + p^{\overline{\lambda}+1} \left(\frac{12n^2}{p}\right) c(n^2) + p^{2\overline{\lambda}+3} c\left(\frac{n^2}{p^2}\right) \equiv \left(\frac{12}{p}\right) (p^{\overline{\lambda}+2} + p^{\overline{\lambda}+1}) c(n^2) \pmod{v},$$

and an induction argument on a then implies that

$$c(p^{2a}n_0^2) \equiv \left(\frac{12}{p}\right)^a p^{a(\overline{\lambda}+2)} c(n_0^2) \pmod{v}.$$

Thus, we have

$$c(n^2) \equiv \left(\frac{12}{n}\right) n^{\overline{\lambda}+2} c(1) \equiv \left(\frac{12}{n}\right) \frac{n^{\overline{\lambda}+2}}{24} a(1) \equiv b(n^2) \pmod{v}.$$

This shows that the coefficients $c(n^2)$ and $b(n^2)$ agree whenever n is a positive integer such that $(n, 6) = 1$ which is divisible only by primes $p \geq 5$ with $p \not\equiv 0, 1 \pmod{\ell}$.

Now define

$$k := \max\{\lambda + \ell + 1, (\ell + 1)\frac{\overline{\lambda} + 2}{2}\}.$$

These numbers agree modulo $\ell - 1$ by virtue of λ being even, so by multiplying g or h by an appropriate power of $E_{\ell-1} \equiv 1 \pmod{\ell}$, we see that there exist forms g_1 and h_1 in $S_{k+\frac{1}{2}}(1, \nu_\eta)$ such that $g_1 \equiv g \pmod{v}$ and $h_1 \equiv h \pmod{v}$. Thus, to prove the theorem, it suffices to show that $g_1 \equiv h_1 \pmod{v}$. Note that $c(n^2) \equiv b(n^2) \equiv 0 \pmod{v}$ for positive integers n such that $(n, 6) \neq 1$ by (2.2), and that $c(n^2)$ and $b(n^2)$ vanish modulo ℓ whenever n is divisible by ℓ. Thus, $\eta^{-1}(g_1 - h_1) \in S_k(1)$ is of the form

$$\eta^{-1}(g_1 - h_1) \equiv c q^{\frac{\ell^2+\ell}{6}} + \cdots \pmod{v}$$

for some $c \in \mathcal{O}_v$. To prove that $g_1 \equiv h_1 \pmod{v}$, it suffices to show by [Stu87, Thm 1] that

$$\frac{k}{12} < \frac{\ell^2 + \ell}{6}.$$

Since $\overline{\lambda} < \ell$, we have

$$\frac{(\ell + 1)(\overline{\lambda} + 2)}{24} < \frac{\ell^2 + \ell}{6}.$$

Since $\lambda < 2\ell^2 + \ell - 1$, we have

$$\frac{\lambda + \ell + 1}{12} < \frac{\ell^2 + \ell}{6}.$$

The result follows. □

We now consider what happens when λ is odd.

Proposition 5.3. *Suppose that $\ell \geq 5$ is prime, that K is a number field, and that v is a prime of K above ℓ. Suppose that λ is a non-negative integer, that r is a positive integer with $(r, 6) = 1$, and that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta^r)$ has the form*

$$f \equiv \sum_{n=1}^{\infty} a(n^2) q^{\frac{n^2}{24}} + \sum_{n=1}^{\infty} a(\ell n^2) q^{\frac{\ell n^2}{24}} \not\equiv 0 \pmod{v}.$$

If λ is odd, then $\Theta(f) \equiv 0 \pmod{v}$.

Proof. Suppose by way of contradiction that $\Theta(f) \not\equiv 0 \pmod{v}$. By Lemma 3.1, there exists $g \in S_{\lambda+\ell+1+\frac{1}{2}}(1, \nu_\eta^r)$ such that

$$g \equiv \sum_{n=1}^{\infty} \frac{n^2}{24} a(n^2) q^{\frac{n^2}{24}} \not\equiv 0 \pmod{v},$$

so there exists n_0 such that $a(n_0^2) \neq 0$. By (2.2), we have $r \equiv 1 \pmod{24}$. By Lemma 2.1, we then have $\eta^{-1} f \in M_\lambda(1) = \{0\}$, which is a contradiction. Thus, $\Theta(f) \equiv 0 \pmod{v}$. □

We require one more result before proving Theorem 1.3.

Proposition 5.4. *Suppose that $\ell \geq 5$ is prime, that K is a number field which is Galois over $\mathbb{Q}$, and that v is a prime of K above ℓ. Suppose that r is a positive integer with $(r, 6) = 1$ and that $g \in S_{\lambda'+\frac{1}{2}}(1, \nu_\eta^r)$ satisfies*

$$g \equiv \sum_{n=1}^{\infty} a(n^2) q^{\frac{n^2}{24}} + \sum_{n=1}^{\infty} a(\ell n^2) q^{\frac{\ell n^2}{24}} \not\equiv 0 \pmod{v}.$$

If $\lambda' < \frac{\ell-1}{2}$, then $\lambda' = 0$, $r = 1$, and $g = c\eta$ for some $c \in \mathcal{O}_v$.

Proof. First assume that $\lambda' = 0$. Assume without loss of generality that $0 < r < 24$. By Lemma 2.1, we have $\eta^{-r} g \in M_{\frac{1}{2}-\frac{r}{2}}(1)$, from which $r = 1$ follows. Thus, $\eta^{-1} g \in M_0(1) = \mathcal{O}_v$, and the result follows.

Now assume that $1 \leq \lambda' < \frac{\ell-1}{2}$. Suppose first that λ' is odd. Proposition 5.3 implies that $\Theta(g) \equiv 0 \pmod{v}$, so we have

$$g \equiv \sum_{n=1}^{\infty} a(\ell n^2) q^{\frac{\ell n^2}{24}} + \sum_{n=1}^{\infty} a(\ell^2 n^2) q^{\frac{\ell^2 n^2}{24}} \pmod{v}.$$

By Lemma 3.2, there exists $f \in S_{\lambda^*+\frac{1}{2}}(1, \nu_\eta^{r\ell})$ with

$$f \equiv \sum_{n=1}^{\infty} a(\ell n^2) q^{\frac{n^2}{24}} + \sum_{n=1}^{\infty} a(\ell^2 n^2) q^{\frac{\ell n^2}{24}} \pmod{v},$$

and $\lambda^* + \frac{1}{2} \leq \frac{1}{\ell}(\lambda' + \frac{1}{2})$. Since $\lambda^* \geq 0$, we have $\frac{1}{2} \leq \frac{1}{\ell}(\lambda' + \frac{1}{2})$, which would imply that $\lambda' \geq \frac{\ell-1}{2}$, so λ' cannot be odd.

Now assume that $2 \leq \lambda' < \frac{\ell-1}{2}$ and λ' is even. Applying Theorem 5.1, we see that

$$\Theta^{\ell-1}(g) \equiv a(1) \sum_{\ell \nmid n} \left(\frac{12}{n}\right) n^{\lambda'} q^{\frac{n^2}{24}} \pmod{v},$$

which implies that

$$a(n^2) \equiv a(1) \left(\frac{12}{n}\right) n^{\lambda'} \pmod{v} \quad \text{if} \quad \ell \nmid n. \tag{5.2}$$

We show that $a(1) \equiv 0 \pmod{\ell}$. Assume to the contrary that $a(1) \not\equiv 0 \pmod{\ell}$. Consider the Shimura lift

$$G := \mathrm{Sh}_1(g) = \sum_{n=1}^{\infty} b(n) q^n \in S_{2\lambda'}(288) \tag{5.3}$$

from (2.4). If $\ell \nmid n$, then (2.5) and (5.2) give

$$b(n) \equiv a(1) \left(\frac{12}{n}\right) \sum_{d|n} d^{\lambda'-1} \left(\frac{n}{d}\right)^{\lambda'} = a(1) \left(\frac{12}{n}\right) n^{\lambda'-1} \sigma_1(n). \tag{5.4}$$

This is congruent to the n^{th} coefficient of

$$-\frac{a(1)}{24}\Theta^{\lambda'-1}\left(E_{\ell+1}\otimes\left(\frac{12}{\bullet}\right)\right)\equiv-\frac{a(1)}{24}\Theta^{\lambda'-1}\left(E_2\otimes\left(\frac{12}{\bullet}\right)\right)\pmod{\ell}.$$

Note that $E_2\otimes\left(\frac{12}{\bullet}\right)=(E_2-2E_2|V_2)\otimes\left(\frac{12}{\bullet}\right)$. This implies that the filtration of $E_2\otimes\left(\frac{12}{\bullet}\right)$ on level 288 is 2, since $E_2-2E_2|V_2\in M_2(2)$ [DS05, §1.2]. By (2) of Proposition 2.3, we have

$$\omega\left(\overline{\Theta^{\lambda'-1}\left(E_{\ell+1}\otimes\left(\frac{12}{\bullet}\right)\right)}\right)=\ell(\lambda'-1)+\lambda'+1.$$

We also have

$$G\equiv GE_{\ell-1}^{\lambda'-1}\in S_{\ell(\lambda'-1)+\lambda'+1}(288).$$

By (1) of Proposition 2.3, there exists $H\in S_{\ell(\lambda'-1)+\lambda'+1}(288)$ with

$$\overline{H}=\overline{GE_{\ell-1}^{\lambda'-1}+\frac{a(1)}{24}\Theta^{\lambda'-1}\left(E_{\ell+1}\otimes\left(\frac{12}{\bullet}\right)\right)}\in\overline{S_{\ell(\lambda'-1)+\lambda'+1}(288)}.$$

Let $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$ be a Frobenius element for v. Note that $\overline{H}$ has the form $\sum\overline{a(n)}q^{\ell n}$, so we have

$$\overline{H}^{\sigma}=(\overline{H|U_\ell})^{\ell}.$$

By (3) of Proposition 2.3, we have

$$\omega(\overline{H})=\omega(\overline{H}^{\sigma})=\ell\omega(\overline{H|U_\ell}).$$

If $a(1)\not\equiv 0\pmod{\ell}$, then $\omega(\overline{\frac{a(1)}{24}\Theta^{\lambda'-1}\left(E_{\ell+1}\otimes\left(\frac{12}{\bullet}\right)\right)})>\omega(\overline{GE_{\ell-1}^{\lambda'-1}})$ since $\omega(GE_{\ell-1}^{\lambda'-1})\leq 2\lambda'$ and λ' is even. This would imply that $\omega(\overline{H})=\omega(\overline{\Theta^{\lambda'-1}\left(E_{\ell+1}\otimes\left(\frac{12}{\bullet}\right)\right)})=\ell(\lambda'-1)+\lambda'+1$. This contradicts the fact that $\omega(\overline{H})$ is a multiple of ℓ. Thus, $a(1)\equiv 0\pmod{\ell}$. By (5.4), we have $\Theta(g)\equiv 0\pmod{v}$. The result now follows as in the odd case.

□

Now we prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that $\ell\geq 5$ is prime, that K is a number field, and that v is a prime of K above ℓ. We may assume that K is Galois over $\mathbb{Q}$. Suppose that r is a positive integer satisfying $(r,6)=1$, that λ is a non-negative integer satisfying $\lambda+\frac{1}{2}<\frac{\ell^2}{2}$, and that $f\in S_{\lambda+\frac{1}{2}}(1,\nu_\eta^r)$ has the property that

$$f\equiv\sum_{i=1}^{m}\sum_{n=1}^{\infty}a(t_in^2)q^{\frac{t_in^2}{24}}\not\equiv 0\pmod{v}.$$

By Proposition 4.1 and Lemma 4.3, we may assume that

$$f\equiv\sum_{n=1}^{\infty}a(n^2)q^{\frac{n^2}{24}}+\sum_{n=1}^{\infty}a(\ell n^2)q^{\frac{\ell n^2}{24}}\not\equiv 0\pmod{v}$$

and that either $r\equiv 1\pmod{24}$ or $r\equiv\ell\pmod{24}$. So, we need only consider the cases when $f\in S_{\lambda+\frac{1}{2}}(1,\nu_\eta)$ and when $f\in S_{\lambda+\frac{1}{2}}(1,\nu_\eta^\ell)$ with $\ell\not\equiv 1\pmod{24}$.

Suppose that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta)$. Assume that λ is even. If $\lambda = 0$, then $f = c\eta$ for some $c \in \mathcal{O}_v$. This implies that

$$f = a(1) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{n^2}{24}},$$

which has the form of case (1) of Theorem 1.3, so assume that $\lambda > 0$. Theorem 5.1 then implies that

$$\Theta^{\ell-1}(f) = \sum_{\ell \nmid n} a(n^2) q^{\frac{n^2}{24}} \equiv a(1) \sum_{\ell \nmid n} \left(\frac{12}{n}\right) n^\lambda q^{\frac{n^2}{24}} \pmod{v}. \tag{5.5}$$

Define $\overline{\lambda} := \lambda \pmod{\ell - 1}$. By Lemma 3.1, we have

$$\overline{\Theta^{\ell-1}(f)} = \overline{24^{\overline{\lambda}} a(1) \Theta^{\frac{\overline{\lambda}}{2}}(\eta)} \in \overline{S_{\frac{\overline{\lambda}}{2}(\ell+1)+\frac{1}{2}}(1, \nu_\eta)}. \tag{5.6}$$

The fact that

$$\left(\frac{\overline{\lambda}}{2}\right)(\ell+1) + \frac{1}{2} < \frac{\ell^2}{2}$$

implies that $\overline{f - \Theta^{\ell-1}(f)} \in \overline{S_{\lambda'+\frac{1}{2}}(1, \nu_\eta)}$, where $\lambda' + \frac{1}{2} < \frac{\ell^2}{2}$. Since

$$f - \Theta^{\ell-1}(f) \equiv \sum_{n=1}^{\infty} a(\ell n) q^{\frac{\ell n^2}{24}} + \sum_{n=1}^{\infty} a(\ell^2 n^2) q^{\frac{\ell^2 n^2}{24}} \pmod{v},$$

we apply Lemma 3.2 to conclude that there exists $g \in S_{\lambda^*+\frac{1}{2}}(1, \nu_\eta^\ell)$ with $\lambda^* + \frac{1}{2} < \frac{\ell}{2}$ satisfying

$$g \equiv \left(f - \Theta^{\ell-1}(f)\right) \big| U_\ell \equiv \sum_{n=1}^{\infty} a(\ell n^2) q^{\frac{n^2}{24}} + \sum_{n=1}^{\infty} a(\ell^2 n^2) q^{\frac{\ell n^2}{24}} \pmod{v}.$$

If $g \equiv 0 \pmod{v}$, then

$$f \equiv \Theta^{\ell-1}(f) \pmod{v},$$

and, by (5.5), this proves that

$$f \equiv a(1) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) n^\lambda q^{\frac{n^2}{24}} \pmod{v}.$$

This has the form of case (1) of Theorem 1.3.

If $g \not\equiv 0 \pmod{v}$, then Proposition 5.4 implies that $\lambda' = 0$ and $g = c\eta$ for some $c \in \mathcal{O}_v$, which means that

$$f - \Theta^{\ell-1}(f) \equiv a(\ell) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{v}.$$

Thus, we have

$$f \equiv a(1) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) n^\lambda q^{\frac{n^2}{24}} + a(\ell) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{v}. \tag{5.7}$$

Since $g \not\equiv 0 \pmod{v}$, we have $a(\ell) \not\equiv 0 \pmod{v}$. Proposition 5.4 applied to f implies that $\ell \equiv 1 \pmod{24}$. If $a(1) \equiv 0 \pmod{v}$, then (5.7) has the form of case (2) of Theorem 1.3. This is equivalent to the congruence

$$f \equiv a(\ell)\eta^\ell \pmod{v}.$$

By (4) of Proposition 2.3, we have $\lambda \equiv \omega(\overline{\eta^{-1}f}) \equiv \omega(\overline{\eta^{\ell-1}}) \equiv \frac{\ell-1}{2} \pmod{\ell-1}$.

If $a(1) \not\equiv 0 \pmod{v}$, then (5.7) has the form of case (3) of Theorem 1.3. This is equivalent to the congruence

$$f \equiv 24^{\lambda} a(1)\Theta^{\frac{\lambda}{2}}(\eta) + a(\ell)\eta^{\ell} \pmod{v}.$$

By (4) of Proposition 2.3, we have $\omega(\overline{\eta^{-1}f}) \equiv \omega(\overline{\eta^{-1}\Theta^{\frac{\lambda}{2}}(\eta)}) \equiv \lambda \pmod{\ell-1}$. This implies that $\omega(\overline{\eta^{\ell-1}}) \equiv \lambda \pmod{\ell-1}$. Since $\omega(\overline{\eta^{\ell-1}}) = \frac{\ell-1}{2}$, we have $\lambda \equiv \frac{\ell-1}{2} \pmod{\ell-1}$.

Now assume that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta)$ and that λ is odd. By Proposition 5.3, we have

$$f \equiv \sum a(\ell n) q^{\frac{\ell n}{24}} \pmod{\ell}.$$

By Lemma 3.2 (since $\lambda + \frac{1}{2} < \frac{\ell^2}{2}$), there exists $g \in S_{\lambda'+\frac{1}{2}}(1, \nu_\eta^{\ell})$ such that $g \equiv f\big|U_\ell \pmod{v}$ and $\lambda' + \frac{1}{2} < \frac{\ell}{2}$. Proposition 5.4 implies that $\lambda' = 0$ and $g = c\eta$ for some $c \in \mathcal{O}_v$. Thus,

$$f \equiv a(\ell) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{v}.$$

Since $f \not\equiv 0 \pmod{v}$ has a Fourier expansion of the form (2.2), we have $r \equiv \ell \equiv 1 \pmod{24}$ in this case. As above, this implies that $\lambda \equiv \frac{\ell-1}{2} \pmod{\ell-1}$, which implies that λ is even. This is a contradiction, so λ cannot be odd in this case.

Finally, suppose that $\ell \not\equiv 1 \pmod{24}$ and that $f \in S_{\lambda+\frac{1}{2}}(1, \nu_\eta^{\ell})$. By (2.2), we have

$$f \equiv \sum_{n=1}^{\infty} a(\ell n^2) q^{\frac{\ell n^2}{24}} \pmod{v}.$$

By Lemma 3.2, there is $g \in S_{\lambda'+\frac{1}{2}}(1, \nu_\eta)$ with $\lambda' + \frac{1}{2} < \frac{\ell}{2}$ such that $g \equiv f\big|U_\ell \pmod{v}$. Proposition 5.4 implies that $\lambda' = 0$ and that $g = c\eta$ for some $c \in \mathcal{O}_v$. Thus,

$$f \equiv a(\ell) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{\ell n^2}{24}} \pmod{v}.$$

This has the form of case (2) of Theorem 1.3. As above, we have $\lambda \equiv \frac{\ell-1}{2} \pmod{\ell-1}$. □

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References

[AB05] Scott Ahlgren and Matthew Boylan, *Coefficients of half-integral weight modular forms modulo l^j*, Math. Ann. **331** (2005), no. 1, 219–239. MR 2107445

[AB07] ———, *Central critical values of modular L-functions and coefficients of half-integral weight modular forms modulo l*, Amer. J. Math. **129** (2007), no. 2, 429–454. MR 2306041

[ABR20] Scott Ahlgren, Olivia Beckwith, and Martin Raum, *Scarcity of congruences for the partition function*, Preprint, `https://arxiv.org/pdf/2006.07645.pdf` (2020).

[ACR09] Scott Ahlgren, Dohoon Choi, and Jeremy Rouse, *Congruences for level four cusp forms*, Math. Res. Lett. **16** (2009), no. 4, 683–701. MR 2525033

[AR10] Scott Ahlgren and Jeremy Rouse, *Critical L-values of level p newforms (mod p)*, Int. Math. Res. Not. IMRN (2010), no. 9, 1575–1606. MR 2643576

[BGS18] Joël Bellaïche, Ben Green, and Kannan Soundararajan, *Nonzero coefficients of half-integral weight modular forms* $\bmod \ell$, Res. Math. Sci. **5** (2018), no. 1, Paper No. 6, 10. MR 3757164

[BO03] Jan H. Bruinier and Ken Ono, *Coefficients of half-integral weight modular forms*, J. Number Theory **99** (2003), no. 1, 164–179. MR 1957250

[Bru98] J. H. Bruinier, *On a theorem of Vignéras*, Abh. Math. Sem. Univ. Hamburg **68** (1998), 163–168. MR 1658385

[Bru99] Jan Hendrik Bruinier, *Nonvanishing modulo l of Fourier coefficients of half-integral weight modular forms*, Duke Math. J. **98** (1999), no. 3, 595–611. MR 1695803

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196

[Gro90] Benedict H. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61** (1990), no. 2, 445–517. MR 1074305

[Kno70] Marvin I. Knopp, *Modular functions in analytic number theory*, Markham Publishing Co., Chicago, Ill., 1970. MR 0265287

[Niw75] Shinji Niwa, *Modular forms of half integral weight and the integral of certain theta-functions*, Nagoya Math. J. **56** (1975), 147–161. MR 364106

[OS98] Ken Ono and Christopher Skinner, *Fourier coefficients of half-integral weight modular forms modulo l*, Ann. of Math. (2) **147** (1998), no. 2, 453–470. MR 1626761

[Shi73] Goro Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481. MR 332663

[Stu87] Jacob Sturm, *On the congruence of modular forms*, Number theory (New York, 1984–1985), Lecture Notes in Math., vol. 1240, Springer, Berlin, 1987, pp. 275–280. MR 894516

[Vig77] M.-F. Vignéras, *Facteurs gamma et équations fonctionnelles*, Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 1977, pp. 79–103. Lecture Notes in Math., Vol. 627. MR 0485739

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