A HIGHER WEIGHT ANALOGUE OF OGG’S THEOREM ON WEIERSTRASS POINTS

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Abstract. For a positive integer $N$, we say that $\infty$ is a Weierstrass point on the modular curve $X_0(N)$ if there is a non-zero cusp form of weight 2 on $\Gamma_0(N)$ which vanishes at $\infty$ to order greater than the genus of $X_0(N)$. If $p$ is a prime with $p \nmid N$, Ogg proved that $\infty$ is not a Weierstrass point on $X_0(pN)$ if the genus of $X_0(N)$ is 0. We prove a similar result for even weights $k \geq 4$. We also study the space of weight $k$ cusp forms on $\Gamma_0(N)$ vanishing to order greater than the dimension.

1. Introduction

If $k$ and $N$ are positive integers, let $S_k(N)$ be the rational vector space of cusp forms of weight $k$ on $\Gamma_0(N)$ with rational Fourier coefficients. These forms have a Fourier expansion at $\infty$ of the form

$$f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \text{ with } a(n_0) \neq 0,$$

and we define $\text{ord}_\infty(f) := n_0$. Let $g(N) = \dim(S_2(N))$ be the genus of $X_0(N)$. We say that $\infty$ is a Weierstrass point on the modular curve $X_0(N)$ if there exists $0 \neq f \in S_2(N)$ such that $\text{ord}_\infty(f) > g(N)$. Ogg [Ogg78] proved the following theorem.

**Theorem 1.1.** If $p$ is a prime such that $p \nmid N$, and if $g(N) = 0$, then $\infty$ is not a Weierstrass point on $X_0(pN)$.

A non-geometric proof of Theorem 1.1 was given in [AMR09] (previously, certain cases of level $p\ell$ for distinct primes $p$ and $\ell$ were considered in [Koh04], [Kil08]). To state our first result, when $N$ is a positive integer and $p$ is a prime such that $p \nmid N$, we require the Atkin-Lehner operator $W_p^{kN}$ on $S_k(pN)$ defined in [2.3]. Furthermore, if

$$f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \in S_k(N),$$

define

$$v_p(f) := \inf\{v_p(a(n))\}.$$

With this notation, we prove the following theorem.

**Theorem 1.2.** Let $N$ be a positive integer and $k$ be a positive even integer. Let $p$ be a prime with $p \geq \max(5, k+1)$ and $p \nmid N$. Suppose that $0 \neq f \in S_k(pN)$ satisfies

$$v_p(f) = 0, \quad v_p(f|_{W_p^{kN}}) \geq 1 - k/2.$$

Then

$$\text{ord}_\infty(f) \leq \dim(S_k(pN)).$$

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As a corollary, we prove an analogue of Ogg’s theorem.

**Corollary 1.3.** Suppose that \( N \) is a positive integer, that \( k \) is a positive even integer, and that \( p \) is a prime with \( p \geq \max(5, k+1) \) and \( p \nmid N \). If \( S_k(N) = \{0\} \), then for \( 0 \neq f \in S_k(pN) \), we have

\[
\text{ord}_\infty(f) \leq \dim(S_k(pN)).
\]

There is a finite list of \( N \) and \( k \) for which \( S_k(N) = \{0\} \). For \( k = 2 \), there are 15 such values of \( N \) [Ono04, pg. 110]. For \( k \geq 4 \), the rest are

- \( k = 4 : N = 1, 2, 3, 4 \)
- \( k = 6 : N = 1, 2 \)
- \( k = 8, 10, 14 : N = 1 \).

It is natural to seek to understand the subspace of forms \( f \in S_k(N) \) which vanish to order greater than the dimension. If \( N \) is a positive integer and \( k \) is a positive even integer, define the subspace

\[
W_k(N) := \{ f \in S_k(N) : \text{ord}_\infty(f) > \dim(S_k(N)) \}.
\]

With this notation, we have \( W_2(N) = \{0\} \) if and only if \( \infty \) is not a Weierstrass point on \( X_0(N) \). As a corollary of Theorem 1.2, we obtain a bound for \( \dim(W_k(pN)) \).

**Corollary 1.4.** Suppose that \( p \geq \max(5, k+1) \) is a prime satisfying \( p \nmid N \). Then we have

\[
\dim(W_k(pN)) \leq \dim(S_k(N)).
\]

Note that this implies Theorem 1.1 in the case \( k = 2 \). It is interesting to note that the bound in Corollary 1.4 is independent of \( p \). Thus, for fixed \( N \), the spaces \( W_k(pN) \) have uniformly bounded dimension as \( p \to \infty \).

**Remark.** For squarefree \( N \), Arakawa and Böcherer [AB03] study the space

\[
S_k(N)^* := \{ f \in S_k(N) : \text{ord}_\infty(f) > \dim(S_k(N)) \}.
\]

We will use a similar subspace to prove Corollary 1.4 for small values of \( N \).

**Example.** For an example which is sharp, set \( N = 1, p = 19, \) and \( k = 16 \). Here, we have \( \dim(S_k(pN)) = 24 \) and \( \dim(S_k(N)) = 1 \). In this case, there is a form \( f \in S_k(pN) \) with \( f = q^{25} + \cdots \).

**Example.** To get an example which is sharp and for which \( pN \) is not prime, set \( N = 2, p = 23, \) and \( k = 12 \). Here, \( \dim(S_k(pN)) = 64 \) and \( \dim(S_k(N)) = 2 \). In this case, there are forms \( f \) and \( g \) with \( f = q^{67} + \cdots \) and \( g = q^{68} + \cdots \).

**Example.** Corollary 1.4 is not always sharp. For example, set \( N = 1, p = 29, \) and \( k = 28 \). Here, \( \dim(S_k(pN)) = 67 \) and \( \dim(S_k(N)) = 3 \). In this case, there is no non-zero \( f \in S_k(N) \) satisfying \( \text{ord}(f) > 67 \).

The paper is organized as follows. Section 2 contains the background necessary to prove these results. Section 3 contains the proof of Theorem 1.2, which uses results from [AMR09]. Finally, Section 4 contains the proofs of Corollary 1.3 and Corollary 1.4.
2. Preliminaries on Modular Forms

The definitions and facts given here can be found in [DS05] and [AMR09]. Let \( N \) and \( k \) be positive integers. Let \( \varepsilon_{\infty}(N) \) denote the number of cusps on \( X_0(N) \), let \( g(N) \) denote its genus, and let \( \varepsilon_2(N), \varepsilon_3(N) \) denote the numbers of elliptic points of orders 2 and 3, respectively. Then we have

\[
g(N) = \frac{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}{12} - \frac{1}{2} \varepsilon_{\infty}(N) - \frac{1}{4} \varepsilon_2(N) - \frac{1}{3} \varepsilon_3(N) + 1, \tag{2.1}
\]

\[
\varepsilon_2(N) = \begin{cases} 0 & \text{if } 4 \mid N, \\ \prod_{p \mid N} \left( 1 + \left( -\frac{4}{p} \right) \right) & \text{otherwise,}
\end{cases}
\]

\[
\varepsilon_3(N) = \begin{cases} 0 & \text{if } 9 \mid N, \\ \prod_{p \mid N} \left( 1 + \left( -\frac{3}{p} \right) \right) & \text{otherwise.}
\end{cases}
\]

We have the well-known formula

\[
[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \left( 1 + \frac{1}{p} \right).
\]

For weights \( k \geq 4 \), we have

\[
\dim(S_k(N)) = (k - 1)(g(N) - 1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2(N) + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3(N) + \left( \frac{k}{2} - 1 \right) \varepsilon_{\infty}(N). \tag{2.2}
\]

A form in \( S_k(N) \) may have forced vanishing at the elliptic points. As in [AMR09], let \( \alpha_2(N,k) \) and \( \alpha_3(N,k) \) count the number of forced complex zeroes of a form \( f \in S_k(N) \) at the elliptic points of order 2 and 3, respectively. These are given by

\[
(\alpha_2(N,k), \alpha_3(N,k)) = \begin{cases} (\varepsilon_2(N), 2\varepsilon_3(N)) & \text{if } k \equiv 2 \pmod{12}, \\ (0, \varepsilon_3(N)) & \text{if } k \equiv 4 \pmod{12}, \\ (\varepsilon_2(N), 0) & \text{if } k \equiv 6 \pmod{12}, \\ (0, 2\varepsilon_3(N)) & \text{if } k \equiv 8 \pmod{12}, \\ (\varepsilon_2(N), \varepsilon_3(N)) & \text{if } k \equiv 10 \pmod{12}, \\ (0, 0) & \text{if } k \equiv 0 \pmod{12}. \end{cases} \tag{2.3}
\]

If \( d = \dim(S_k(N)) \), then \( S_k(N) \) has a basis \( \{f_1, ..., f_d\} \) with integer coefficients with the property

\[
f_i(z) = a_i q^{c_i} + O(q^{c_i+1}), \quad 1 \leq i \leq d, \tag{2.4}
\]

where \( a_i \neq 0 \) and \( c_1 < c_2 < ... < c_d \). This fact implies that every non-zero \( f \in S_k(N) \) has bounded denominators.

For \( f \in S_k(N) \) and

\[
\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{Q}),
\]

define the weight \( k \) slash operator by

\[
f(z) \mid_k \alpha := \det(\alpha)\frac{k}{2}(cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right).
\]
If $p$ is a prime with $p \nmid N$, let $a, b \in \mathbb{Z}$ satisfy $p^2 a - p N b = p$. Define the Atkin-Lehner operator $W_p^{pN}$ on $S_k(pN)$ by
\[ f \mid_k W_p^{pN} := f \mid_k \left( \frac{p a}{p N b}, \frac{1}{p} \right). \tag{2.5} \]
The operator $W_p^{pN}$ preserves the rationality of the coefficients of $f \in S_k(N)$ [Coh19, Thm. 2.6]. For any prime $p$, define the $U_p$ operator by
\[ \left( \sum a(n)q^n \right) \mid U_p := \sum a(p n)q^n. \]

The trace map
\[ \text{Tr}^{pN}_N : S_k(pN) \to S_k(N) \]
is defined by
\[ \text{Tr}^{pN}_N(f) := f + p^{1 - \frac{k}{2}} f \mid_k W_p^{pN} U_p. \]
This map is surjective, since for $f \in S_k(N)$, we have $\text{Tr}^{pN}_N(f) = (p + 1)f$.  

3. Proof of Theorem 1.2

Let $N$ be a positive integer and $k$ be a positive even integer. When $k = 2$, Theorem 1.2 follows from [AMR09, Thm. 1.1]. Therefore, we may assume that $k \geq 4$. Throughout, let
\[ \alpha_2 := \alpha_2(N, (k - 1)p + 1), \]
\[ \alpha_3 := \alpha_3(N, (k - 1)p + 1), \]
and
\[ I(N) := [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]. \]
Suppose that $p \geq \max(5, k + 1)$ is a prime with $p \nmid N$, and that $f \in S_k(pN)$ satisfies $v_p(f) = 0$ and $v_p(f \mid_k W_p^{pN}) \geq 1 - \frac{k}{2}$. By [AMR09, Thm. 4.2], we have
\[ \text{ord}_\infty(f) \leq \frac{(k - 1)p + 1}{12} I(N) - \frac{1}{2} \alpha_2 - \frac{1}{3} \alpha_3 - \varepsilon_\infty(N) + 1. \tag{3.1} \]
Using (2.1), (2.2), and the facts that $I(pN) = (p + 1)I(N)$ and $\varepsilon_\infty(pN) = 2 \varepsilon_\infty(N)$, the proof of Theorem 1.2 reduces to proving that
\[ \frac{k - 2}{12} I(N) + \left( \left\lfloor \frac{k}{4} \right\rfloor - \frac{k - 1}{4} \right) \varepsilon_2(pN) + \left( \left\lfloor \frac{k}{3} \right\rfloor - \frac{k - 1}{3} \right) \varepsilon_3(pN) + \frac{1}{2} \alpha_2 + \frac{1}{3} \alpha_3 \geq 1. \tag{3.2} \]
The proof of (3.2) breaks up into several cases.

3.1. $\alpha_2 = 0$ and $\alpha_3 = 0$. Suppose that $\varepsilon_2(N) = \varepsilon_3(N) = 0$. Then (3.2) simplifies to
\[ \frac{k - 2}{12} I(N) \geq 1. \tag{3.3} \]

The definitions of $\varepsilon_2(N)$ and $\varepsilon_3(N)$ imply that $N \geq 4$. Thus, we have (3.2) because $I(N) \geq 6$ whenever $N \geq 4$.

Assume now that $\varepsilon_2(N) \neq 0$ and $\varepsilon_3(N) = 0$. From (2.3), we have
\[ (k - 1)p + 1 \equiv 0 \pmod{4}, \]
so
\[ (k, p) \equiv (0, 1) \text{ or } (2, 3) \pmod{4}. \]
The definitions of \( \varepsilon_2(N) \) and \( \varepsilon_3(N) \) imply that \( N \geq 2 \), so that \( I(N) \geq 3 \). In the former case, (3.2) reduces to

\[
\frac{k-2}{12}I(N) + \frac{1}{2}\varepsilon_2(N) \geq 1,
\]

which holds since \( k \geq 4 \). In the latter case, (3.2) reduces to (3.3), which holds since \( k \geq 6 \).

Now assume that \( \varepsilon_2(N) = 0 \) and \( \varepsilon_3(N) \neq 0 \). From (2.3), we have \((k-1)p+1 \equiv 0 \pmod{3}\), so

\[
(k, p) \equiv (0, 1) \text{ or } (2, 2) \pmod{3}.
\]

In the first case, (3.2) reduces to

\[
\frac{k-2}{12}I(N) + \frac{2}{3}\varepsilon_3(N) \geq 1,
\]

which holds since \( k \geq 6 \). If \( k \equiv 2 \pmod{3} \), then (3.2) reduces to (3.3), which holds since \( k \geq 8 \).

Finally, assume that \( \varepsilon_2(N) \neq 0 \) and \( \varepsilon_3(N) \neq 0 \). By (2.3) we have

\[
(k-1)p+1 \equiv 0 \pmod{12}.
\]

Consider the 4 possible classes of \((k, p) \pmod{12}\). If \((k, p) \equiv (2, 11) \pmod{12}\), then we have \( \varepsilon_2(pN) = \varepsilon_3(pN) = 0 \), so (3.2) reduces to (3.3). Here, we have \( k \geq 14 \), so (3.3) holds. If \((k, p) \equiv (6, 7) \pmod{12}\), then (3.2) becomes (3.5), which holds because \( k \geq 6 \) and \( \varepsilon_3(N) \geq 1 \). If \((k, p) \equiv (8, 5) \pmod{12}\), then (3.2) becomes (3.4). We have \( k \geq 8 \) and \( \varepsilon_2(N) \geq 1 \), so (3.4) follows. Finally, if \((k, p) \equiv (0, 1) \pmod{12}\), then (3.2) becomes

\[
\frac{k-2}{12}I(N) + \frac{1}{3}\varepsilon_2(N) + \frac{2}{3}\varepsilon_3(N) \geq 1,
\]

which holds since \( \varepsilon_2(N) \geq 1 \) and \( \varepsilon_3(N) \geq 1 \). This finishes the proof when \( \alpha_2 = \alpha_3 = 0 \). The remaining cases use similar ideas; fewer details will be given.

3.2. \( \alpha_2 \neq 0 \) and \( \alpha_3 = 0 \). In this case, (3.2) becomes

\[
\frac{k-2}{12}I(N) + \left(\left[\frac{k}{4}\right] - \frac{k-1}{4}\right)\varepsilon_2(pN) + \left(\left[\frac{k}{3}\right] - \frac{k-1}{3}\right)\varepsilon_3(pN) + \frac{1}{2}\alpha_2 \geq 1.
\]

If \( \varepsilon_3(N) = 0 \), then \( N \geq 2 \). Since \( I(N) \geq 3 \) and \( k \geq 4 \), (3.7) holds. So, assume that \( \varepsilon_3(N) \neq 0 \). By (2.3), we have \((k-1)p+1 \equiv 6 \pmod{12}\). The strategy is then to consider the 4 possibilities for \((k, p) \pmod{12}\). We illustrate this only when \((k, p) \equiv (6, 1) \pmod{12}\). In this case, the quantity in (3.7) is at least \( \frac{1}{3}I(N) + \frac{2}{3}\varepsilon_3(N) \geq 1 \).

3.3. \( \alpha_2 = 0 \) and \( \alpha_3 \neq 0 \). In this case, (3.2) reduces to

\[
\frac{k-2}{12}I(N) + \left(\left[\frac{k}{4}\right] - \frac{k-1}{4}\right)\varepsilon_2(pN) + \left(\left[\frac{k}{3}\right] - \frac{k-1}{3}\right)\varepsilon_3(pN) + \frac{1}{3}\alpha_3 \geq 1.
\]

If \( \varepsilon_2(N) = 0 \), then \( N \geq 3 \), so (3.8) holds. So, assume that \( \varepsilon_2(N) \neq 0 \). By (2.3), we have

\[
(k-1)p+1 \equiv 8 \pmod{12},
\]

so that \( \alpha_3 \geq 2 \). We illustrate only the case \((k, p) \equiv (8, 1) \pmod{12}\). In this case, (3.8) reduces to (3.4), which holds since \( k \geq 8 \).

3.4. \( \alpha_2 \neq 0 \) and \( \alpha_3 \neq 0 \). By (2.3), we have \((k-1)p+1 \equiv 2 \text{ or } 10 \pmod{12}\). We illustrate only the case \((k, p) \equiv (2, 1) \pmod{12}\). In this case, \( \alpha_3 = 2\varepsilon_3(N) \), so (3.2) reduces to (3.3), which holds since \( k \geq 14 \).
4. Proofs of Corollary 1.3 and Corollary 1.4

**Proof of Corollary 1.3.** Suppose that $N$ is a positive integer, that $k$ is a positive even integer, and that $p$ is a prime with $p \geq \max(5, k + 1)$ and $p \nmid N$. Since every non-zero $f \in S_k(N)$ has bounded denominators, we may assume that $v_p(f) = 0$. Since $S_k(N) = \{0\}$, we have

$$\text{Tr}_{N}(f|_k W_p^N) = f|_k W_p^N + p^{1 - \frac{k}{2}} f|_p U_p = 0.$$ 

Thus, $v_p(f|_k W_p^N) \geq 1 - \frac{k}{2}$, so Corollary 1.3 follows from Theorem 1.2.

**Proof of Corollary 1.4.** Define the subspace

$$S := \{ f \in S_k(pN) : f|_k W_p^N + p^{1 - \frac{k}{2}} f|_p U_p = 0 \}.$$ 

Suppose that $0 \neq f \in S$. We apply Theorem 1.2 after clearing denominators to conclude that $\text{ord}_\infty(f) \leq \dim(S_k(pN))$. Thus, $S \cap W_k(pN) = \{0\}$. We also have $S \cong \ker(\text{Tr}_p^{pN})$, since the Atkin-Lehner operator is an isomorphism. Since $\text{Tr}_p^{pN}$ is surjective, we have

$$\dim(S) = \dim(S_k(pN)) - \dim(S_k(N)).$$

Since $S_k(pN)$ contains $S \oplus W_k(pN)$, we have $\dim(W_k(pN)) \leq \dim(S_k(N))$. 

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References


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