Course Descriptions
Spring 2021

Department of Mathematics
University of Illinois

Math 500

Professor Dutta

Basic facts about groups, rings, vector spaces, such as those covered in Math 416 and Math 417 courses, are assumed. Instructors should not spend time on elementary material: the syllabus is quite full. Books that could be used include "Abstract Algebra" by Dummit and Foote, "Algebra" by Hungerford, and "Advanced Modern Algebra" by Rotman.

1. Group Theory. [Approximately 4.5 weeks]
(a) Isomorphism theorems for groups.
(b) Group actions on sets; orbits, stabilizers. Application to conjugacy classes, centralizers, normalizers.
(c) The class equation with application to finite p-groups and the simplicity of A_5.
(d) Composition series in a group. Refinement Theorem and Jordan-Hölder Theorem. Solvable and nilpotent groups.
(e) Sylow Theorems and applications.

2. Commutative rings and Modules.[Approximately 5 weeks]
(a) Review of subrings, ideals and quotient rings. Integral domains and fields. Polynomial rings over a commutative ring.
(b) Euclidean rings, PID's, UFD's.
(c) Brief introduction to modules (over commutative rings), submodules, quotient modules.
(d) Free modules, invariance of rank. Torsion modules, torsion free modules. Primary decomposition theorem for torsion modules over PID's.
(e) Structure theorem for nitely generated modules over a PID. Application to finitely generated Abelian groups and to canonical form of matrices.
(f) Zorn's lemma and Axiom of Choice (no proofs). Application to maximal ideals, bases of vector spaces.

3. Field Theory. [Approximately 5 weeks]
(a) Prime fields, characteristic of a field.
(b) Algebraic and transcendental extensions, degree of an extension. Irreducible polynomial of an algebraic element.
(c) Normal extensions and splitting fields. Galois group of an extension.
(d) Algebraic closure, existence and uniqueness via Zorn's Lemma. Finite fields.
(e) Fundamental theorem of Galois theory.
(f) Examples of Galois extensions. Cyclotomic extensions.
(g) If time permits, application of Galois theory to solution of polynomial equations, symmetric functions and ruler and compass constructions.

Math 501

Professor Duursma

1. Modules over non-commutative rings.
Review of left and right ideals in rings. Submodules and quotient modules, homomorphisms. Direct sums and products. Limits and colimits.

2. Categories.
Examples of categories and functors. Natural transformations. Products and coproducts. Free objects in a category.

3. More module theory.
Free modules and the Invariant Basis Property. Exact sequences of modules. Left exactness of Hom. Projective and injective modules. Chain conditions and composition series. Hilbert's Basis Theorem.

4. Semisimple rings.
Semisimple rings and modules. Group rings and Maschke's theorem. Wedderburn's theorem on semisimple artinian rings. Application to representations of finite groups.

5. Tensor products and multilinear algebra.
Tensor products of modules. Mapping property of tensor products. Right exactness of tensor products. Flat modules. Exterior and symmetric products.

As time allows, either 6 or 7 below or another topic at instructor's choice.

6. Introduction to homological algebra.
Complexes and resolutions. Denitions of Ext and Tor. Long exact sequences for Ext and Tor.

7. Introduction to algebraic geometry.
Affine varieties, the Nullstellensatz, Zariski topology, irreducible varieties.

Note. Since Math 501 is no longer a comp course, instructors have some latitude in their choice of class material. However, the major topics in 1-5 above should be covered. Instructors may substitute for 6 and 7 another topic of their choice in the general area.

Books that could be used include "Abstract Algebra" by Dummitt and Foote, "Algebra" by Hungerford, and "Advanced Modern Algebra" by Rotman.

Math 511

Professor Heleodoro

The course will prepare students for research in algebraic geometry and related areas including number theory, algebraic combinatorics, logic, and complex geometry.

Topics include:

- Plane Algebraic Curves
    - Smooth curves, local parameter
    - Function field of a curve
    - Rational curves
    - Bezout's Theorem
- Affine algebraic sets and varieties
- Irreducible components
- Regular and rational functions
- Regular and rational maps
- Projective and quasiprojective varieties
    - Projective spaces
    - Hypersurfaces and complete intersections
    - Grassmannians
- Products of varieties
- Normal varieties; normalization
- Dimension
- Singular and nonsingular points; tangent spaces
- Birational maps; blowups
- Divisors
    - Cartier and Weil divisors
    - Linear systems
    - Maps to projective space
- The group law on a plane cubic

Possible texts:
Shafarevich, Basic Algebraic Geometry, Volume 1

Math 519

Professor Fernandes

Syllabus:

- **Riemannian Geometry.** Riemannian metrics, covariant derivatives, parallel transport, geodesics, Hopf-Rinow Theorem. Curvature tensors, first and second variation formulas, Jacobi fields, Myers' and Hadamard's Theorems, Gauss-Bonet Theorem. Hodge-star operator, Laplace operator, harmonica forms, Hodge-de Rham theorem.
- **Fiber Bundles.** Vector bundles, principal bundles, connections, parallel transport, curvature, Chern-Weyl theory, Poincaré-Hopf Theorem.

Recommended Textbooks:

- S. Gallot, D. Hulin and J. Lafontaine, Riemannian Geometry third edition. Springer-Verlag, Universitytext 2004.
- J. Jost, Riemannian Geometry and Geometric Analysis 6th Edition, Springer-Verlag, Universitytext 2011.
- S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol 1 and 2 New York, Interscience Publishers, 1963.
- C. Taubes, Differential Geometry: Bundles, Connections, Metrics and Curvature Oxford University Press, 2011.

Grading Policy

- **Homework:** There will be 5 homework assignments. The grade of the course will be based on these homework sets.
- **Problem Sessions:** I will hold a problem session before each homework assignment is due.

Math 522

Professor Haboush

(1) A summary of the theory of finite group representations, including the symmetric group. Review of multilinear algebra.

(2) Linear Lie groups - definition and examples. Lie groups as manifolds.

(3) Lie algebras: As the tangent space at the identity to the Lie group, and also the definition as abstract objects. The exponential map. Examples, esp. *sln*, definition of simple Lie algebras.

(4) Representations of Lie groups and Lie algebras: Definition and some examples.

(5) Integration on groups, Peter-Weyl theorem.

ALGEBRAIC TOPOLOGY I

Igor Mineyev. Math 525, Spring 2021.
`https://faculty.math.illinois.edu/~mineyev/class/21s/525/` (to appear)
Textbook: *Algebraic topology*, by Allen Hatcher. Freely available online at
`www.math.cornell.edu/~hatcher/AT/ATpage.html`

The tentative syllabus. This is the official syllabus, we will roughly follow it in the course. Fundamental group and covering spaces.

(1) Definition of the fundamental group.
(2) Covering spaces and lifts of maps.
(3) Computing the fundamental group via covering spaces.
(4) Applications, such as the Fundamental Theorem of Algebra and the Brouwer fixed point theorem in 2d.
(5) Deforming spaces: retraction and homotopy equivalence.
(6) Quotient topology and cell complexes.
(7) Homotopy extension property and applications to homotopy equivalence.
(8) Fundamental groups of CW complexes.
(9) Van Kampen's Theorem.
(10) Covering spaces and subgroups of the fundamental group.
(11) Universal covers.
(12) The definitive lifting criterion, classification of covering spaces.
(13) Covering transformations and regular covers.

Homology.

(14) Delta complexes and their cellular homology.
(15) Singular homology.
(16) Homotopic maps and homology.
(17) The long exact sequence of the pair.
(18) Relative homology and excision.
(19) Equality of cellular and singular homology.
(20) Applications, such as degree of maps of spheres, invariance of dimension, and the Brouwer fixed point theorem.
(21) Homology of CW complexes.
(22) Homology and the fundamental group: the Hurewicz theorem.
(23) Euler characteristic.
(24) Homology with coefficients.
(25) Intro to categories and axiomatic characterization of homology theories.
(26) Further applications, such as the Jordan curve theorem, wild spheres, invariance of domain.

MATH 527 Homotopy Theory
Spring 2021

Instructor: Jeremiah Heller (jbheller@illinois.edu)

Course description: This is an introduction to homotopy theory. Topics will include

- classical homotopy theory of spaces
- cofiber, fiber sequences
- homotopy limits and colimits
- spectral sequences
- Brown representability

Other possible topics (depending on interest) include: simplicial methods, model categories, introduction to ∞-categories, spectra and stable homotopy theory.

Prerequisites: MATH 526 or consent of instructor.

Course logistics: All aspects of this course will be carried out virtually. Lectures and student presentations will be conducted (synchronously) via Zoom. Materials will also be recorded and posted for students who can't attend.

Required technology: To participate in this course you will need:

- computer or tablet,
- webcam, speakers/headphone and microphone,
- reasonable internet connection for downloading course content and participating in course discussions and office hours.

Textbook: We'll use the following text:

- *Modern Classical Homotopy Theory*, by J. Strom. (freely available through our library)

A couple more useful texts:

- *Algebraic Topology*, by Hatcher. Free pdf available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`.
- *Geometry and Topology*, by Bredon. Free pdf available through our library.

Math 530
Shiang Tang

This course is an introduction to algebraic number theory. You will learn the basic concepts and fundamental objects in algebraic number theory. Topics include Number Fields and Number Rings, Arithmetic of Dedekind Domains, Minkowski Theory and Dirichlet Unit Theorem, Local Fields, Global Fields and Galois Theory. We will also discuss additional topics if time permits.

SPRING 2021, MATH 532, ANALYTIC THEORY OF NUMBERS II : MULTIPLICATIVE NUMBER THEORY

INSTRUCTOR: ALEXANDRU ZAHARESCU

Math 532, D1, TR 12:30 - 1:50 PM

In this course we will discuss ideas from multiplicative number theory. The largest part of the course will cover classical material focussing on the Riemann zeta function and Dirichlet $L-$functions. For this part we will follow Davenport's book. In the last part of the course we will study some recent papers on the distribution of zeros of the Riemann zeta function and more general $L-$functions.

Prerequisite: MATH 531.

Recommended Textbook:

Harold Davenport, Multiplicative number theory. Third edition. Graduate Texts in Mathematics, 74. Springer-Verlag, New York, 2000. xiv+177 pp. ISBN: 0-387-95097-4

There will be no exams. Students registered for this course will be expected to give a couple of lectures on some topics related to the content of the course. In addition some homework problems will be assigned.

Office hours : Tuesday - Thursday 1:50 - 3pm.

Office: 449 Altgeld Hall.

E-mail: *zaharesc@illinois.edu*

Math 553: Partial Differential Equations

Zhao Yang

spring, 2021

Course Info

Online classes: Zoom meetings on MWF 1:00 pm - 1:50 pm.
Office Hours: by appointment.

Course Description

We will use the Evans' PDEs book throughout the semester. Particularly, we will cover the first four chapters of the book to understand:

i. some principal theoretical issues concerning the solving of PDEs;

ii. four fundamental linear PDEs for which various explicit formulas for solutions are available;

iii. methods used in solving general nonlinear first-order PDEs;

iv. some other useful techniques used in finding explicit solutions of certain PDEs.

Required Materials

- Partial Differential Equations, 2^{nd} edition, by Lawrence C. Evans

Prerequisites

Undergraduate knowledge in linear algebra, multi-variable calculus, and ordinary/ partial differential equations.
Background in real analysis & functional analysis is not required but would help.

Math 540 Professor Erdogan

1. Measures on the line

Abstract measure theory, outer measure, Lebesgue measure on the real line, measurable sets, Borel sets, Cantor sets and functions, non-measurable sets.
(**Optional**: *Baire's category theorem.*)

2. Measurable functions

Structure of measurable sets, approximation of measurable functions by simple functions, Littlewood's three principles, Egorov and Lusin's theorems.

3. Integration

Lebesgue theory of integration, convergence theorems (Monotone Convergence, Fatou's Lemma, little Fubini, Dominated Convergence), comparison of the Riemann and Lebesgue integrals, modes of convergence, approximation of integrable functions by continuous functions, Fubini's theorem for the plane.
(**Optional**: *product measures, the general Fubini-Tonelli theorem, applications to probability, the convolution product*.)

4. Differentiability

Functions of bounded variation (structure and differentiability), absolutely continuous functions, maximal functions, fundamental theorem of calculus.
(**Optional**: *the Radon-Nikodym theorem.*)

5. L^p **spaces on intervals and** l^p **spaces**

Jensen's inequality, Hölder and Minkowski's inequalities, class of L^p functions, completeness, duals of L^p; spaces, inclusions of L^p spaces.

6. Hilbert spaces and Fourier series

Elementary Hilbert space theory, orthogonal projections, Riesz representation theorem, Bessel's inequality, Riemann-Lebesgue lemma, Parseval's identity, completeness of trigonometric spaces.

Optional topics are not required for the comp exam.

Textbooks used in past semesters:
G. B. Folland, Real Analysis, John Wiley & Sons.
H.L. Royden and Patrick Fitzpatrick, Real Analysis (4th edition), Pearson, 2010.
W. Rudin, Real and Complex Analysis, McGraw-Hill.

Math 541

Professor Junge

1. Review of abstract measure theory.

2. Basic topics on Banach spaces, linear and bounded maps on Banach spaces, open mapping theorem, closed graph theorem. Examples and connection to measure theory.

3. Hahn-Banach theorem, Extreme points, Krein-Milman and Caratheodory. Examples.

4. Compact operators, spectrum and spectral theorem for compact operators on Hilbert spaces.

Further topics: Fredholm alternative, unbounded operators, Riesz representation theorem, Haar measure for locally compact groups, non-linear functional analysis, distributions and Sobolev spaces.

Recommended textbooks:

1. J.B. Conway, A Course in Functional Analysis.
2. W. Rudin, Functional Analysis.
3. G.B. Folland, Real Analysis. Modern Techniques and their Applications.
4. Y. Benyamini and J. Lindenstrauss: Geometric Nonlinear Functional Analysis.

Math 561
Professor Dey

Course Description:
This is the first half of the basic graduate course in probability theory. The goal of this course is to understand the basic tools and language of modern probability theory. We will start with the basic concepts of probability theory: random variables, distributions, expectations, variances, independence and convergence of random variables. Then we will cover the following topics: (1) limit theorems (law of large numbers, central limit theorem and large deviation principle); (2) martingales and their applications c) Brownian motion and d) Stein's method for normal distribution.

Recommended Textbook:
Richard Durrett: Probability: Theory and Examples (4th edition), Cambridge University Press, 2010 (ISBN-13: 978-0521765398, ISBN-10: 0521765390).

Version 4.1 of the book is freely available from the Author's website and course notes will be provided.

Prerequisite:
The prerequisite for Math 561 is Math 540 - Real Analysis I. We will review measure theory topics as needed. Math 541 is nice to have, but not necessary.

Grading Policy:
40% of your grade will depend on homework assignment, 30% will depend on the in class midterm test and 30% on the take home final exam.

Math 580
Professor Balog

Syllabus: This is a rigorous, graduate level introduction to combinatorics. It does not assume prior study, but requires mathematical maturity; it moves at a fast pace. The first half of the course is on enumeration. The second half covers graph theory. There are some topics that are treated more in depth in advanced graduate courses (Math 581, 582, 583, 584, 585): Ramsey theory, partially ordered sets, the probabilistic method and combinatorial designs (as time permits).

Textbook: The FALL 2020 edition of the text COMBINATORIAL MATHEMATICS (by Douglas West).

REQUIREMENTS: A raw score of 80% or higher guarantees an A while a score of 60% or higher guarantees a B- (grade drops by 5%). Additionally, for an A, in the final exam minimum 50%, for B+ (passing com) 40% required. (Near) weekly assignments. Each assignment will have 6 problems of your choice of 5/6 are graded.
There are 9 homework assignments, each worth 4.44%, two tests, each 15%, and a final exam for 30%.
The gradings: 80%− : A, 75%− : A−, 70%− : B+, 65%− : B, etc.
Note that the writings of the solutions must have a high quality and typed, if the argument is messy or not typed then even if the solution is correct it could be returned without grading with 0 points.

Late homework policy: In case the homework is not submitted on time, it could be submitted for the next class, with losing 10% of the score. If there is offical or medical reason then try to notify me in advance via e-mail.
RESOURCES: Electronic mail is a medium for announcements and questions.
PREREQUISITES: There are no official prerequisites, but students need the mathematical maturity and background for graduate-level mathematics.

University of Illinois
Department of Mathematics
273 Altgeld Hall
244-0539
****

C O U R S E D E S C R I P T I O N

****

Spring 2021

MATH 595

L O C A L   C O H O M O L O G Y

Prof. S. P. Dutta
9:30-10:50 Tu-Th

This course will be a study of Local Cohomology, introduced by Grothendieck, with various applications. The main topics will include: Cohen-Macaulay Rings and Modules, Injective Modules over Noethierian rings, Gorenstein rings, local Cohomology -- connection with dimension and depth, local duality theorem of Grothendieck, Cohomology of quasi-coherent and coherent sheaves, Serre's Theorem on coherent sheaves on projective spaces, classification of Line-bundles on P^n, Hartshorne - Lichtenbaum Theorem and Faltings Connectedness Theorem.

<u>Prerequisite</u>: Math 502

<u>Recommended Text</u>: 1. Local Cohomology by R. Hartshorne; 2. Local Cohomology by Brodmann and Sharp, Cambridge University Press.

MATH 595: Algebraic Geometry and String Theory
Spring 2021
Instructor: Sheldon Katz
Format: online
Scheduled time: 11-12:20 TR

The purpose of this first-half minicourse is twofold:

1. To introduce some foundational ideas of string theory, together with additional background from physics, in mathematical language.

2. To demonstrate how modern physics provides deep insights and new objects of study in pure geometry.

While there are no formal prerequisites, familiarity with algebraic geometry will be invaluable, especially transcendental methods for complex projective or Kähler manifolds. Familiarity with the topology and differential geometry of manifolds will also be helpful. No familiarity with physics will be assumed.

The course will begin with an overview. I intend to be responsive to the interests and backgrounds of students. I am initially planning to include a subset of the following topics (since the entire list of topics is almost certainly too much to fit into a minicourse):

Crash course in relevant physics: mechanics, quantum mechanics, path integral, quantum field theory, supersymmetry

Quick review of relevant complex algebraic geometry

Supersymmetric quantum mechanics and cohomology of manifolds

String Theory, its field content, and moduli. The main focus will be on type II string theory (IIA and IIB), but the other flavors of string theory, or M-theory, may be discussed as well

Compactification; dimensional reduction of fields; effective theories; topological twisting

Calabi-Yau compactification

D-Branes (and possibly M2-branes in M-theory)

Dualities, such as mirror symmetry of M-theory/Type IIA duality

Gromov-Witten theory and string theory

Moduli of sheaves/derived category/DT invariants and connections to string theory

Homological mirror symmetry

Geometric Langlands duality

Math 595

Quantum channels I: representations and properties

Spring term 2021 (weeks 1-8)

Lecturer: Felix Leditzky

This course gives an introduction to the theory of quantum channels in the finite-dimensional setting of quantum information theory. We discuss the various mathematically equivalent representations of quantum channels, focus on some important subclasses of channels, and make connections to the theory of majorization and covariant channels.

Class time Weeks 1–8, Tue & Thu 12:30–1:50

Prerequisites MATH 415 or MATH 416.
Throughout the course I will draw connections to quantum information theory, in particular the subfield of "quantum Shannon theory" that is concerned with the study of capacities of quantum channels. However, no prior knowledge in this area is necessary to follow the course.

Grading policy There will be no homework assignments or written exams for this course. Grading will be based on active class participation.

Table of contents

- Representations of quantum channels:
  Isometric picture, Unitary evolution, Kraus representation, Linear operator representation
- Classes of quantum channels:
  Mixed-unitary and unital channels, Entanglement-breaking channels, Symmetric and antidegradable channels, PPT channels
- Unital channels and majorization:
  Majorization for real vectors, Majorization for Hermitian operators, Schur-Horn Theorem
- Covariant quantum channels:
  Definition, properties, examples, Holevo information and minimum output entropy

Literature

- J. Watrous. *Theory of Quantum Information*. Cambridge University Press, 2018. Available online at: https://cs.uwaterloo.ca/~watrous/TQI
- M. M. Wolf. *Quantum Channels & Operations: Guided Tour*. Lecture notes. 2012. Available online at: https://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf
- M. M. Wilde. *Quantum Information Theory*. Cambridge University Press, 2017. Available online at: https://arxiv.org/abs/1106.1445

Math 595

Quantum channels II:
data-processing, recovery channels, and quantum Markov chains

Spring term 2021 (weeks 9-16)

Lecturer: Felix Leditzky

This course gives an introduction to the theory of quantum Markov chains in the finite-dimensional setting of quantum information theory. We first discuss the quantum relative entropy and its fundamental property, the data-processing inequality, and give a proof of this inequality that naturally leads to equality conditions and the concept of recovery channels. Specializing this analysis to the partial trace, we obtain the strong subadditivity property of the von Neumann entropy, as well as a natural definition of quantum Markov chains. We then review a structure theorem for quantum Markov chains, the fundamental differences to classical Markov chains, and—time permitting—venture into the active research topic of approximate quantum Markov chains.

Class time Weeks 9–16, Tue & Thu 12:30–1:50

Prerequisites MATH 415 or MATH 416.
Throughout the course I will draw connections to quantum information theory, in particular the subfield of "quantum Shannon theory" that is concerned with the study of capacities of quantum channels. However, no prior knowledge in this area is necessary to follow the course.

Grading policy There will be no homework assignments or written exams for this course. Grading will be based on active class participation.

Table of contents

- Quantum relative entropy:
  Definition and operational interpretation, Joint convexity and data-processing inequality
- Equality conditions for data-processing:
  Petz's proof of the equality condition, Formulation in terms of recovery channels
- Quantum Markov chains:
  Strong subadditivity of von Neumann entropy, Quantum conditional mutual information and its operational interpretations, Structure theorem for exact quantum Markov chains
- Approximate quantum Markov chains:
  Classical vs. quantum setting, Approximate recovery channels

Literature

- D. Sutter. *Approximate quantum Markov chains.* Vol. 28. SpringerBriefs in Mathematical Physics, 2018. Available online at: https://arxiv.org/abs/1802.05477
- D. Petz. *Quantum Information Theory and Quantum Statistics.* Theoretical and Mathematical Physics. Springer, 2008
- M. A. Nielsen and D. Petz. "A Simple Proof of the Strong Subadditivity Inequality". *Quantum Information & Computation* 5.6 (2005), pp. 507–513. Available online at: https://arxiv.org/abs/quant-ph/0408130

- D. Petz. "Monotonicity of quantum relative entropy revisited". *Reviews in Mathematical Physics* 15.01 (2003), pp. 79–91. Available online at: https://arxiv.org/abs/quant-ph/0209053
- M. B. Ruskai. "Inequalities for quantum entropy: A review with conditions for equality". *Journal of Mathematical Physics* 43.9 (2002), pp. 4358–4375. Available online at: https://arxiv.org/abs/quant-ph/0205064
- B. Ibinson et al. "Robustness of quantum Markov chains". *Communications in Mathematical Physics* 277.2 (2008), pp. 289–304. Available online at: https://arxiv.org/abs/quant-ph/0611057