A HIGHER WEIGHT ANALOGUE OF OGG'S THEOREM ON WEIERSTRASS POINTS

ROBERT DICKS

ABSTRACT. For a positive integer N, we say that ∞ is a Weierstrass point on the modular curve $X_0(N)$ if there is a non-zero cusp form of weight 2 on $\Gamma_0(N)$ which vanishes at ∞ to order greater than the genus of $X_0(N)$. If p is a prime with $p \nmid N$, Ogg proved that ∞ is not a Weierstrass point on $X_0(pN)$ if the genus of $X_0(N)$ is 0. We prove a similar result for even weights $k \geq 4$. We also study the space of weight k cusp forms on $\Gamma_0(N)$ vanishing to order greater than the dimension.

1. Introduction

If k and N are positive integers, let $S_k(N)$ be the rational vector space of cusp forms of weight k on $\Gamma_0(N)$ with rational Fourier coefficients. These forms have a Fourier expansion at ∞ of the form

$$f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \quad \text{with} \quad a(n_0) \neq 0,$$

and we define $\operatorname{ord}_\infty(f) := n_0$. Let $g(N) = \dim(S_2(N))$ be the genus of $X_0(N)$. We say that ∞ is a Weierstrass point on the modular curve $X_0(N)$ if there exists $0 \neq f \in S_2(N)$ such that $\operatorname{ord}_\infty(f) > g(N)$. Ogg [Ogg78] proved the following theorem.

Theorem 1.1. *If p is a prime such that $p \nmid N$, and if $g(N) = 0$, then ∞ is not a Weierstrass point on $X_0(pN)$.*

A non-geometric proof of Theorem 1.1 was given in [AMR09] (previously, certain cases of level $p\ell$ for distinct primes p and ℓ were considered in [Koh04], [Kil08]). To state our first result, when N is a positive integer and p is a prime such that $p \nmid N$, we require the Atkin-Lehner operator W_p^{pN} on $S_k(pN)$ defined in (2.5). Furthermore, if

$$f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \in S_k(N),$$

define

$$v_p(f) := \inf\{v_p(a(n))\}.$$

With this notation, we prove the following theorem.

Theorem 1.2. *Let N be a positive integer and k be a positive even integer. Let p be a prime with $p \geq \max(5, k+1)$ and $p \nmid N$. Suppose that $0 \neq f \in S_k(pN)$ satisfies*

$$v_p(f) = 0, \quad v_p(f|_k W_p^{pN}) \geq 1 - k/2.$$

Then

$$\operatorname{ord}_\infty(f) \leq \dim(S_k(pN)).$$

Date: June 16, 2020.

As a corollary, we prove an analogue of Ogg's theorem.

Corollary 1.3. *Suppose that N is a positive integer, that k is a positive even integer, and that p is a prime with $p \geq \max(5, k+1)$ and $p \nmid N$. If $S_k(N) = \{0\}$, then for $0 \neq f \in S_k(pN)$, we have*

$$\mathrm{ord}_\infty(f) \leq \dim(S_k(pN)).$$

There is a finite list of N and k for which $S_k(N) = \{0\}$. For $k = 2$, there are 15 such values of N [Ono04, pg. 110]. For $k \geq 4$, the rest are

$$k = 4 : N = 1, 2, 3, 4$$
$$k = 6 : N = 1, 2$$
$$k = 8, 10, 14 : N = 1.$$

It is natural to seek to understand the subspace of forms $f \in S_k(N)$ which vanish to order greater than the dimension. If N is a positive integer and k is a positive even integer, define the subspace

$$W_k(N) := \{f \in S_k(N) : \mathrm{ord}_\infty(f) > \dim(S_k(N))\}.$$

With this notation, we have $W_2(N) = \{0\}$ if and only if ∞ is not a Weierstrass point on $X_0(N)$. As a corollary of Theorem 1.2, we obtain a bound for $\dim(W_k(pN))$.

Corollary 1.4. *Suppose that $p \geq \max(5, k+1)$ is a prime satisfying $p \nmid N$. Then we have*

$$\dim(W_k(pN)) \leq \dim(S_k(N)).$$

Note that this implies Theorem 1.1 in the case $k = 2$. It is interesting to note that the bound in Corollary 1.4 is independent of p. Thus, for fixed N, the spaces $W_k(pN)$ have uniformly bounded dimension as $p \to \infty$.

Remark. For squarefree N, Arakawa and Böcherer [AB03] study the space

$$S_k(N)^* := \{f \in S_k(N) : f\big|_k W_p^{pN} + p^{1-\frac{k}{2}}\big|U_p = 0 \text{ for all } p \mid N\}.$$

We will use a similar subspace to prove Corollary 1.4.

The following examples, which we computed with Magma, illustrate Corollary 1.4 for small values of N.

Example. For an example which is sharp, set $N = 1$, $p = 19$, and $k = 16$. Here, we have $\dim(S_k(pN)) = 24$ and $\dim(S_k(N)) = 1$. In this case, there is a form $f \in S_k(pN)$ with $f = q^{25} + \cdots$.

Example. To get an example which is sharp and for which pN is not prime, set $N = 2$, $p = 23$, and $k = 12$. Here, $\dim(S_k(pN)) = 64$ and $\dim(S_k(N)) = 2$. In this case, there are forms f and g with $f = q^{67} + \cdots$ and $g = q^{68} + \cdots$.

Example. Corollary 1.4 is not always sharp. For example, set $N = 1$, $p = 29$, and $k = 28$. Here, $\dim(S_k(pN)) = 67$ and $\dim(S_k(N)) = 3$. In this case, there is no non-zero $f \in S_k(N)$ satisfying $\mathrm{ord}(f) > 67$.

The paper is organized as follows. Section 2 contains the background necessary to prove these results. Section 3 contains the proof of Theorem 1.2, which uses results from [AMR09]. Finally, Section 4 contains the proofs of Corollary 1.3 and Corollary 1.4.

2. Preliminaries on Modular Forms

The definitions and facts given here can be found in [DS05] and [AMR09]. Let N and k be positive integers. Let $\varepsilon_\infty(N)$ denote the number of cusps on $X_0(N)$, let $g(N)$ denote its genus, and let $\varepsilon_2(N)$, $\varepsilon_3(N)$ denote the numbers of elliptic points of orders 2 and 3, respectively. Then we have

$$g(N) = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}{12} - \frac{1}{2}\varepsilon_\infty(N) - \frac{1}{4}\varepsilon_2(N) - \frac{1}{3}\varepsilon_3(N) + 1, \tag{2.1}$$

$$\varepsilon_2(N) = \begin{cases} 0 & \text{if } 4 \mid N, \\ \prod_{p \mid N} \left(1 + \left(\frac{-4}{p}\right)\right) & \text{otherwise,} \end{cases}$$

$$\varepsilon_3(N) = \begin{cases} 0 & \text{if } 9 \mid N, \\ \prod_{p \mid N} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{otherwise.} \end{cases}$$

We have the well-known formula

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right).$$

For weights $k \geq 4$, we have

$$\dim(S_k(N)) = (k-1)(g(N)-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2(N) + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3(N) + \left(\frac{k}{2} - 1\right) \varepsilon_\infty(N). \tag{2.2}$$

A form in $S_k(N)$ may have forced vanishing at the elliptic points. As in [AMR09], let $\alpha_2(N,k)$ and $\alpha_3(N,k)$ count the number of forced complex zeroes of a form $f \in S_k(N)$ at the elliptic points of order 2 and 3, respectively. These are given by

$$(\alpha_2(N,k), \alpha_3(N,k)) = \begin{cases} (\varepsilon_2(N), 2\varepsilon_3(N)) & \text{if } k \equiv 2 \pmod{12}, \\ (0, \varepsilon_3(N)) & \text{if } k \equiv 4 \pmod{12}, \\ (\varepsilon_2(N), 0) & \text{if } k \equiv 6 \pmod{12}, \\ (0, 2\varepsilon_3(N)) & \text{if } k \equiv 8 \pmod{12}, \\ (\varepsilon_2(N), \varepsilon_3(N)) & \text{if } k \equiv 10 \pmod{12}, \\ (0,0) & \text{if } k \equiv 0 \pmod{12}. \end{cases} \tag{2.3}$$

If $d = \dim(S_k(N))$, then $S_k(N)$ has a basis $\{f_1, ..., f_d\}$ with integer coefficients with the property

$$f_i(z) = a_i q^{c_i} + O(q^{c_i+1}), \qquad 1 \leq i \leq d, \tag{2.4}$$

where $a_i \neq 0$ and $c_1 < c_2 < ... < c_d$. This fact implies that every non-zero $f \in S_k(N)$ has bounded denominators.

For $f \in S_k(N)$ and

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}),$$

define the weight k slash operator by

$$f(z)\big|_k \alpha := \det(\alpha)^{\frac{k}{2}} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

If p is a prime with $p \nmid N$, let $a, b \in \mathbb{Z}$ satisfy $p^2a - pNb = p$. Define the Atkin-Lehner operator W_p^{pN} on $S_k(pN)$ by

$$f\big|_k W_p^{pN} := f\big|_k \begin{pmatrix} pa & 1 \\ pNb & p \end{pmatrix}. \tag{2.5}$$

The operator W_p^{pN} preserves the rationality of the coefficients of $f \in S_k(N)$ [Coh19, Thm. 2.6]. For any prime p, define the U_p operator by

$$\left(\sum a(n)q^n\right) \mid U_p := \sum a(pn)q^n.$$

The trace map

$$\mathrm{Tr}_N^{pN} : S_k(pN) \to S_k(N)$$

is defined by

$$\mathrm{Tr}_N^{pN}(f) := f + p^{1-\frac{k}{2}} f\big|_k W_p^{pN}\big|U_p.$$

This map is surjective, since for $f \in S_k(N)$, we have $\mathrm{Tr}_N^{pN}(f) = (p+1)f$.

3. Proof of Theorem 1.2

Let N be a positive integer and k be a positive even integer. When $k = 2$, Theorem 1.2 follows from [AMR09, Thm. 1.1]. Therefore, we may assume that $k \geq 4$. Throughout, let

$$\alpha_2 := \alpha_2(N, (k-1)p + 1),$$
$$\alpha_3 := \alpha_3(N, (k-1)p + 1),$$

and

$$I(N) := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)].$$

Suppose that $p \geq \max(5, k+1)$ is a prime with $p \nmid N$, and that $f \in S_k(pN)$ satisfies $v_p(f) = 0$ and $v_p(f\big|_k W_p^{pN}) \geq 1 - \frac{k}{2}$. By [AMR09, Thm. 4.2], we have

$$\mathrm{ord}_\infty(f) \leq \frac{(k-1)p+1}{12} I(N) - \frac{1}{2}\alpha_2 - \frac{1}{3}\alpha_3 - \varepsilon_\infty(N) + 1. \tag{3.1}$$

Using (2.1), (2.2), and the facts that $I(pN) = (p+1)I(N)$ and $\varepsilon_\infty(pN) = 2\varepsilon_\infty(N)$, the proof of Theorem 1.2 reduces to proving that

$$\frac{k-2}{12} I(N) + \left(\left\lfloor \frac{k}{4} \right\rfloor - \frac{k-1}{4}\right) \varepsilon_2(pN) + \left(\left\lfloor \frac{k}{3} \right\rfloor - \frac{k-1}{3}\right) \varepsilon_3(pN) + \frac{1}{2}\alpha_2 + \frac{1}{3}\alpha_3 \geq 1. \tag{3.2}$$

The proof of (3.2) breaks up into several cases.

3.1. $\alpha_2 = 0$ and $\alpha_3 = 0$.

Suppose that $\varepsilon_2(N) = \varepsilon_3(N) = 0$. Then (3.2) simplifies to

$$\frac{k-2}{12} I(N) \geq 1. \tag{3.3}$$

The definitions of $\varepsilon_2(N)$ and $\varepsilon_3(N)$ imply that $N \geq 4$. Thus, we have (3.2) because $I(N) \geq 6$ whenever $N \geq 4$.

Assume now that $\varepsilon_2(N) \neq 0$ and $\varepsilon_3(N) = 0$. From (2.3), we have

$$(k-1)p + 1 \equiv 0 \pmod 4,$$

so

$$(k, p) \equiv (0, 1) \text{ or } (2, 3) \pmod 4.$$

The definitions of $\varepsilon_2(N)$ and $\varepsilon_3(N)$ imply that $N \geq 2$, so that $I(N) \geq 3$. In the former case, (3.2) reduces to

$$\frac{k-2}{12} I(N) + \frac{1}{2}\varepsilon_2(N) \geq 1, \tag{3.4}$$

which holds since $k \geq 4$. In the latter case, (3.2) reduces to (3.3), which holds since $k \geq 6$.

Now assume that $\varepsilon_2(N) = 0$ and $\varepsilon_3(N) \neq 0$. From (2.3), we have $(k-1)p+1 \equiv 0 \pmod 3$, so

$$(k,p) \equiv (0,1) \text{ or } (2,2) \pmod 3.$$

In the first case, (3.2) reduces to

$$\frac{k-2}{12} I(N) + \frac{2}{3}\varepsilon_3(N) \geq 1, \tag{3.5}$$

which holds since $k \geq 6$. If $k \equiv 2 \pmod 3$, then (3.2) reduces to (3.3), which holds since $k \geq 8$.

Finally, assume that $\varepsilon_2(N) \neq 0$ and $\varepsilon_3(N) \neq 0$. By (2.3) we have

$$(k-1)p + 1 \equiv 0 \pmod{12}.$$

Consider the 4 possible classes of $(k,p) \pmod{12}$. If $(k,p) \equiv (2,11) \pmod{12}$, then we have $\varepsilon_2(pN) = \varepsilon_3(pN) = 0$, so (3.2) reduces to (3.3). Here, we have $k \geq 14$, so (3.3) holds. If $(k,p) \equiv (6,7) \pmod{12}$, then (3.2) becomes (3.5), which holds because $k \geq 6$ and $\varepsilon_3(N) \geq 1$. If $(k,p) \equiv (8,5) \pmod{12}$, then (3.2) becomes (3.4). We have $k \geq 8$ and $\varepsilon_2(N) \geq 1$, so (3.4) follows. Finally, if $(k,p) \equiv (0,1) \pmod{12}$, then (3.2) becomes

$$\frac{k-2}{12} I(N) + \frac{1}{2}\varepsilon_2(N) + \frac{2}{3}\varepsilon_3(N) \geq 1, \tag{3.6}$$

which holds since $\varepsilon_2(N) \geq 1$ and $\varepsilon_3(N) \geq 1$. This finishes the proof when $\alpha_2 = \alpha_3 = 0$. The remaining cases use similar ideas; fewer details will be given.

3.2. $\alpha_2 \neq 0$ **and** $\alpha_3 = 0$**.** In this case, (3.2) becomes

$$\frac{k-2}{12} I(N) + \left(\left\lfloor \frac{k}{4} \right\rfloor - \frac{k-1}{4}\right) \varepsilon_2(pN) + \left(\left\lfloor \frac{k}{3} \right\rfloor - \frac{k-1}{3}\right) \varepsilon_3(pN) + \frac{1}{2}\alpha_2 \geq 1. \tag{3.7}$$

If $\varepsilon_3(N) = 0$, then $N \geq 2$. Since $I(N) \geq 3$ and $k \geq 4$, (3.7) holds. So, assume that $\varepsilon_3(N) \neq 0$. By (2.3), we have $(k-1)p + 1 \equiv 6 \pmod{12}$. The strategy is then to consider the 4 possibilities for $(k,p) \pmod{12}$. We illustrate this only when $(k,p) \equiv (6,1) \pmod{12}$. In this case, the quantity in (3.7) is at least $\frac{1}{3}I(N) + \frac{2}{3}\varepsilon_3(N) \geq 1$.

3.3. $\alpha_2 = 0$ **and** $\alpha_3 \neq 0$**.** In this case, (3.2) reduces to

$$\frac{k-2}{12} I(N) + \left(\left\lfloor \frac{k}{4} \right\rfloor - \frac{k-1}{4}\right) \varepsilon_2(pN) + \left(\left\lfloor \frac{k}{3} \right\rfloor - \frac{k-1}{3}\right) \varepsilon_3(pN) + \frac{1}{3}\alpha_3 \geq 1. \tag{3.8}$$

If $\varepsilon_2(N) = 0$, then $N \geq 3$, so (3.8) holds. So, assume that $\varepsilon_2(N) \neq 0$. By (2.3), we have

$$(k-1)p + 1 \equiv 8 \pmod{12},$$

so that $\alpha_3 \geq 2$. We illustrate only the case $(k,p) \equiv (8,1) \pmod{12}$. In this case, (3.8) reduces to (3.4), which holds since $k \geq 8$.

3.4. $\alpha_2 \neq 0$ **and** $\alpha_3 \neq 0$**.** By (2.3), we have $(k-1)p + 1 \equiv 2 \text{ or } 10 \pmod{12}$. We illustrate only the case $(k,p) \equiv (2,1) \pmod{12}$. In this case, $\alpha_3 = 2\varepsilon_3(N)$, so (3.2) reduces to (3.3), which holds since $k \geq 14$.

4. Proofs of Corollary 1.3 and Corollary 1.4

Proof of Corollary 1.3. Suppose that N is a positive integer, that k is a positive even integer, and that p is a prime with $p \geq \max(5, k+1)$ and $p \nmid N$. Since every non-zero $f \in S_k(N)$ has bounded denominators, we may assume that $v_p(f) = 0$. Since $S_k(N) = \{0\}$, we have

$$\mathrm{Tr}_N^{pN}(f\big|_k W_p^{pN}) = f\big|_k W_p^{pN} + p^{1-\frac{k}{2}} f\big|U_p = 0.$$

Thus, $v_p(f\big|_k W_p^N) \geq 1 - \frac{k}{2}$, so Corollary 1.3 follows from Theorem 1.2. □

Proof of Corollary 1.4. Define the subspace

$$S := \{f \in S_k(pN) : f\big|_k W_p^{pN} + p^{1-\frac{k}{2}} f\big|U_p = 0\}.$$

Suppose that $0 \neq f \in S$. We apply Theorem 1.2 after clearing denominators to conclude that $\mathrm{ord}_\infty(f) \leq \dim(S_k(pN))$. Thus, $S \cap W_k(pN) = \{0\}$. We also have $S \cong \ker(\mathrm{Tr}_p^{pN})$, since the Atkin-Lehner operator is an isomorphism. Since Tr_p^{pN} is surjective, we have

$$\dim(S) = \dim(S_k(pN)) - \dim(S_k(N)).$$

Since $S_k(pN)$ contains $S \oplus W_k(pN)$, we have $\dim(W_k(pN)) \leq \dim(S_k(N))$. □

5. Acknowledgements

The author would like to thank Scott Ahlgren for suggesting this project and making many helpful comments.

References

[AB03] Tsuneo Arakawa and Siegfried Böcherer, *Vanishing of certain spaces of elliptic modular forms and some applications*, J. Reine Angew. Math. **559** (2003), 25–51. MR 1989643

[AMR09] Scott Ahlgren, Nadia Masri, and Jeremy Rouse, *Vanishing of modular forms at infinity*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1205–1214. MR 2465641

[Coh19] Henri Cohen, *Expansions at cusps and Petersson products in Pari/GP*, Elliptic integrals, elliptic functions and modular forms in quantum field theory, Texts Monogr. Symbol. Comput., Springer, Cham, 2019, pp. 161–181. MR 3889557

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196

[Ogg78] A. P. Ogg, *On the Weierstrass points of $X_0(N)$*, Illinois J. Math. **22** (1978), no. 1, 31–35. MR 0463178

[Ono04] Ken Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004. MR 2020489

Department of Mathematics, University of Illinois, Urbana, IL 61801
Email address: rdicks2@illinois.edu