

# Color isomorphic even cycles and a related Ramsey problem

Gennian Ge<sup>a,\*</sup>, Yifan Jing<sup>b,†</sup>, Zixiang Xu<sup>a,‡</sup> and Tao Zhang<sup>c,§</sup>

<sup>a</sup> School of Mathematics Sciences, Capital Normal University, Beijing 100048, China.

<sup>b</sup> Department of Mathematics, University of Illinois at Urbana Champaign, Urbana, Illinois, 61801, USA.

<sup>c</sup> School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China.

## Abstract

In this paper, we first study a new extremal problem recently posed by Conlon and Tyomkyn (arXiv: 2002.00921). Given a graph  $H$  and an integer  $k \geq 2$ , let  $f_k(n, H)$  be the smallest number of colors  $c$  such that there exists a proper edge-coloring of the complete graph  $K_n$  with  $c$  colors containing no  $k$  vertex-disjoint color-isomorphic copies of  $H$ . Using algebraic properties of polynomials over finite fields, we give an explicit proper edge-coloring of  $K_n$  and show that  $f_k(n, C_4) = \Theta(n)$  when  $k \geq 3$  and  $n \rightarrow \infty$ . The methods we used in the edge-coloring may be of some independent interest. We also consider a related generalized Ramsey problem. For given graphs  $G$  and  $H$ , let  $r(G, H, q)$  be the minimum number of edge-colors (not necessarily proper) of  $G$ , such that the edges of every copy of  $H \subseteq G$  together receive at least  $q$  distinct colors. Establishing the relation to the Turán number of specified bipartite graphs, we obtain some general lower bounds for  $r(K_{n,n}, K_{s,t}, q)$  with a broad range of  $q$ .

*Key words and phrases:* Generalized Ramsey number, edge-coloring, algebraic construction.

*AMS subject classifications:* 05C15, 05C35, 11T06.

---

\*e-mail: gnge@zju.edu.cn. Research supported by the National Natural Science Foundation of China under Grant No. 11971325, National Key Research and Development Program of China under Grant No. 2018Y-FA0704703, and Beijing Scholars Program.

†e-mail: yifanjing17@gmail.com. Research partially supported by a Wind Information Scholarship.

‡e-mail: zxxu8023@qq.com.

§e-mail: zhant220@163.com. Research supported by the National Natural Science Foundation of China under Grant No. 11801109.

# 1 Introduction

The problems of finding rainbow structures in proper edge-colorings of complete graphs have been widely studied in recent years. For example, in a recent breakthrough by Montgomery, Pokrovskiy and Sudakov [24], they confirmed the famous Ringel’s conjecture, one of whose statements involves finding a rainbow copy of any tree with  $n$  edges in a particular proper edge-coloring of  $K_{2n+1}$ . There have also been many papers written on finding large or spanning structures in proper edge-colorings, see, e.g. [9, 23, 25, 27]. Very recently, Conlon and Tyomkyn [8] studied a new Ramsey type problem, which aims to find two or more vertex-disjoint color-isomorphic copies of some given graph in proper edge-colorings of complete graphs.

For  $k, n \geq 2$  and a fixed graph  $H$ , define  $f_k(n, H)$  to be the smallest integer  $c$  such that there exists a proper edge-coloring of  $K_n$  with  $c$  colors containing no  $k$  vertex-disjoint color-isomorphic repeats of  $H$ . One may ask the following natural question.

**Problem 1.1.** *Given  $k \geq 2$  and a fixed graph  $H$ , determine the order of growth of  $f_k(n, H)$  as  $n \rightarrow \infty$ .*

In [8], Conlon and Tyomkyn studied this problem systematically. They first made many useful observations on the properties of  $f_k(n, H)$ . For instance,  $f_k(n, H)$  is monotone increasing in  $n$ , but decreasing in  $k$ . Moreover,  $f_k(n, H)$  is monotone decreasing in  $H$  with respect to taking subgraph, i.e,  $f_k(n, H) \leq f_k(n, H')$  when  $H'$  is a subgraph of  $H$ . Also, since every proper coloring of  $K_n$  uses at least  $n - 1$  colors, then  $n - 1 \leq f_k(n, H) \leq \binom{n}{2}$ . Using the Lovász Local Lemma and Bukh’s random algebraic method [3], they proved the following results.

**Theorem 1.2** ([8]). *The followings hold.*

(i) *For any graph  $H$  with  $v$  vertices and  $e$  edges,*

$$f_k(n, H) = O(\max\{n, n^{\frac{kv-2}{(k-1)e}}\}).$$

(ii) *For every graph  $H$  containing a cycle, there exists  $k = k(H)$  such that*

$$f_k(n, H) = \Theta(n).$$

Conlon and Tyomkyn also suggested to study  $f_k(n, H)$  when  $H$  is an even cycle. Theorem 1.2 implies  $f_k(n, C_4) = O(n^{\frac{2k-1}{2k-2}})$  (using the Lovász Local Lemma), and there is an integer  $k$  such that  $f_k(n, C_4) = \Theta(n)$  (using the random algebraic method). The constant  $k$  obtained by the random algebraic method is likely very large due to the Lang-Weil bound [22], and they asked whether  $f_2(n, C_4) = \Theta(n)$ .

Our first result in this paper studies  $f_k(n, C_4)$ . We try to estimate the smallest integer  $k$  such that  $f_k(n, C_4) = \Theta(n)$ , and we give the following result via an algebraic construction.

**Theorem 1.3.**  $f_3(n, C_4) = \Theta(n)$ .

This result improves the best known upper bound  $O(n^{\frac{5}{4}})$  obtained by Theorem 1.2 (i), and it perhaps gives some evidence that  $f_2(n, C_4)$  is also of order  $\Theta(n)$ .

As the authors mentioned in [8], the problem of studying  $f_k(n, H)$  was motivated by a generalized Ramsey problem raised by Krueger [21]. Our next result in this paper studies this generalized Ramsey problem. The classical graph Ramsey problem asks for the minimum number  $n$  such that every  $k$ -coloring of the edges of  $K_n$  forces a monochromatic copy of  $K_p$ . By fixing  $n$ , the inverse problem asks for the minimum  $k$  such that there exists an edge-coloring of  $K_n$  with  $k$  colors, and each copy of  $K_p$  receives at least  $q$  colors. For general graphs  $G$  and  $H$ , the generalized Ramsey number  $r(G, H, q)$  denotes the minimum number of edge-colors of  $G$ , such that the edges of every copy of  $H \subseteq G$  together receive at least  $q$  distinct colors. This function was first studied by Elekes, Erdős and Füredi (see Section 9 of [10]). Later, Erdős and Gyárfás [11] systematically studied the function  $r(K_n, K_p, q)$  and showed many improved results. After that, a number of wonderful papers [1, 6, 13, 15, 26, 28] studied this problem, and obtained lots of interesting results of this fashion. In recent years, some questions about distinct distances and difference sets with similar flavors have also been studied in [13, 14, 26].

In this paper, we are interested in the bipartite version of generalized Ramsey number  $r(K_{n,n}, K_{s,t}, q)$ , which has been studied in [1, 5, 29]. In particular, Axenovich, Füredi, and Mubayi [1] obtained a series of improved results via many different methods such as Lovász Local Lemma and algebraic methods. We list some results of [1] in Tables 1 and 2.

In our studies of  $r(K_{n,n}, K_{s,t}, q)$ , we will always assume that  $s, t$  and  $q$  are fixed integers and  $n \rightarrow \infty$ . Axenovich, Füredi, and Mubayi [1] determined the linear and quadratic thresholds of the function  $r(K_{n,n}, K_{s,s}, q)$ . More precisely, they determined the smallest integers  $q_1(s)$  and  $q_2(s)$ , where  $q_1(s) = s^2 - 2s + 3$  and  $q_2(s) = s^2 - s + 2$ , such that  $r(K_{n,n}, K_{s,s}, q_1(s)) = \Theta(n)$  and  $r(K_{n,n}, K_{s,s}, q_2(s)) = \Omega(n^2)$ . Up to now, nothing has been shown about  $r(K_{n,n}, K_{s,s}, q)$  beyond the trivial lower bound  $\Omega(n)$  when  $s^2 - 2s + 3 < q < s^2 - s + 2$ .

Our next results give some general lower bounds for  $r(K_{n,n}, K_{s,t}, q)$  with a broad range of  $q$ . Recall that  $\text{ex}(n, H)$  denotes the maximum number of edges in an  $H$ -free graph  $G$  with  $n$  vertices.

**Theorem 1.4.** *For given integers  $t \geq s \geq 4$ , we have*

$$r(K_{n,n}, K_{s,t}, st - e(H) + 1) = \Omega\left(\frac{n^4}{\text{ex}(n^2, H)}\right),$$

where  $H$  is a bipartite graph with bipartition  $H = H_1 \cup H_2$  such that  $|H_1| \leq \lfloor \frac{s}{2} \rfloor$  and  $|H_2| \leq \lfloor \frac{t}{2} \rfloor$ .

When  $s = t$  is even, let  $H$  be the even cycle of length  $s$ . Using the upper bound  $\text{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}})$  by Bondy [2], we obtain the following corollary.

**Corollary 1.5.** *When  $s \geq 4$  is an even integer, we have*

$$r(K_{n,n}, K_{s,s}, s^2 - s + 1) = \Omega(n^{2 - \frac{4}{s}}).$$

As we see in Table 2, our lower bound is not far from the best known upper bound  $r(K_{n,n}, K_{s,s}, s^2 - s + 1) = O(n^{2 - \frac{2}{s}})$ , and Corollary 1.5 gives the first non-trivial lower bound when  $s > 4$  is an even integer and  $q = s^2 - s + 1$ . Note that the choice of  $H$  in Theorem 1.4 is flexible, which leads to a broad range of  $q$  in function  $r(K_{n,n}, K_{s,t}, q)$ . Let  $q_1(s, t) = st - s - t + 3$  and  $q_2(s, t) = st - \frac{s+t}{2} + 2$ . The results in [21] imply  $r(K_{n,n}, K_{s,t}, q_1(s, t)) = \Theta(n)$  and  $r(K_{n,n}, K_{s,t}, q_2(s, t)) = \Theta(n^2)$ . By choosing different graphs  $H$  satisfying the conditions in Theorem 1.4, as corollaries, we are able to get super-linear lower bounds for  $r(K_{n,n}, K_{s,t}, q)$  with many distinct parameters  $q$ .

We give two corollaries in this fashion for the asymmetric version, that is, when  $t > s$ . The corollaries are obtained by choosing  $H$  as 1-subdivision of complete graphs and 1-subdivision of complete bipartite graphs, respectively. We utilise known extremal numbers of 1-subdivision of complete graphs  $\text{ex}(n, \text{sub}(K_t)) = O(n^{\frac{3}{2} - \frac{1}{4t-6}})$ , and 1-subdivision of complete bipartite graphs  $\text{ex}(n, \text{sub}(K_{s,t})) = O(n^{\frac{3}{2} - \frac{1}{2s}})$  (see [7, 17]).

**Corollary 1.6.** *When  $s \geq 6$  is an even integer and  $t = 2\binom{s/2}{2}$ , we have*

$$r(K_{n,n}, K_{s,t}, st - t + 1) = \Omega(n^{1 + \frac{1}{s-3}}).$$

**Corollary 1.7.** *When  $s \geq 8$  is an even integer and  $t = \frac{s^2}{8}$ , we have*

$$r(K_{n,n}, K_{s,t}, st - t + 1) = \Omega(n^{1 + \frac{4}{s}}).$$

Also, there are not many lower bounds for  $r(K_{n,n}, K_{s,t}, q)$  that have been found when  $q < q_1(s, t) = st - s - t + 3$ . The only known case is when  $q = 2$ ,  $r(K_{n,n}, K_{s,s}, 2) = \Omega(n^{\frac{1}{s}})$  [1]. We obtain some new sub-linear lower bounds by using Theorem 1.4 and the famous Kövari-Sós-Turán bound [20]  $\text{ex}(n, K_{m,\ell}) = O(n^{2 - \frac{1}{m}})$  with  $m \leq \ell$ .

**Corollary 1.8.** *When  $t \geq s \geq 4$ , let  $m \leq \frac{s}{2}$  and  $\ell \leq \frac{t}{2}$  be integers with  $m \leq \ell$ , we have*

$$r(K_{n,n}, K_{s,t}, st - m\ell + 1) = \Omega(n^{\frac{2}{m}}).$$

*In particular, set  $s = t$  and  $m = \ell = \frac{s}{2}$ , we have*

$$r(K_{n,n}, K_{s,s}, \frac{3s^2}{4} + 1) = \Omega(n^{\frac{4}{s}}).$$

The paper is organized as follows. In Section 2, we prove Theorem 1.3 by giving an algebraic construction. In Section 3, we prove Theorem 1.4. Finally we conclude with some remarks and further questions in Section 4.

Table 1:  $r(K_{n,n}, K_{s,s}, q)$  with  $s = 2, 3$

$q$	$r(K_{n,n}, K_{2,2}, q)$	$r(K_{n,n}, K_{3,3}, q)$
2	$(1 + o(1))\sqrt{n}$	$(1 + o(1))n^{\frac{1}{3}}$
3	$> \lfloor \frac{2n}{3} \rfloor; \leq n - 1$	$O(n^{\frac{4}{7}})$
4	$n^2$	$O(n^{\frac{2}{3}})$
5	$n^2$	$O(n^{\frac{4}{5}})$
6	$n^2$	$\Theta(n)$
7	$n^2$	$\Omega(n); O(n^{\frac{4}{3}})$
8	$n^2$	$\lceil \frac{n}{2} \rceil \lceil \frac{3n}{2} \rceil$
9	$n^2$	$n^2$

Table 2:  $r(K_{n,n}, K_{s,s}, q)$  with  $s \geq 4$

$q$	$r(K_{n,n}, K_{s,s}, q)$
2	$\Omega(n^{\frac{1}{s}})$
$s^2 - 2s + 2$	$O(n^{1 - \frac{1}{2s-1}})$
$s^2 - 2s + 3$	$\Theta(n)$
$s^2 - s + 1$	$O(n^{2 - \frac{2}{s}})$
$s^2 - s + 2$	$\geq C_s(n^2 - n); < (1 - c_s)n^2$
$s^2 - \lfloor \frac{2s-1}{3} \rfloor + 1$	$> n^2 - 2\lfloor \frac{s-2}{3} \rfloor(n-1)$
$s^2 - \lfloor \frac{s}{2} \rfloor + 1$	$n^2 - \lfloor \frac{s}{2} \rfloor + 1$
$s^2$	$n^2$

## 2 Even cycle $C_4$

In this section, we are going to prove Theorem 1.3. The proof goes as follows. We first choose a field  $\mathcal{K}$ , and construct a map  $\pi : V \rightarrow \mathcal{K}$ , where  $V = V(K_n)$ . Then we choose a symmetric polynomial  $P \in \mathcal{K}[x, y]$ , and color the edge  $ab$  by  $P(\pi(a), \pi(b))$ . We aim to show that under this construction, the edge-coloring we obtained has bounded maximum degree in each color class (thus by a standard application of Vizing's theorem, we are able to get a proper edge coloring), the image  $|P(\pi(V), \pi(V))|$ <sup>1</sup> is  $O(n)$ , and we cannot find too many color isomorphic copies of  $C_4$  in this coloring.

One may choose  $\mathcal{K}$  a field of characteristic 0. Thus, by the symmetric Elekes–Ronyai theorem [19], our symmetric polynomial  $P(x, y)$  has the form  $f(u(x) + u(y))$  or  $f(u(x)u(y))$ , where  $f, u$  are some one variable polynomials in  $\mathcal{K}[x]$ . However, constant many color isomorphic copies of  $C_4$  in this case would imply the set  $u(\pi(V))$  has low additive or multiplicative energy, and this gives us that the image  $|P(\pi(V), \pi(V))|$  is  $\Theta(\pi(V)^2)$ , providing  $\pi(V) = O(n^{\frac{1}{2}})$ . Hence the maximum degree of the color class is  $\Omega(n^{\frac{1}{2}})$ , we will no longer have a proper edge coloring.

We fix this problem by choosing  $\mathcal{K}$  to be a finite field  $\mathbb{F}_p$ , and we choose our polynomial  $P$  to be an expanding polynomial over  $\mathbb{R}$ . This means, if we view our polynomial as an element in

<sup>1</sup>Given  $A, B \subseteq \mathcal{K}$ , let  $P(A, B)$  denote  $\{P(a, b) : a \in A, b \in B\}$ .

$\mathbb{R}[x, y]$ , and  $\pi : V \rightarrow \mathbb{R}$ , by taking an injection  $\pi$ , the edge coloring is proper, we will not have many color isomorphic copies of  $C_4$ , but the image  $|P(\pi(V), \pi(V))|$  is quadratic in  $n$ . Now, by taking  $p = O(n)$ , mapping everything down to  $\mathbb{F}_p$  will help us to decrease  $|P(\pi(V), \pi(V))|$  to  $O(n)$ . By choosing the polynomial  $P(x, y)$ , the vertex map  $\pi$ , and the character  $p$  carefully, and using the ideas from the properties of the resultants of polynomials, we manage to show that, we can still get a proper coloring, and taking this projection down to  $\mathbb{F}_p$  will not create too many color isomorphic copies of  $C_4$ .

*Proof of Theorem 1.3.* Let  $p \equiv 5 \pmod{6}$  be a sufficiently large prime, and

$$A = \left\{1, 2, \dots, \left\lfloor \frac{p-1}{3} \right\rfloor\right\}$$

be a subset of  $\mathbb{F}_p$ .

We remark that the purpose of choosing  $p \equiv 5 \pmod{6}$  is to make  $-3$  a non-residue modulo  $p$ . This follows from the law of quadratic reciprocity, which states that if  $r$  and  $s$  are odd primes, and we define  $r^*$  to be  $r$  if  $r \equiv 1 \pmod{4}$  or  $-r$  if  $r \equiv 3 \pmod{4}$ , then  $s$  is a quadratic residue modulo  $r$  if and only if  $r^*$  is a quadratic residue modulo  $s$ . Applying this with  $r = 3$  and  $s = p$ , we see that  $-3$  is a quadratic residue modulo  $p$  if and only if  $p$  is a quadratic residue modulo  $3$ , i.e.  $p \equiv 1 \pmod{3}$ .

Now, let  $G$  be a complete graph with vertex set  $A$ . Let  $P(x, y) = x^2 + xy + y^2$  be an element in  $\mathbb{F}_p[x, y]$ . Let  $P^*(A, A)$  be the restricted image, that is

$$P^*(A, A) = \{P(a, b) : a, b \in A, a \neq b\}.$$

Define the edge coloring

$$\chi : E(G) \rightarrow P^*(A, A),$$

such that for every  $x, y \in A$  with  $x \neq y$ , we assign color  $P(x, y)$  to the edge  $xy$ . Note that  $x^2 + xy + y^2 = (x + \frac{y}{2})^2 + \frac{3}{4}y^2$ , and  $-3$  is a non-quadratic residue modulo  $p$ , then  $x^2 + xy + y^2 \neq 0$ . Hence  $P^*(A, A) \subseteq \mathbb{F}_p^*$ .

Next, we claim that the edge coloring  $\chi$  is proper. Suppose there is  $a \in \mathbb{F}_p^*$ , such that two edges  $xy$  and  $xz$  are assigned to the same color  $a$ . That is, we have  $x^2 + xy + y^2 = a$  and  $x^2 + xz + z^2 = a$ . Hence  $(y - z)(x + y + z) = 0$ . By the way we construct the vertex set  $A$ ,  $x + y + z \neq 0$ . This implies  $y = z$ . Therefore, two distinct edges  $xy$  and  $xz$  can not be assigned the same color, then  $\chi$  is proper.

Let  $a, b, c, d \in \mathbb{F}_p^*$ , and  $a \neq b, b \neq c, c \neq d, d \neq a$ . Assume the colors  $a, b, c, d$  are incident to a four-cycle  $x, y, z, w \in A$ . Then we have

$$x^2 + xy + y^2 = a, \tag{1}$$

$$y^2 + yz + z^2 = b, \tag{2}$$

$$z^2 + zw + w^2 = c, \tag{3}$$

$$w^2 + wx + x^2 = d. \tag{4}$$

From Equations (1) and (2), we obtain

$$y = \frac{a-b}{x-z} - x - z. \quad (5)$$

Similarly, from Equations (3) and (4), we obtain

$$w = \frac{c-d}{z-x} - z - x. \quad (6)$$

Substituting Equations (5) and (6) into Equations (1) and (4), respectively, we have

$$x^2 + x\left(\frac{a-b}{x-z} - x - z\right) + \left(\frac{a-b}{x-z} - x - z\right)^2 = a, \quad (7)$$

$$x^2 + x\left(\frac{c-d}{z-x} - z - x\right) + \left(\frac{c-d}{z-x} - z - x\right)^2 = d. \quad (8)$$

Making the change of variables  $u \mapsto x+z$  and  $v \mapsto x-z$  in Equations (7) and (8), we get  $v \neq 0$ , and

$$v^4 + 3u^2v^2 - 6(a-b)uv - 2(a+b)v^2 + 4(a-b)^2 = 0, \quad (9)$$

$$v^4 + 3u^2v^2 - 6(c-d)uv - 2(c+d)v^2 + 4(c-d)^2 = 0. \quad (10)$$

Assume that  $a-b = c-d$ . Then from Equations (9) and (10), we have  $a+b = c+d$ , hence  $a=c, b=d$ . By Equations (5) and (6), we have  $y+w+2x+2z=0$ , and by the symmetry, we can also obtain  $x+z+2y+2w=0$ . Therefore,  $x+z=y+w=0$ , which contradicts the construction of  $A$ .

Now we have that  $a-b \neq c-d$ . From Equations (9) and (10) we can obtain

$$uv = \frac{(a+b-c-d)v^2}{3(c-d-a+b)} + \frac{2}{3}(c-d+a-b). \quad (11)$$

Substituting Equation (11) into Equation (9), and then multiplying by  $3(c-d-a+b)^2$ , we obtain

$$k_2v^4 + k_1v^2 + k_0 = 0, \quad (12)$$

where

$$\begin{aligned} k_2 &= 4(a^2 - ab - 2ac + ad + b^2 + bc - 2bd + c^2 - cd + d^2), \\ k_1 &= 4(a-b-c+d)(a^2 - 2ac - 3ad - b^2 + 3bc + 2bd + c^2 - d^2), \\ k_0 &= 4(a-b-c+d)^2(a^2 - 2ab - ac + ad + b^2 + bc - bd + c^2 - 2cd + d^2). \end{aligned}$$

Assume first that  $k_2 = k_1 = k_0 = 0$ . Since  $a - b - c + d \neq 0$ , we have

$$a^2 - ab - 2ac + ad + b^2 + bc - 2bd + c^2 - cd + d^2 = 0, \quad (13)$$

$$a^2 - 2ab - ac + ad + b^2 + bc - bd + c^2 - 2cd + d^2 = 0. \quad (14)$$

From Equations (13) and (14), we obtain  $(a - d)(b - c) = 0$ , which contradicts the choice of  $a, b, c, d$ .

Recall that  $v = x - z$  and  $x, z \in A$ . Thus if at least one of  $k_i$  ( $i = 0, 1, 2$ ) is not 0, then there are at most 4 solutions for  $v$  in  $(A - A) \setminus \{0\}$ . Also, if the number of solutions for  $v$  is at least 3, then there exist two solutions  $v_1, v_2$  such that  $v_1 + v_2 = 0$ . In this case, let  $x_1, z_1, u_1$  ( $x_2, z_2, u_2$ ) be the corresponding solutions with respect to  $v_1$  ( $v_2$ , respectively). Then  $v_1^2 = v_2^2$ , and by Equation (11), we have  $u_1 v_1 = u_2 v_2$ . Hence  $u_1 = -u_2$ . Note that since  $v_i = x_i - z_i$  and  $u_i = x_i + z_i$ , we have  $x_1 - z_1 = -x_2 + z_2$  and  $x_1 + z_1 = -x_2 - z_2$ . Then we can get  $x_1 + x_2 = 0$ , which contradicts the construction of  $A$ .

Hence there are at most 2 solutions for  $v$  in  $(A - A) \setminus \{0\}$ . For any fixed  $v$ , by Equation (11), there is a unique solution for  $u$ . Note that  $x = \frac{u+v}{2}$  and  $z = \frac{u-v}{2}$ , and by Equations (5) and (6),  $y$  and  $w$  are uniquely determined by  $x$  and  $z$ , there are at most 2 solutions for  $(x, y, z, w)$ . Therefore, the number of copies of four-cycles with edge colors  $a, b, c, d$  is at most two.  $\square$

### 3 Lower bounds for $r(K_{n,n}, K_{s,t}, q)$

In this section, we are going to prove Theorem 1.4. The ideas used in this proof are mainly inspired by the recent work of Conlon and Tyomkyn [8]. It will often be helpful to think of  $r(K_{n,n}, K_{s,t}, q)$  in terms of repeated colors. Let  $\mathcal{C}$  be the collection of colors, and let  $\chi : E(G) \rightarrow \mathcal{C}$  be an edge coloring of graph  $G$ . Let  $H$  be a subgraph of  $G$ . If a color  $c \in \mathcal{C}$  appears on exactly  $r_c$  edges in  $\chi(E(H))$ , then we say such color  $c$  is repeated  $r_c - 1$  times in  $H$ . We say  $H$  has  $r$  repeats if  $r = \sum_{c \in \chi(E(H))} (r_c - 1)$ , where every color  $c \in \chi(E(H))$  is repeated  $r_c - 1$  times, and the sum is taking over all colors in  $\chi(E(H))$  (hence  $r_c \geq 1$ ).

*Proof of Theorem 1.4.* Suppose that  $n$  is sufficiently large, and let

$$\chi : E(K_{n,n}) \rightarrow \mathcal{C}$$

be an edge coloring of  $K_{n,n}$ , where  $\mathcal{C}$  is the collection of colors. Suppose  $K_{n,n}$  has the vertex bipartition  $A \cup B$ . We label the vertices in  $A$  and  $B$  respectively, such that  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_n\}$ , with  $a_1 < a_2 < \dots < a_n$  and  $b_1 < b_2 < \dots < b_n$ .

Now we construct the auxiliary graph  $F$  as follows.  $F$  is a bipartite graph, with vertex set  $U \cup W$ , such that  $U = \binom{A}{2}$  and  $W = \binom{B}{2}$ . Thus  $|U| = |W| = \binom{n}{2}$ . Moreover, we require the elements in  $U$  to have the form  $(a_i, a_j)$  with  $a_i > a_j$ , and elements in  $W$  to have the form  $(b_k, b_\ell)$





which is denoted by  $R(f, g)$ .

The resultant of two polynomials has the following property, which is crucial.

**Lemma 4.2** ([16]). *Let  $f, g \in \mathcal{K}[x]$  be one variable polynomials. Suppose  $h(x) = \gcd(f(x), g(x))$ , where  $\deg(h(x)) \geq 1$ . Then  $R(f, g) = 0$ . In particular, if  $f$  and  $g$  have a common root in  $\mathcal{K}$ , then  $R(f, g) = 0$ .*

For the multivariable polynomials, the above lemma still holds when we project down to a one variable polynomial ring. For any  $f, g \in \mathcal{K}[x_1, \dots, x_n]$ , let  $R(f, g; x_i)$  denote the resultant of  $f$  and  $g$  with respect to the variable  $x_i$ .

In the proof of Theorem 1.3, Equations (1), (2), (3), and (4) actually give us four polynomials in  $\mathbb{F}_p[x, y, z, w]$

$$\begin{aligned} f_1(x, y, z, w) &= x^2 + xy + y^2 - a = 0, \\ f_2(x, y, z, w) &= y^2 + yz + z^2 - b = 0, \\ f_3(x, y, z, w) &= z^2 + zw + w^2 - c = 0, \\ f_4(x, y, z, w) &= w^2 + wx + x^2 - d = 0. \end{aligned}$$

By computing the resultants  $f_5(x, z) := R(f_1, f_2; y)$ ,  $f_6(x, z) := R(f_3, f_4; w)$ , and  $g(x) := R(f_5, f_6; z)$  (which are actually the similar computations we did in the proof), we will get  $g(x) = 0$ , and  $g(x)$  is a quadratic polynomial on  $x^2$ , which is an analogue of Equation (12). The proof is finished by analyzing the coefficients in  $g(x)$ , as we did for  $k_0, k_1, k_2$  in Equation (12).

Several interesting questions remain about the function  $f_k(n, H)$  when  $H$  is a longer even cycle. One that immediately arises from Theorem 1.2 (ii) and Theorem 1.3.

**Problem 4.3.** *For any integer  $\ell \geq 2$ , estimate the smallest  $k$  such that  $f_k(n, C_{2\ell}) = \Theta(n)$ .*

The  $\ell = 2$  case of Problem 4.3 is the main topic of this paper. Deriving a similar bound for  $f_2(n, C_4)$  is likely to be difficult. The next case, when  $\ell = 3$ , now seems an attractive candidate for further exploration. The idea of using resultants of polynomials mentioned above may be useful, and we suspect our method could be used to obtain some good upper bounds on  $k$  for the general  $\ell \geq 3$ . We are also interested in the problem of  $f_2(n, C_{2\ell})$  and we provide the following conjecture.

**Conjecture 4.4.** *For any  $\ell \geq 3$ ,  $f_2(n, C_{2\ell}) = \Omega(n^{2-\frac{2}{\ell}})$ .*

Conlon and Tyomkyn [8] verified this conjecture when  $\ell = 3$ . The proof relies on the upper bound for  $\text{ex}(n, \theta_{\ell, t})$  [12] and the observation that the endpoints of theta graph  $\theta_{\ell, t}$  cannot be in the same part when  $\ell$  is odd (this key observation is also useful in [4]).

**Note added:** Very recently, Janzer [18] developed a method for finding suitable cycles of given length and then proved Conjecture 4.4 in a more general form.

## Acknowledgements

Yifan Jing would like to thank József Balogh for helpful discussions. The authors express their gratitude to the anonymous reviewer for the detailed and constructive comments which are very helpful for the improvement of the presentation of this paper.

## References

- [1] M. Axenovich, Z. Füredi, and D. Mubayi. On generalized Ramsey theory: the bipartite case. *J. Combin. Theory Ser. B*, 79(1):66–86, 2000.
- [2] A. Bondy and M. Simonovits. Cycles of even length in graphs. *J. Combinatorial Theory Ser. B*, 16:97–105, 1974.
- [3] B. Bukh. Random algebraic construction of extremal graphs. *Bull. Lond. Math. Soc.*, 47(6):939–945, 2015.
- [4] B. Bukh and M. Tait. Turán number of theta graphs. *arXiv preprint*, arXiv: 1804.10014, 2018.
- [5] F. Chung and R. Graham. On multicolor Ramsey numbers for complete bipartite graphs. *J. Combinatorial Theory Ser. B*, 18:164–169, 1975.
- [6] D. Conlon, J. Fox, C. Lee, and B. Sudakov. The Erdős-Gyárfás problem on generalized Ramsey numbers. *Proc. Lond. Math. Soc. (3)*, 110(1):1–18, 2015.
- [7] D. Conlon, O. Janzer, and J. Lee. More on the extremal number of subdivisions. *arXiv preprint*, arXiv: 1903.10631, 2019.
- [8] D. Conlon and M. Tyomkyn. Repeated patterns in proper colourings. *arXiv preprint*, arXiv: 2002.00921, 2020.
- [9] S. Ehard, S. Glock, and F. Joos. A rainbow blow-up lemma for almost optimally bounded edge-colourings. *arXiv preprint*, arXiv: 1907.09950, 2019.
- [10] P. Erdős. Solved and unsolved problems in combinatorics and combinatorial number theory. *Congr. Numer.*, 32:49–62, 1981.
- [11] P. Erdős and A. Gyárfás. A variant of the classical Ramsey problem. *Combinatorica*, 17(4):459–467, 1997.
- [12] R. Faudree and M. Simonovits. On a class of degenerate extremal graph problems. *Combinatorica*, 3(1):83–93, 1983.

- [13] S. Fish, C. Pohoata, and A. Sheffer. Local properties via color energy graphs and forbidden configurations. *SIAM J. Discrete Math.*, 34(1):177–187, 2020.
- [14] J. Fox, J. Pach, and A. Suk. More distinct distances under local conditions. *Combinatorica*, 38(2):501–509, 2018.
- [15] J. Fox and B. Sudakov. Ramsey-type problem for an almost monochromatic  $K_4$ . *SIAM J. Discrete Math.*, 23(1):155–162, 2008/09.
- [16] P. A. Fuhrmann. *A polynomial approach to linear algebra*. Universitext. Springer, New York, second edition, 2012.
- [17] O. Janzer. Improved bounds for the extremal number of subdivisions. *Electron. J. Combin.*, 26(3):Paper 3.3, 6, 2019.
- [18] O. Janzer. Rainbow Turán number of even cycles, repeated patterns and blow-ups of cycles. *arXiv preprint*, arXiv: 2006.01062, 2020.
- [19] Y. Jing, S. Roy, and C.-M. Tran.  $\alpha$ -Minimal method and generalized sum-product phenomena. *arXiv preprint*, arXiv: 1910.04904, 2019.
- [20] T. Kövári, V. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.
- [21] R. Krueger. Generalized ramsey numbers: forbidding paths with few colors. *arXiv preprint*, arXiv: 1906.06935, 2019.
- [22] S. Lang and A. Weil. Number of points of varieties in finite fields. *Amer. J. Math.*, 76:819–827, 1954.
- [23] R. Montgomery, A. Pokrovskiy, and B. Sudakov. Decompositions into spanning rainbow structures. *Proc. Lond. Math. Soc. (3)*, 119(4):899–959, 2019.
- [24] R. Montgomery, A. Pokrovskiy, and B. Sudakov. A proof of Ringel’s conjecture. *arXiv preprint*, arXiv: 2001.02665, 2020.
- [25] R. Montgomery, A. Pokrovskiy, and B. Sudakov. Embedding rainbow trees with applications to graph labelling and decompositions. *J. Eur. Math. Soc.*, to appear.
- [26] C. Pohoata and A. Sheffer. Local properties in colored graphs, distinct distances, and difference sets. *Combinatorica*, 39(3):705–714, 2019.
- [27] A. Pokrovskiy and B. Sudakov. Linearly many rainbow trees in properly edge-coloured complete graphs. *J. Combin. Theory Ser. B*, 132:134–156, 2018.

- [28] G. N. Sárközy and S. Selkow. On edge colorings with at least  $q$  colors in every subset of  $p$  vertices. *Electron. J. Combin.*, 8(1):Research Paper 9, 6, 2001.
- [29] G. N. Sárközy and S. Selkow. On bipartite generalized Ramsey theory. *Ars Combin.*, 68:57–64, 2003.