STABLE SUBGROUPS OF THE GENUS TWO HANDLEBODY GROUP

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ABSTRACT. We show that a finitely generated subgroup of the genus two handlebody group is stable if and only if the orbit map to the disk graph is a quasi-isometric embedding. To this end, we prove that the genus two handlebody group is a hierarchically hyperbolic group, and that the maximal hyperbolic space in the hierarchy is quasi-isometric to the disk graph of a genus two handlebody by appealing to a construction of Hamenstdt-Hensel. We then utilize the characterization of stable subgroups of hierarchically hyperbolic groups provided by Abbott-Behrstock-Durham. We also provide a counterexample for the higher genus analogue of the main theorem.

1. INTRODUCTION

In the setting of hyperbolic groups, quasiconvex subgroups are particularly well-behaved subgroups. Specifically, quasiconvex subgroups of hyperbolic groups are precisely the subgroups that are finitely generated and quasi-isometrically embedded, (see for instance [BH99, Corollary III.Γ.3.6]). However, unlike the situation for hyperbolic groups, in the setting of arbitrary finitely generated groups, quasiconvexity is not a quasi-isometric invariant.

One generalization of a quasiconvex subgroup to arbitrary finitely generated groups that is a quasi-isometric invariant is a stable subgroup. Stable subgroups were introduced by Durham and Taylor in [DT15] as a way of characterizing convex cocompact subgroups of mapping class groups, in the sense of Farb-Mosher [FM02]. Another useful characterization of convex cocompact subgroups of the mapping class group is that the orbit map to the curve graph is a quasi-isometric embedding, which was proven independently in [Ham05] and [KL08]. In this paper, we prove an analogous result for the handlebody group of genus two, i.e. the group of isotopy classes of orientation preserving homeomorphisms of a genus two handlebody.

Theorem 1.1. *Let V_2 be a genus two handlebody and suppose G is a finitely generated subgroup of the handlebody group of genus two, $\mathcal{H}_2$. Then the following are equivalent.*

1. *G is a stable subgroup of $\mathcal{H}_2$.*
2. *Any orbit map of G into the disk graph $\mathcal{D}(V_2)$ is a quasi-isometric embedding.*

The disk graph is a δ-hyperbolic graph akin to the curve graph whose vertices correspond to disk-bounding curves on the boundary of the handlebody, (called meridians), and whose edges correspond to disjointness.

The equivalence of stability and quasi-isometrically embedding in the curve graph is particularly notable because analogous characterizations have been proven in a number of other settings closely related to mapping class groups.

1. In the setting of right angled Artin groups, stability is equivalent to quasi-isometrically embedding in the extension graph, and is equivalent to being purely loxodromic, [KMT17].

(2) In the setting of relatively hyperbolic groups, stability is equivalent to quasi-isometrically embedding in the cusped space or the coned off Cayley graph, under mild assumptions on the peripheral subgroups, [ADT17].
(3) In the setting of $\mathrm{Out}(F_n)$, quasi-isometrically embedding in the free factor graph implies stability, [ADT17].
(4) In the setting of hierarchically hyperbolic groups (HHGs), stability is equivalent to quasi-isometrically embedding in the maximal δ-hyperbolic space, and is equivalent to having uniformly bounded projections, [ABB+17]. We include in Section 2.5 a version of this theorem that will be used in this paper.

Our main theorem provides yet another instance of this type of characterization of stable subgroups, at least for genus two. Indeed, in the final section of this paper, we provide a counterexample showing that the higher dimensional analogue of Theorem 1.1 does not hold.

1.1. Methodology.

Let V_g be a genus g handlebody, and $\mathcal{H}_g$ the genus g handlebody group. As the boundary of a handlebody is homeomorphic to a surface of genus g, one can view $\mathcal{H}_g$ as a subgroup of the surface mapping class group $MCG(\partial V_g)$. Similarly, one can view the disk graph $\mathcal{D}(V_g)$ as a subgraph of the curve graph $\mathcal{C}(\partial V_g)$. Given this natural relationship between surface mapping class groups and handlebody groups, one might expect the characterization of stable subgroups of $\mathcal{H}_g$ to be straightforward from the characterization of stable subgroups of surface mapping class groups. However, for genus $g \geq 2$, $\mathcal{H}_g$ is exponentially distorted in $MCG(\partial V_g)$ [HH12]. Furthermore, though $\mathcal{D}(V_g)$ is quasiconvex in $\mathcal{C}(\partial V_g)$ [MM04], the inclusion $\mathcal{D}(V_g) \hookrightarrow \mathcal{C}(\partial V_g)$ is in general not a quasi-isometric embedding [MS13]. This means that much of the toolkit used in the surface mapping class group setting cannot be easily utilized in the handlebody group setting. Indeed, even though subgroups $G \leq \mathcal{H}_2 \leq MCG(\partial V_2)$ that are stable in $MCG(\partial V_2)$ must be stable in $\mathcal{H}_2$, via [ADT17, Theorem 1.6], even for cyclic subgroups of $\mathcal{H}_2$, being stable in $\mathcal{H}_2$ does not necessarily imply stability in $MCG(\partial V_2)$, (see [Hen18, Example 10.2]).

The main method employed to prove Theorem 1.1 is to use the machinery of hierarchically hyperbolic spaces. In particular, we show that the CAT(0) cube complex $\mathcal{M}$ constructed in [HH18], on which $\mathcal{H}_2$ acts properly, cocompactly, and by isometries, is a hierarchically hyperbolic space (HHS). It has been conjectured that any group acting properly, comcompactly, and by isometries on a CAT(0) cube complex is in fact an HHG, but as of yet, this has not been proven, (see for instance the discussion in the introduction of [HS20]). Thus, in order to prove that $\mathcal{M}$ is an HHS, (and hence that $\mathcal{H}_2$ is an HHG), we construct a factor system for $\mathcal{M}$ as described in [BHS17] using techniques developed in [HS20].

We furthermore show that the maximal δ-hyperbolic space in the HHS structure of $\mathcal{M}$ is quasi-isometric to the disk graph. The maximal δ-hyperbolic space in the setting of CAT(0) cube complexes is the factored contact graph of the entire cube complex. The factored contact graph is an augmentation of the contact graph, which is the incidence graph of hyperplane carriers in the cube complex. To show that the factored contact graph is quasi-isometric to the disk graph, we characterize the hyperplanes of $\mathcal{M}$ and demonstrate how these hyperplanes correspond to specific meridians. The above leads us to the second theorem.

Theorem 1.2. *The handlebody group of genus two, $\mathcal{H}_2$, is an HHG with top level hyperbolic space coarsely $\mathcal{H}_2$-equivariantly quasi-isometric to the disk graph, $\mathcal{D}(V_2)$.*

Here, coarsely equivariantly means that the quasi-isometry fails to be equivariant by some uniformly bounded distance.

Theorem 1.2 allows us to use the characterization of stable subgroups in the context of HHGs afforded by [ABB+17], and to replace "quasi-isometric embedding into the maximal δ-hyperbolic space" with "quasi-isometric embedding into the disk graph". The characterization of stable subgroups of HHGs also gives us the following additional characterization of stable subgroups of $\mathcal{H}_2$.

Corollary 1.3. *Suppose G is a finitely generated subgroup of the handlebody group of genus two, $\mathcal{H}_2$. Then the following are equivalent.*

1. *G is a stable subgroup of $\mathcal{H}_2$.*
2. *G is undistorted in $\mathcal{H}_2$ and has uniformly bounded projections.*

Lastly, we note that we can now fully answer Question C posed in [BHS19] which asks whether handlebody groups are HHGs. For genus 0 and 1, the answer is yes because $\mathcal{H}_0$ is trivial and $\mathcal{H}_1 \cong \mathbb{Z}$, (generated by the Dehn twist about the only merdian). Theorem 1.2 tells us that $\mathcal{H}_2$ is also an HHG. For $g \geq 3$, we know by [HH18, Theorem 1.1] that $\mathcal{H}_g$ has exponential Dehn function. Since HHGs have quadratic Dehn functions by [BHS19, Corollary 7.5], $\mathcal{H}_g$ cannot be an HHG for $g \geq 3$.

1.2. **Outline of the paper.** In Section 2, we provide the necessary background on the handlebody group, the disk graph, meridian surgeries, coarse geometry, stability of subgroups, the geometry of CAT(0) cube complexes, and the characterization of CAT(0) cube complexes as HHSs.

In Section 3, we describe the CAT(0) cube complex $\mathcal{M}$ constructed by Hamenstädt and Hensel that will serve as the model for the handlebody group of genus two. This includes a description of the overall structure of $\mathcal{M}$, as well as an in depth account of the two types of (combinatorial) hyperplanes found in $\mathcal{M}$, a classification of the parallelism classes of these hyperplanes, and a discussion of the ways these hyperplanes can contact one another.

Section 4 starts with an explicit characterization of the convex subcomplexes that are included in our factor system. Following this, we use this characterization of the factor system to prove that $\mathcal{H}_2$ is an HHG with unbounded products.

In section 5, we prove that the factored contact graph of $\mathcal{M}$ is quasi-isometric to the disk graph $\mathcal{D}(V_2)$, and prove Theorems 1.2 and 1.1.

Finally, in Section 6, we provide a counterexample to the higher dimensional analogue of Theorem 1.1.

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2. Background

2.1. **The handlebody group and the disk graph.** A *handlebody V_g of genus g* is a three-manifold constructed by attaching g one-handles to the boundary of a three-ball. The boundary ∂V_g is homeomorphic to a surface of genus g. We will occasionally refer to a *handlebody with spots*, which is a handlebody V_g along with a collection of disjoint, embedded disks $D_i \subset \partial V_g$, referred to as *spots*. Note that we define the boundary surface of a spotted handlebody to be the complement of the interior of the disks D_i, so the boundary of a spotted handlebody is a surface with boundary components.

The *handlebody group*, $\mathcal{H}_g$, is the mapping class group of a handlebody; that is, the group of isotopy classes of orientation preserving self-homeomorphisms of V_g. We can view the handlebody group as a subgroup of a surface mapping class group via the injective restriction homomorphism

$$\iota : \mathcal{H}_g \to MCG(\partial V_g).$$

Similarly to surface mapping class groups, handlebody groups are finitely generated, (see [Waj98], [Suz77]). We will view the handlebody group $\mathcal{H}_g$ as a metric space by fixing some finite generating set and equipping $\mathcal{H}_g$ with the word metric. Early investigation of handlebody groups was conducted by Birman [Bir75] and Masur [Mas86]. A survey of properties of handlebody groups can be found in [Hen18].

A essential curve α on ∂V_g is called a *meridian* if it bounds an embedded disk in V_g. A *multimeridian* is a finite collection of pairwise disjoint, pairwise nonhomotopic meridians. Note that whenever we discuss multiple curves in relation to one another, we assume they are in pairwise minimal position. In the setting of handlebody groups, the *disk graph*, denoted $\mathcal{D}(V_g)$, is a graph whose vertices correspond to isotopy classes of meridians, and for which there is an edge between two vertices when the corresponding isotopy classes of meridians have disjoint representatives. The disk graph can be viewed as a subgraph of the curve graph of ∂V_g, denoted $\mathcal{C}(\partial V_g)$, which is a graph whose vertices correspond to isotopy classes of essential simple closed curves on ∂V_g, and for which there is an edge between two vertices when the corresponding isotopy classes of curves can be made disjoint. Like the curve graph, the disk graph is δ-hyperbolic [MS13].

Just as there is a natural action of a surface mapping class group on the corresponding curve graph, there is a natural action of the handlebody group $\mathcal{H}_g$ on the disk graph $\mathcal{D}(V_g)$. In particular, for an element $h \in \mathcal{H}_g$ and a meridian α, the image $h(\alpha)$ is also a meridian, (see for instance [Hen18, Corollary 5.11]). Additionally, because homeomorphisms of a surface preserve disjointness, if α and β are two disjoint meridians, $h(\alpha)$ and $h(\beta)$ will also be disjoint. Thus the action of $\mathcal{H}_g$ on the vertices $\mathcal{D}(V_g)^{(0)}$ will preserve edges.

One particular class of elements in the handlebody group that will be relevant in this paper are Dehn twists along meridians. Intuitively, a Dehn twist along a meridian α, denoted in this paper by T_α, corresponds to cutting V_g along a disk bounded by α, twisting the handle one full twist, and then re-gluing. Clearly this restricts to a Dehn twist in the typical sense on ∂V_g.

2.2. **Meridian surgeries.** Given a handlebody V_g, a *cut system* on V_g is a collection $\alpha_1, \ldots, \alpha_g$ of disjoint, non-isotopic meridians such that $\partial V_g - (\alpha_1 \cup \cdots \cup \alpha_g)$ is connected. Equivalently, a cut system $\alpha_1, \ldots, \alpha_g$ are the boundary curves of a collections of disks $D_1, \ldots, D_g \subset V_g$ such that $V_g - (D_1 \cup \cdots \cup D_g)$ is a single 3-ball.

The following lemma demonstrates a way to construct a sequence of cut systems $\{Z_i\}$ such that each Z_i has two fewer intersections with some (multi)meridian β than Z_{i-1}. The version of this lemma listed below comes from [HH18, Proposition 4.1], though this lemma is well-known and is true in higher genus. Versions of this lemma in higher genus cases can be found, for instance, in [Mas86, Lemma 1.1] and [HH12, Lemma 5.2].

Lemma 2.1 ([HH18, Proposition 4.1])**.** *Let $Z = \{\alpha_1, \alpha_2\}$ be a cut system on V_2, and suppose β is some (multi)meridian. Then either $\alpha_1 \cup \alpha_2$ is disjoint from β, or there exists a subarc $b \subset \beta$ with the following properties.*

1. *The arc b intersects $\alpha_1 \cup \alpha_2$ only in its endpoints, and both endpoints lie on the same curve, say α_1.*
2. *The endpoints of b approach α_1 from the same side.*
3. *Let a, a' be the two components of $\alpha_1 - b$. Then one of $a \cup b$ or $a' \cup b$ is a meridian, say $a \cup b$. Furthermore, $(a \cup b, \alpha_2)$ is a cut system that we call the* surgery of Z defined by b in the direction of β.
4. *The surgery defined by b has two fewer intersections with β than Z.*

Given an initial cut system Z and (multi)meridian β, Lemma 2.1 allows us to construct a sequence $\{Z_i\}_{i=1}^n$ of cut systems such that $Z_1 = Z$, Z_i is a surgery of Z_{i-1} in the direction of β for $i \in (1, n]$, and Z_n is disjoint from β. Furthermore, consecutive cut systems in $\{Z_i\}_{i=1}^n$ have no transverse intersections. We call a sequence of cut systems constructed in this way a *surgery sequence starting at Z in the direction of β.*

2.3. **Geometry of** CAT(0) **cube complexes.** An *n-cube* for $0 \leq n < \infty$ is a copy of the Euclidean cube $[-\frac{1}{2}, \frac{1}{2}]^n$. A *cube complex* is a cell complex in which the n-cells are n-cubes, and in which the attaching maps are isometries. A cube complex $\mathcal{X}$ is CAT(0) if every triangle in $\mathcal{X}$ is at least as thin as a comparison triangle in Euclidean space. For a more detailed definition and other properties of CAT(0) cube complexes, see for example [BH99, Sections I.7 and II.1]. For the remainder of this section, let $\mathcal{X}$ be a CAT(0) cube complex.

A *midcube* of a cube c is a subspace obtained by restricting exactly one coordinate of c to 0. A *hyperplane* H is a connected union of midcubes of $\mathcal{X}$ such that for any finite dimensional cube c of $\mathcal{X}$, either $H \cap c = \emptyset$ or $H \cap c$ is a midcube. The *carrier* $N(H)$ of a hyperplane H is the union of cubes in $\mathcal{X}$ which have non-empty intersection with H. There is a cubical isometric embedding $H \times [-\frac{1}{2}, \frac{1}{2}] \simeq N(H) \hookrightarrow \mathcal{X}$, and we denote by $H^{\pm}$ the images of $H \times \{\pm\frac{1}{2}\}$. We call each of these $H^{\pm}$ *combinatorial hyperplanes.*

A subcomplex F of $\mathcal{X}$ is *convex* if $F^{(1)}$ is metrically convex in $\mathcal{X}^{(1)}$, using the induced path metric, and if every cube whose 0-skeleton is contained in F is also contained in F. This notion of convexity agrees with the CAT(0)-metric convexity for subcomplexes, though not for arbitrary subspaces of $\mathcal{X}$ [HS20].

We say that two convex subcomplexes F_1 and F_2 are *parallel* if for every hyperplane H in $\mathcal{X}$, $F_1 \cap H \neq \emptyset$ if and only if $F_2 \cap H \neq \emptyset$. Note that parallelism is an equivalence relation, and we will denote the parallelism class of a convex subcomplex F via $[F]$. Also note that combinatorial hyperplanes H^+ and H^- are convex, and are always parallel to one another. We say that a convex subcomplex F_1 is *parallel into* a convex subcomplex F_2 if for every hyperplane H in $\mathcal{X}$, if $F_1 \cap H \neq \emptyset$, then $F_2 \cap H \neq \emptyset$. Occasionally it will be useful to talk about hyperplanes H_1 and H_2 being parallel (into). By this we mean that the associated combinatorial hyperplanes $H_1^{\pm}$ and $H_2^{\pm}$ are parallel (into). The following lemma provides a useful characterization of parallel subcomplexes.

Lemma 2.2 ([BHS17, Lemma 2.4])**.** *Let $F, F' \subset \mathcal{X}$ be convex subcomplexes. The following are equivalent:*

1. *F and F' are parallel.*
2. *There is a cubical isometric embedding $F \times [0, a] \to \mathcal{X}$ whose restrictions to $F \times \{0\}$ and $F \times \{a\}$ factor as $F \times \{0\} \cong F \hookrightarrow \mathcal{X}$ and $F \times \{a\} \cong F' \hookrightarrow \mathcal{X}$, respectively,*

and for every vertex $x \in F$, $\{x\} \times [0,a]$ is a combinatorial geodesic segment crossing exactly those hyperplanes that separate F from F'.

Hence, there exists a convex subcomplex E_F such that there is a cubical embedding $F \times E_F \to \mathcal{X}$ with convex image such that for each F' in the parallelism class of F, there exists a 0-cube $e \in E_F$ such that $F \times \{e\} \to \mathcal{X}$ factors as $F \times \{e\} \xrightarrow{id} F' \hookrightarrow F$.

Given any subset $A \subset \mathcal{X}$ and a hyperplane H in $\mathcal{X}$, we will say H *crosses* A if $A \cap H \neq \emptyset$. Additionally, two hyperplanes H_1 and H_2 are said to *osculate* if $H_1 \cap H_2 = \emptyset$ but $N(H_1) \cap N(H_2) \neq \emptyset$.

Given a convex subcomplex $F \subset \mathcal{X}$, we define the *gate map* $\mathfrak{g}_F : \mathcal{X}^{(0)} \to F^{(0)}$ between 0-skeleta to be the map such that $\mathfrak{g}_F(x)$ is the unique closest 0-cube in $F^{(0)}$ to x. It is proven in [BHS17] that this map extends to a cubical map $\mathfrak{g}_F : \mathcal{X} \to F$ such that an n-cube c is collapsed to the unique m-cube whose 0-cubes are the images of the 0-cubes of c under the gate map, where $0 \leq m \leq n$,. We include here a lemma from [HS20] and another lemma from [BHS17] regarding the gate map that we will make use of throughout the paper.

Lemma 2.3 ([HS20, Lemma 1.5])**.** *For any convex subcomplexes $F, F' \subset \mathcal{X}$, the hyperplanes crossing $\mathfrak{g}_F(F')$ are precisely the hyperplanes crossing both F and F'.*

Lemma 2.4 ([BHS17, Lemma 2.6])**.** *If $F, F' \subset \mathcal{X}$ are convex subcomplexes, then $\mathfrak{g}_F(F')$ and $\mathfrak{g}_{F'}(F)$ are parallel subcomplexes. Moreover, if $F \cap F' \neq \emptyset$, then $\mathfrak{g}_F(F') = \mathfrak{g}_{F'}(F) = F \cap F'$.*

Let $F \subseteq \mathcal{X}$ be a convex subcomplex; F is a CAT(0) cube complex. The *contact graph* $\mathcal{C}F$ is a δ-hyperbolic graph, (actually a quasi-tree), originally defined by Hagen [Hag14]. Each vertex in the contact graph corresponds to a hyperplane in F, and there is an edge between two vertices if the carriers of the corresponding hyperplanes have non-empty intersection, i.e. if the hyperplanes either cross or osculate. If $K \subseteq F$ is a convex subcomplex, then the hyperplanes of K can be described as $K \cap H$ where H is a hyperplane of F; this is because every hyperplane is determined by a single midcube contained in it. The definition of convexity implies that the inclusion $K \hookrightarrow F$ induces an injective graph homomorphism $\mathcal{C}K \to \mathcal{C}F$ sending a hyperplane $H \cap K$ of K to the hyperplane H of F. Furthermore, via the definition of parallel subcomplexes, if $K_1, K_2 \subseteq F$ are parallel, convex subcomplexes, then $\mathcal{C}K_1$ and $\mathcal{C}K_2$ are the same subcomplexes of $\mathcal{C}F$.

2.4. CAT(0) **cube complexes as hierarchically hyperbolic spaces.** Here we describe what is needed in order to show that a CAT(0) cube complex $\mathcal{X}$ is a hierarchically hyperbolic space. Because this paper is only concerned with a specific CAT(0) cube complex, we omit the complete definition of an HHS and refer the reader to [BHS17] for full details.

For the remainder of this subsection, let $\mathcal{X}$ denote some arbitrary CAT(0) cube complex. In order to show that $\mathcal{X}$ is an HHS, one must demonstrate the existence of a *factor system.*

Definition 2.5 ([BHS17, Definition 8.1])**.** A *factor system* $\mathfrak{F}$ is a collection of non-empty, convex subcomplexes of $\mathcal{X}$ satisfying the following properties:

(1) $\mathcal{X} \in \mathfrak{F}$.
(2) There exists a number $N \geq 1$ such that every $x \in \mathcal{X}^{(0)}$ is contained in at most N subcomplexes in $\mathfrak{F}$. We refer to this as the *finite multiplicity* property.
(3) If F is a non-trivial subcomplex of $\mathcal{X}$ that is parallel to a combinatorial hyperplane of $\mathcal{X}$, then $F \in \mathfrak{F}$.
(4) There is some number $\xi \geq 0$ such that if $F_1, F_2 \in \mathfrak{F}$ and $\operatorname{diam}(\mathfrak{g}_{F_1}(F_2)) \geq \xi$, then $\mathfrak{g}_{F_1}(F_2) \in \mathfrak{F}$.

Given a factor system $\mathfrak{F}$ for $\mathcal{X}$ and a subcomplex $F \in \mathfrak{F}$, [BHS17, Definition 8.14] defines the *factored contact graph* $\hat{\mathcal{C}}F$, which is constructed from the contact graph $\mathcal{C}F$ in the following way. Let $F' \in \mathfrak{F} - F$ such that $F' \subseteq F$ and such that $\operatorname{diam}(F') \geq \xi$ or F' is parallel to a combinatorial hyperplane of $\mathcal{X}$, (or both). Given a subcomplex F' with these properties, we add one vertex $v_{[F']}$ to $\mathcal{C}F$ corresponding to parallelism class of F', and we connect $v_{[F']}$ by an edge to each vertex in $\mathcal{C}F' \subseteq \mathcal{C}F$. This means that if F'' is parallel to F', then we only add one vertex $v_{[F']} = v_{[F'']}$ to $\mathcal{C}F$. Note that the contact graph $\mathcal{C}F$ is an induced subgraph of $\hat{\mathcal{C}}F$, i.e. $\mathcal{C}F^{(0)} \subset \hat{\mathcal{C}}F^{(0)}$ and the edges of $\mathcal{C}F$ consist of all edges from $\hat{\mathcal{C}}F$ whose endpoints are in $\mathcal{C}F^{(0)}$.

If $\mathcal{X}$ contains a factor system $\mathfrak{F}$, then via [BHS17, Remark 13.2], $\mathcal{X}$ is an HHS whose set of *domains* $\mathfrak{S}$, (sometimes called an *index set*), is a subset of $\mathfrak{F}$ containing one representative $F \in \mathfrak{F}$ of each parallelism class in $\mathfrak{F}$, (except single points). The set $\mathfrak{S}$ is equipped with a partial order $\sqsubseteq$ such that $F_1 \sqsubseteq F_2$ if and only if F_1 is parallel into F_2; the maximal element is the cube complex $\mathcal{X}$ itself. The δ-hyperbolic space associated to a subcomplex F is the factored contact graph $\hat{\mathcal{C}}F$. If G is a group acting properly, cocompactly, and by isometries on $\mathcal{X}$, then $(G, \mathfrak{S})$ is a hierarchically hyperbolic group. Note that this is not the definition of an HHG, but rather a specific example of an HHG. For the full definition, see for instance [BHS19, Definition 1.21].

One way to construct a factor system for $\mathcal{X}$ is via the hyperclosure. Let $\mathfrak{C}$ denote the set of combinatorial hyperplanes in $\mathcal{X}$.

Definition 2.6 ([HS20, Definition 1.14])**.** The *hyperclosure of* $\mathcal{X}$ is the intersection $\mathfrak{F}$ of all sets $\mathfrak{G}$ of convex subcomplexes of $\mathcal{X}$ that satisfy the following properties:

(1) $\mathcal{X} \in \mathfrak{G}$.
(2) If $C \in \mathfrak{C}$, then $C \in \mathfrak{G}$.
(3) If $F, F' \in \mathfrak{G}$, then $\mathfrak{g}_F(F') \in \mathfrak{G}$.
(4) If $F \in \mathfrak{G}$ and F' is parallel to F, then $F' \in \mathfrak{G}$.

If the hyperclosure has the finite multiplicity property, then $\mathfrak{F}$ is a factor system in the sense of Definition 2.5. This is clear because properties (1), (2), and (4) of Definition 2.6 satisfy properties (1) and (3) of Definition 2.5, and property (3) of Definition 2.6 satisfies property (4) of Definition 2.5 with $\xi = 0$.

The following lemma will be useful in our analysis of the hyperclosure.

Lemma 2.7 ([HS20, Lemma 2.2])**.** *Let $\mathfrak{F}$ be the hyperclosure of $\mathcal{X}$. Let $\mathfrak{F}_0 = \{\mathcal{X}\}$, and let $\mathfrak{F}_n$, for $n \geq 1$, be the subset of $\mathfrak{F}$ consisting of subcomplexes that can be written in the form $\mathfrak{g}_C(F)$ where $C \in \mathfrak{C}$ and $F \in \mathfrak{F}_{n-1}$. Then $\mathfrak{F} = \cup_{n \geq 0} \mathfrak{F}_n$.*

Notice that $\mathfrak{F}_1 = \mathfrak{C}$.

A set of convex subcomplexes closely related to the hyperclosure is the set $\mathfrak{M}$ that we will refer to as the *closure of* $\mathcal{X}$. Here we define $\mathfrak{M}$ to be the closure of $\mathfrak{C} \cup \{\mathcal{X}\}$ under projections with diameter $\geq \xi$, for a chosen ξ. If $\mathfrak{F}$ is the hyperclosure of $\mathcal{X}$, then $\mathfrak{M} \subset \mathfrak{F}$ when we choose $\xi = 0$. This is because $\mathfrak{M}$ satisfies properties (1)-(3) of Definition 2.6, implying that any $\mathfrak{G}$ as in Definition 2.6 must contain $\mathfrak{M}$. We have equality when $\mathfrak{M}$ is closed under parallelism. As with the hyperclosure, $\mathfrak{M}$ will be a factor system when it has finitely multiplicity; it is clear that $\mathfrak{M}$ satisfies properties (1) and (4) of Definition 2.5, using the chosen ξ. Moreover, $\mathfrak{M}$ satisfies property (3) of Definition 2.5 because if $C \in \mathfrak{C}$ and $C' \subset \mathcal{X}$ is a convex subcomplex parallel to C, then by [BHS17, Lemma 2.5], C is contained in a combinatorial hyperplane H, and $C' = \mathfrak{g}_H(C) \in \mathfrak{M}$. When the closure is a factor system, [BHS17] refer to it as the *minimal* *factor system.* The authors note that any factor system for $\mathcal{X}$ with projections closed under a chosen ξ must contain $\mathfrak{M}$, using the same ξ. For the remainder of this paper, we let $\xi = 0$ so that $\mathfrak{M} \subset \mathfrak{F}$.

Associated to every HHS are sets known as *standard product regions.* For the purposes of this paper, we will only need to understand what the product regions look like when our HHS is a CAT(0) cube complex; for details regarding standard product regions for a general HHS, see for instance [BHS17, Section 13.1]. For a CAT(0) cube complex, [BHS17, Remark 13.5] describes the standard product regions as subcomplexes of $\mathcal{X}$ that are of the form $F \times E_F$, where F is any subcomplex in a factor system $\mathfrak{F}$, and E_F is an associated subcomplex as described in Lemma 2.2.

Suppose $(G, \mathfrak{S})$ is an HHG for which the HHG structure comes from an action on the CAT(0) cube complex $\mathcal{X}$. We say that $(G, \mathfrak{S})$ has *unbounded products* if the following holds: for every $F \in \mathfrak{S} - \{\mathcal{X}\}$, whenever $\operatorname{diam}(F) = \infty$, then $\operatorname{diam}(E_F) = \infty$. For the more general definition of unbounded products for any HHS, see [ABB$^+$17, Section 3.1].

2.5. **Coarse geometry and stability.** Suppose (X, d_X) and (Y, d_Y) are metric spaces and that $f : X \to Y$ is a (not necessarily continuous) map. If there exists $K \geq 1$, $C \geq 0$ such that for every $a, b \in X$

$$\frac{1}{K} d_X(a, b) - C \leq d_Y(f(a), f(b)) \leq K d_X(a, b) + C,$$

then we say that f is a (K, C)-*quasi-isometric embedding.* If it is also true that there exists $D \geq 0$ such that Y is contained in a D-neighborhood of $f(X)$, then we call f a (K, C)-*quasi-isometry*, and we say that $f(X)$ is a *D-dense subset of Y.* When there exists a quasi-isometry $f : X \to Y$, we say that X and Y are *quasi-isometric.* For any quasi-isometry $f : X \to Y$, there exists a quasi-isometry $g : Y \to X$ and a constant $k \in \mathbb{N}$ such that for all $x \in X$ and $y \in Y$, $d_X(gf(x), x) \leq k$ and $d_Y(fg(y), y) \leq k$; we call g a *quasi-inverse* for f. If $f : I \to Y$ is a quasi-isometric embedding with I an interval of the real line, we call f a (K, C)-*quasi-geodesic.*

Let $f : X \to Y$ be a function on two G-spaces X and Y. We say that the function f is *coarsely G-equivariant* if there exists $N \in \mathbb{N}$ such that for all $x \in X$ and $\gamma \in G$,

$$d_Y(\gamma \cdot f(x), f(\gamma \cdot x)) \leq N.$$

In other words, the function f fails to be G-equivariant by some bounded distance.

Let G be a finitely generated group, and H a finitely generated subgroup of G. We say that H is *undistorted* in G if the inclusion $i : H \hookrightarrow G$ is a quasi-isometric embedding for some (any) word metrics on H and G. If H is undistorted in G then it is a *stable subgroup* of G if for any finite generating set S for G with associated word metric $|\cdot|_S$, and for every $K \geq 1$, $C \geq 0$, there is some $D = D(S, K, C)$ such that any two (K, C)-quasi-geodesics in $(G, |\cdot|_S)$ with common endpoints in $H \subseteq (G, |\cdot|_S)$ remain in the D-neighborhoods of each other. Durham and Taylor show in [DT15] that stability of subgroups is a quasi-isometric invariant.

As mentioned in the introduction, we will be using using the characterization of stable subgroups of HHGs provided in [ABB$^+$17]. The authors of that paper present two characterizations of stable subgroups of HHGs. In [ABB$^+$17, Theorem B], the authors provide a characterization of stable subgroups for any HHG, but this characterization requires alterations to the HHS structure. Alternatively, they produce a characterization of stable subgroups of HHGs that does not require any alteration of the HHS structure, but that adds the additional requirement that product regions are unbounded. In this paper, we will utilize the latter characterization of stable subgroups of HHGs, recorded below, with the wording changed slightly to better fit our setting.

Theorem 2.8 ([ABB+17, Corollary 6.2])**.** *Suppose $(G, \mathfrak{S})$ is a hierarchically hyperbolic group with unbounded products, and that $H < G$ is a finitely generated subgroup. Then the following are equivalent.*

1. *H is a stable subgroup of G.*
2. *H is undistorted in G and has uniformly bounded projections.*
3. *Any orbit map $H \to \mathcal{C}S$ is a quasi-isometric embedding, where S is maximal in $\mathfrak{S}$.*

Note that here $\mathcal{C}S$ refers to the δ-hyperbolic space associated to the maximal domain $S \in \mathfrak{S}$. Additionally, uniformly bounded projections refers to the following notion. Suppose that $(\mathcal{X}, \mathfrak{S})$ is any HHS with associated δ-hyperbolic spaces $\{\mathcal{C}F : F \in \mathfrak{S}\}$. Also associated to $\mathcal{X}$ space is a collection of projection maps $\{\pi_F : \mathcal{X} \to 2^{\mathcal{C}F} : F \in \mathfrak{S}\}$ sending points in $\mathcal{X}$ to sets of bounded diameter in $\mathcal{C}F$. Suppose $\mathcal{Y} \subset \mathcal{X}$ is any subset, and suppose S is the maximal element of $\mathfrak{S}$. We say that $\mathcal{Y}$ has *D-bounded projections* if there exists some $D > 0$ such that, $\mathrm{diam}(\pi_F(\mathcal{Y})) < D$ for all $F \in \mathfrak{S} - \{S\}$. If the constant D does not matter, we say $\mathcal{Y}$ has *uniformly bounded projections.* In the case that $\mathcal{X}$ is a CAT(0) cube complex, the maps $\pi_F : \mathcal{X} \to \hat{\mathcal{C}}F$ send each point $x \in \mathcal{X}^{(0)}$ to the clique of vertices corresponding to hyperplanes whose carriers containing $\mathfrak{g}_F(x)$.

3. A Model for $\mathcal{H}_2$

In [HH18], Hamenstädt and Hensel construct a CAT(0) cube complex on which the handlebody group of genus two acts properly, comcompactly, and by isometries, and which we will refer to as $\mathcal{M}$. In this section, we will take a detailed look at this cube complex. In particular, we summarize their construction in Section 3.1, classify the hyperplanes of $\mathcal{M}$ in Section 3.2, determine the parallelism classes of the combinatorial hyperplanes of $\mathcal{M}$ in Section 3.3, discuss some properties of the contact graph $\mathcal{CM}$ in Section 3.4, and classify the non-empty intersections of combinatorial hyperplanes in Section 3.5.

3.1. **The model.** Let V be a handlebody of genus two, and let $X = \{\alpha_1, \alpha_2, \alpha_3\}$ be a pants decomposition on ∂V consisting only of non-separating meridians. Hamenstädt and Hensel ([HH18, Lemma 6.1]) show that for each such X, one can construct a *dual system* $\Delta = \{\delta_1, \delta_2, \delta_3\}$ of non-separating meridians satisfying the following properties:

(i) δ_i is disjoint from α_j for $i \neq j$.
(ii) δ_i intersects α_i exactly twice.
(iii) A dual system Δ is uniquely defined up to Dehn twists about curves in X. In particular, if δ_i and δ_i' are two different dual curves to α_i, then $\delta_i = T_{\alpha_i}^{n_i}(\delta_i')$ for some integer n_i.

The 0-skeleton of $\mathcal{M}$ is comprised of all pairs (X, Δ) as above.

There are two types of edges in the 1-skeleton of $\mathcal{M}$. Two vertices (X, Δ) and (X', Δ') will be connected by a *twist edge* if $X = X'$ and $\Delta' = T_{\alpha_i}(\Delta)$ for some $\alpha_i \in X$, i.e. the vertices share a pants decomposition and the dual systems differ by a Dehn twist about one of the pants curves.

To describe the second type of edges, suppose $X = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\Delta = \{\delta_1, \delta_2, \delta_3\}$ are a pants decomposition and dual system pair. Switching some α_i, say α_1, with the corresponding dual curve δ_1 produces another pants decomposition $X' = \{\delta_1, \alpha_2, \alpha_3\}$. The collection $\{\alpha_1, \delta_2, \delta_3\}$ will not be a dual system to X' because δ_2 and δ_3 will both intersect δ_1 twice, but by applying a canonical cleanup process c to $\{\alpha_1, \delta_2, \delta_3\}$, (as described in [HH18, Section 6] and as illustrated in Figure 1), we can obtain a dual system $\Delta' = \{\alpha_1, c(\delta_2), c(\delta_3)\}$ for X'. We connect any two such vertices (X, Δ) and (X', Δ') via an edge, called a *switch edge*, and we say (X', Δ') is *obtained from* (X, Δ) *by switching* α_1. The canonical cleanup function c commutes with Dehn twists ([HH18, Lemma 6.3]).

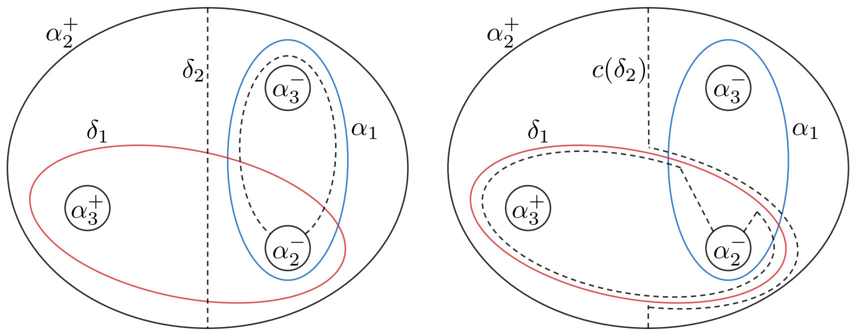


FIGURE 1. This image shows ∂V_2 cut along α_2 and α_3, where $\alpha_i^\pm$ refers to the two sides of the curve after cutting. This figure illustrates the cleanup of δ_2 after switching α_1 and δ_1.

We glue in 3-dimensional Euclidean cubes $[-\frac{1}{2}, \frac{1}{2}]^3$ wherever we see its 1-skeleton. There are two types of 3-cubes that we see in $\mathcal{M}^{(1)}$, which we now describe. First, fix some pants decomposition $X = \{\alpha_1, \alpha_2, \alpha_3\}$. Consider the subgraph of the 1-skeleton of $\mathcal{M}$ containing only vertices whose pants decomposition is X. Because dual systems for a given pants decomposition are unique up to Dehn twists about the pants curves α_i, this subgraph contains only twist edges, and is in fact isomorphic to the Cayley graph of $\mathbb{Z}^3$, with respect to a basis generating set. We glue in Euclidean 3-cubes $[-\frac{1}{2}, \frac{1}{2}]^3$, which we call *twist cubes*, to this subgraph so that the resulting subcomplex is isomorphic to $\mathbb{R}^3$, with the standard integral cube complex structure. We denote this subcomplex by $\mathcal{M}(X)$, and we call such subcomplexes *twist flats.*

To describe the second type of 3-cubes contained in $\mathcal{M}$, again fix a pants decomposition $X = \{\alpha_1, \alpha_2, \alpha_3\}$, as well as a dual system $\Delta = \{\delta_1, \delta_2, \delta_3\}$. Let (X', Δ') be obtained from (X, Δ) by switching α_2. Notice that the following four vertices form the four vertices of a 2-cube $k(X)$ whose edges are all twist edges:

$$\{(X, \Delta), (X, T_{\alpha_1}(\Delta)), (X, T_{\alpha_3}(\Delta)), (X, T_{\alpha_1}T_{\alpha_3}(\Delta))\}.$$

Similarly, the following four vertices are contained in a 2-cube $k(X')$ whose edges are all twist edges:

$$\{(X', \Delta'), (X', T_{\alpha_1}(\Delta')), (X', T_{\alpha_3}(\Delta')), (X', T_{\alpha_1}T_{\alpha_3}(\Delta'))\}.$$

Furthermore, $k(X)$ and $k(X')$ are connected to one another via switch edges. In particular, there are switch edges connecting (X, Δ) to (X', Δ'), $(X, T_{\alpha_1}(\Delta))$ to $(X', T_{\alpha_1}(\Delta'))$, $(X, T_{\alpha_3}(\Delta))$ to $(X', T_{\alpha_3}(\Delta'))$, and $(X, T_{\alpha_1}T_{\alpha_3}(\Delta))$ to $(X', T_{\alpha_1}T_{\alpha_3}(\Delta'))$. These switch edges, along with the twist edges in $k(X)$ and $k(X')$ form the 1-skeleton of a 3-cube. We thus glue in a Euclidean 3-cube $[-\frac{1}{2}, \frac{1}{2}]^3$, and we refer to such cubes as *switch cubes*. The subcomplex containing all switch cubes connecting the twist flats $\mathcal{M}(X)$ and $\mathcal{M}(X')$ is isomorphic to $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$. We denote this subcomplex by $\mathcal{M}(X, X')$ and refer to such subcomplexes as *switch bridges*. An illustration of how a switch bridge connects two twist flats can be seen in Figure 2.

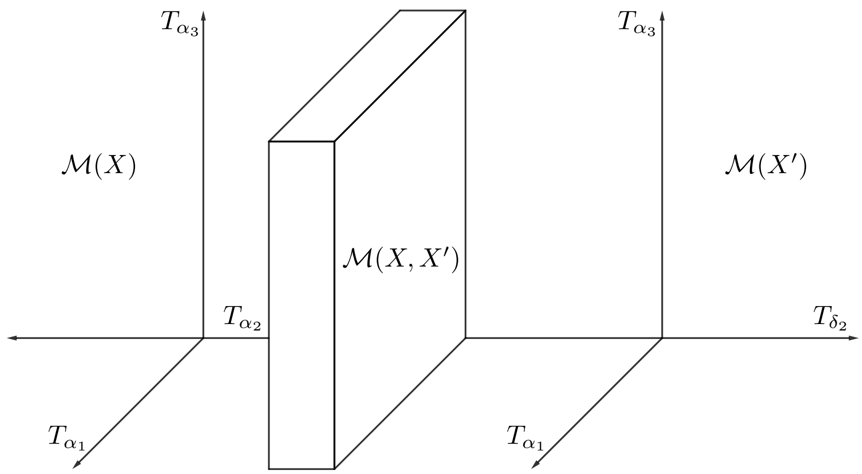


FIGURE 2. Two twist flats $\mathcal{M}(X)$ and $\mathcal{M}(X')$, which are copies of $\mathbb{R}^3$, are glued to the switch bridge $\mathcal{M}(X, X')$, which is a copy of $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$, along copies of $\mathbb{R}^2$ contained in the 2-skeletons of $\mathcal{M}(X)$ and $\mathcal{M}(X')$. Here $X = \{\alpha_1, \alpha_2, \alpha_3\}$ and $X' = \{\alpha_1, \delta_2, \alpha_3\}$.

Hamenstädt and Hensel show that the underlying structure of $\mathcal{M}$ is a tree called the *non-separating meridional pants graph*, which we refer to as $\mathcal{P}$. The vertices of $\mathcal{P}$ correspond to pants decompositions X of non-separating meridians, and there is an edge between two pants decompositions if they intersect minimally, i.e. if they intersect twice. There is a surjective map $p : \mathcal{M} \to \mathcal{P}$ that maps each twist flat $\mathcal{M}(X)$ to the vertex $X \in \mathcal{P}^{(0)}$, and that maps each switch bridge $\mathcal{M}(X, X')$ to the edge between X and X' in $\mathcal{P}$.

Additionally, given a cut system Z on V_2, Hamenstädt and Hensel define the induced subgraph $\mathcal{P}(Z)$ of $\mathcal{P}$ as the subgraph with vertices corresponding to pants decompositions containing the cut system Z. They show in [HH18, Corollary 5.11] that $\mathcal{P}(Z)$ is a tree, and that for any distinct cut systems $Z \neq Z'$, the subtrees $\mathcal{P}(Z)$ and $\mathcal{P}(Z')$ intersect in at most a single point.

Let us define a similar subgraph $\mathcal{P}(\alpha)$ of $\mathcal{P}$ as the induced subgraph with vertices corresponding to pants decompositions that contain the non-separating meridian α. The following lemma regarding $\mathcal{P}(\alpha)$ will be useful in the discussion of parallelism classes of hyperplanes in $\mathcal{M}$.

Lemma 3.1. *Let α a non-separating meridian on V_2. Then $\mathcal{P}(\alpha)$ is a subtree of $\mathcal{P}$.*

Proof. Since $\mathcal{P}$ is a tree, it suffices to show that $\mathcal{P}(\alpha)$ is connected. Suppose $X = \{\alpha, \beta_1, \beta_2\}$ and $X' = \{\alpha, \delta_1, \delta_2\}$ are two distinct pants decompositions of non-separating meridians. Since the pants decompositions are distinct but contain a common curve α, one of the β_i must intersect some δ_i. Say $\beta_1 \cap (\delta_1 \cup \delta_2) \neq \emptyset$. Let $Z = \{\alpha, \beta_1\}$, which is a cut system. There is a surgery sequence $(Z_i)_{i=1}^n$ starting from $Z_1 = Z$ in the direction X'. The final cut system Z_n is disjoint from X', and since X' is a pants decomposition, it must be that $Z_n \subset X'$.

Since $\alpha \cap \beta_i = \emptyset$ and $\alpha \cap \delta_i = \emptyset$ for each i, the meridian surgeries performed to attain $(Z_i)_{i=1}^n$ will never be performed on α. In particular, every cut system Z_i must contain the meridian α. For a fixed $i \in [2, n-1]$, the unions $Z_i \cup Z_{i-1}$ and $Z_i \cup Z_{i+1}$ are vertices in $\mathcal{P}(Z_i)$, since consecutive cut systems have no transverse intersections. Because $\mathcal{P}(Z_i)$ is a tree and because $\alpha \in Z_i$, there is a path $\gamma_i \subset \mathcal{P}(Z_i)$ connecting $Z_i \cup Z_{i-1}$ to $Z_i \cup Z_{i+1}$ such that every vertex $\gamma_i(j)$ contains α.

Furthermore, both X and $Z_1 \cup Z_2$ are vertices in $\mathcal{P}(Z_1)$, so there is a path $\gamma_1 \subset \mathcal{P}(Z_1)$ connecting X to $Z_1 \cup Z_2$ such that each vertex $\gamma_1(j)$ contains α. Similarly, since $Z_n \subset X'$, it follows that $Z_{n-1} \cup Z_n$ and X' are vertices in $\mathcal{P}(Z_n)$, and hence there is a path $\gamma_n \subset \mathcal{P}(Z_n)$ connecting $Z_{n-1} \cup Z_n$ to X' such that every vertex $\gamma_n(j)$ contains α.

The path γ constructed by concatenating the paths γ_i for $i \in [1, n]$ is a path from X to X' contained entirely in $\mathcal{P}(\alpha)$. This implies that $\mathcal{P}(\alpha)$ is a connected subgraph of the tree $\mathcal{P}$, and hence $\mathcal{P}(\alpha)$ is a subtree. □

3.2. **Hyperplanes.** In this subsection, we examine the variants of hyperplanes found in $\mathcal{M}$, and their associated combinatorial hyperplanes. Understanding the hyperplanes will prove useful in the discussion of the contact graph $\mathcal{CM}$, and understanding the combinatorial hyperplanes is necessary to prove that $\mathcal{M}$ is a hierarchically hyperbolic space.

There are two distinct types of hyperplanes that can be found in $\mathcal{M}$. The first type are those that are contained entirely in switch bridges. In particular, these are the hyperplanes in $\mathcal{M}$ whose carriers are switch bridges. For this reason we will refer to them as *switch hyperplanes*. We denote the switch hyperplane contained in $\mathcal{M}(X, X')$ by $H(X, X')$.

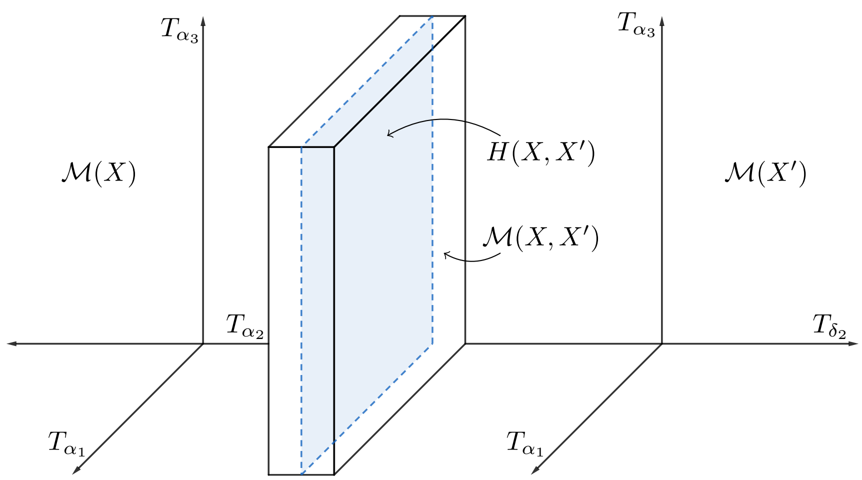


FIGURE 3. A switch hyperplane $H(X, X')$ is entirely contained in the switch bridge $\mathcal{M}(X, X')$. Here $X = \{\alpha_1, \alpha_2, \alpha_3\}$ and $X' = \{\alpha_1, \delta_2, \alpha_3\}$.

Recall that any hyperplane has two associated combinatorial hyperplanes. Let $X = \{\alpha_1, \alpha_2, \alpha_3\}$ be a pants decomposition with a dual system $\Delta = \{\delta_1, \delta_2, \delta_3\}$, and suppose (X', Δ') is obtained from (X, Δ) by switching α_2. For the switch hyperplane $H(X, X')$, the associated combinatorial hyperplanes will be denoted $C(X, \delta_2)$ and $C(X', \alpha_2)$. Here, $C(X, \delta_2)$ is the combinatorial hyperplane in $\mathcal{M}(X)$ that contains all vertices of the form $(X, \{T_{\alpha_1}^{n_1}(\delta_1), \delta_2, T_{\alpha_3}^{n_3}(\delta_3)\})$. Similarly, $C(X', \alpha_2)$ is the combinatorial hyperplane in $\mathcal{M}(X')$

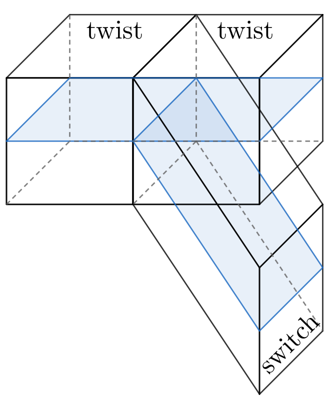
that contains all vertices of the form $(X', \{c(T_{\alpha_1}^{n_1}(\delta_1)), \alpha_2, c(T_{\alpha_3}^{n_3}(\delta_3))\})$. We call such combinatorial hyperplanes *combinatorial switch hyperplanes.*

The second type of hyperplane in $\mathcal{M}$ are those that cross twist flats. We will see that these hyperplanes are uniquely determined by a single pants curve along with a single dual to that pants curve. For now, let us consider one such hyperplane and denote it by H. Suppose the intersection of H with some twist flat $\mathcal{M}(X)$ is non-empty, where $X = \{\alpha_1, \alpha_2, \alpha_3\}$. The intersection $N(H) \cap \mathcal{M}(X)$ is isomorphic to $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$. All the vertices in $N(H) \cap \mathcal{M}(X)$ will be of one of the following forms:

$$(X, T_{\alpha_2}^{n_2} T_{\alpha_3}^{n_3}(\Delta)) \text{ or } (X, T_{\alpha_2}^{n_2} T_{\alpha_3}^{n_3}(T_{\alpha_1}(\Delta))),$$

where $\Delta = \{\delta_1, \delta_2, \delta_3\}$ is a dual system to X, and where $n_2, n_3 \in \mathbb{Z}$. Notice that each of these vertices contains α_1 as a pants curve, and either δ_1 or $T_{\alpha_1}(\delta_1)$ as a dual to α_1. In fact, every vertex in $N(H)$ must contain α_1 as a pants curve because any switch bridge crossed by H corresponds to switching only α_2 or α_3, and moreover, $p(H) = \mathcal{P}(\alpha_1)$. Furthermore, any hyperplane H' for which $p(H') = \mathcal{P}(\alpha_1)$, and for which $N(H')$ contains vertices with only δ_1 or $T_{\alpha_1}(\delta_1)$ as duals to α_1 must actually be H. To see this, suppose (X', Δ') is a vertex of $\mathcal{M}$ such that $X' \in \mathcal{P}(\alpha_1)^{(0)}$ and such that $\delta_1 \in \Delta'$. By the definition of dual curves, we know that $\partial V_2 - (\alpha_1 \cup \delta_1)$ is a disjoint union of two annuli $A_1 \cup A_2$, and we know that the meridians in $X' - \alpha_1$ must be contained in $\partial V_2 - (\alpha_1 \cup \delta_1)$. Since A_1 and A_2 each contain only one meridian, the other curves in $X' - \alpha_1$ are uniquely determined by $\alpha_1 \cup \delta_1$. This tells us that $(X', \Delta') \in \mathcal{M}(X)$. By a similiar argument, we can see that any vertex whose pants decomposition contains α_1 with dual $T_{\alpha_1}(\delta_1)$ must be contained in $\mathcal{M}(X)$. Thus, $N(H') \cap \mathcal{M}(X) = N(H) \cap \mathcal{M}(X')$, and hence $H = H'$. We are therefore justified in denoting H by $H(\alpha_1, \delta_1)$, and we call these hyperplanes *twist hyperplanes.*

In Figure 4, one can see two illustrations of the local structure of a twist hyperplane. Figure 5 illustrates the non-empty intersection of a twist hyperplane with several twist flats.



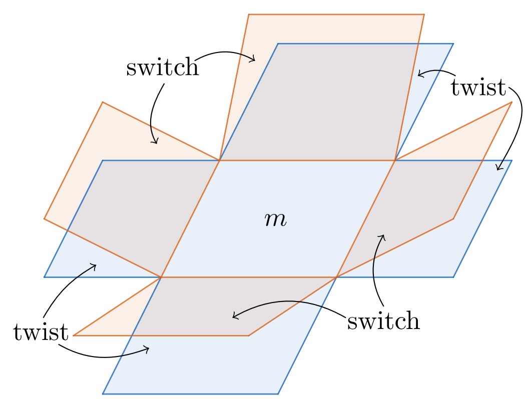


FIGURE 4. Pictured here are two representations of the local structure of a twist hyperplane. Any 2-cube that is crossed by a twist hyperplane is contained in three 3-cubes: two twist cubes contained in a single twist flat, and one switch cube, (pictured left). On the right we see that if m is a midcube of a twist hyperplane that is contained in a twist cube, then it is connected to eight other midcubes: four midcubes contained in disjoint switch bridges, (orange), and four midcubes contained in twist cubes in the same twist flat as m, (blue).

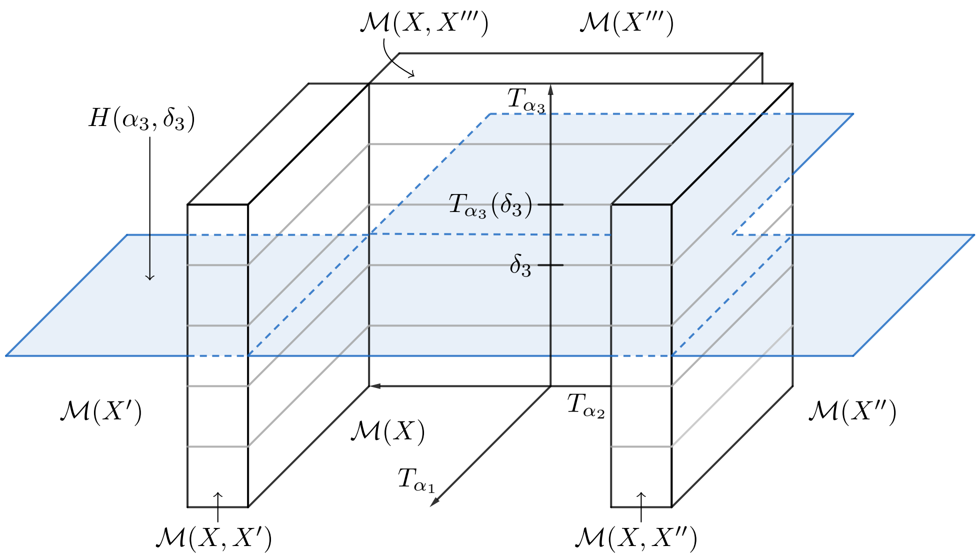


FIGURE 5. A twist hyperplane $H(\alpha_3, \delta_3)$ crossing four twist flats $\mathcal{M}(X)$, $\mathcal{M}(X')$, $\mathcal{M}(X'')$, and $\mathcal{M}(X''')$, and three switch bridges $\mathcal{M}(X, X')$, $\mathcal{M}(X, X'')$, and $\mathcal{M}(X, X''')$. Notice also that $N(H(\alpha_3, \delta_3))$ contains vertices with dual curves δ_3 and $T_{\alpha_3}(\delta_3)$.

The combinatorial hyperplanes associated to a twist hyperplane will be referred to as *combinatorial twist hyperplanes.* We will denote the two combinatorial twist hyperplanes associated to a twist hyperplane $H(\alpha_1, \delta_1)$ by $C(\alpha_1, \delta_1)$ and $C(\alpha_1, T_{\alpha_1}(\delta_1))$. Here, $C(\alpha_1, \delta_1)$ is the combinatorial hyperplane whose vertices all contain the pants curve α_1, and whose intersection with the twist flat $\mathcal{M}(X)$ as in the previous paragraph contain vertices the with dual curve δ_1. Similarly for $C(\alpha_1, T_{\alpha_1}(\delta_1))$.

Remark 3.2. It is worth noting that any twist hyperplane $H(\alpha_1, \delta_1)$ or combinatorial twist hyperplane $C(\alpha_1, \delta_1)$ will have more than one name. This is because we determined the name of our hyperplanes by looking at its intersection with a specific twist flat $\mathcal{M}(X)$. For examples, if we had chosen instead to look at the intersection of the hyperplane with $\mathcal{M}(X')$ where X' is obtained from X by switching α_2, then another name for $H(\alpha_1, \delta_1)$ could be $H(\alpha_1, c(\delta_1))$ where $c(\delta_1)$ is the cleanup of δ_1 after switching α_2.

3.3. **Parallelism classes of combinatorial hyperplanes.** In this section, we determine the parallelism classes for the two types of combinatorial hyperplanes in $\mathcal{M}$. Recall that the parallelism class of a combinatorial hyperplane C is the equivalence class of all convex subcomplexes that are parallel to C. In a general CAT(0) cube complex, it is not necessarily true that all convex subcomplexes that are parallel to a combinatorial hyperplane are themselves combinatorial hyperplanes, but we will see that this is the case in $\mathcal{M}$.

Lemma 3.3. *The parallelism class for a combinatorial twist hyperplane $C(\alpha_1, \delta_1)$ is the collection $[C(\alpha_1, \delta_1)] = \bigcup_{k\in\mathbb{Z}}\{T^k_{\alpha_1}(C(\alpha_1, \delta_1))\}$. In other words, $[C(\alpha_1, \delta_1)]$ consists of all combinatorial twist hyperplanes whose image under the projection $p : \mathcal{M} \to \mathcal{P}$ is $\mathcal{P}(\alpha_1)$.*

Proof. Let $C(\alpha_1, \delta_1)$ be any combinatorial twist hyperplane, and fix any $F \in [C(\alpha_1, \delta_1)]$. One class of hyperplanes that cross $C(\alpha_1, \delta_1)$ are switch hyperplanes that correspond to switching pants curves other than α_1. Specifically, Lemma 3.1 implies that the switch bridges crossed by $C(\alpha_1, \delta_1)$ correspond exactly to the edges of $\mathcal{P}(\alpha_1)$. This means that F must also cross every switch bridge corresponding edges of $\mathcal{P}(\alpha_1)$. Moreover, because F intersects the same switch hyperplanes as $C(\alpha_1, \delta_1)$, it must also intersect all of the same twist flats. In particular, $p(F) = \mathcal{P}(\alpha_1)$.

We also know that $C(\alpha_1, \delta_1)$ does not cross the twist hyperplanes $H(\alpha_1, T_{\alpha_1}^{n_1}(\delta_1))$, where $n_1 \in \mathbb{Z}$. This means that F must be contained in some combinatorial twist hyperplane $C(\alpha_1, T_{\alpha_1}^{n_1}(\delta_1))$. Via the cubical isometric embedding described in Lemma 2.2, F must be an entire twist hyperplane $C(\alpha_1, T_{\alpha_1}^{n_1}(\delta_1))$. Therefore, F is a combinatorial twist hyperplane with $p(F) = \mathcal{P}(\alpha_1)$, and the parallelism class corresponds exactly to the combinatorial twist hyperplanes in $\bigcup_{k\in\mathbb{Z}}\{T_{\alpha_1}^{k}(C(\alpha_1, \delta_1))\}$. □

Corollary 3.4. *The action of $\langle T_{\alpha_1}\rangle$ on $[C(\alpha_1, \delta_1)]$ is simply transitive, (i.e. is transitive and free). Consequently, the set $[C(\alpha_1, \delta_1)]$ has infinite cardinality.*

Proof. By Lemma 3.3, if $C(\alpha_1, \delta_1)$ and $C(\alpha_1, \delta_1')$ are parallel twist hyperplanes, then we know that $C(\alpha_1, \delta_1') = C(\alpha_1, T_{\alpha_1}^{n_1}(\delta_1))$ for exactly one $n_1 \in \mathbb{Z}$. Hence, the action of $\langle T_{\alpha_1}\rangle$ on $[C(\alpha_1, \delta_1)]$ is simply transitive.

Since $\langle T_{\alpha_1}\rangle$ has infinite order and acts simply transitively on $[C(\alpha_1, \delta_1)]$, it follows that $[C(\alpha_1, \delta_1)]$ has infinite cardinality. □

Lemma 3.5. *Let $X = \{\alpha_1, \alpha_2, \alpha_3\}$ be a pants decomposition of non-separating meridians, and let δ_1 be a dual to α_1. The parallelism class for the switch combinatorial hyperplane $C(X, \delta_1)$ consists of all combinatorial switch hyperplanes that correspond to edges in $\mathcal{P}(\{\alpha_2, \alpha_3\})$.*

Proof. Because every switch hyperplane is contained in its own switch bridge and because combinatorial switch hyperplanes do not cross switch bridges, $C(X, \delta_1)$ does not cross any switch hyperplanes. This means that if $F \in [C(X, \delta_1)]$, it must be contained in a single twist flat. Additionally, F does not cross any twist hyperplanes of the form $H(\alpha_1, \delta_1')$, where δ_1' is any dual to α_1. We can see this via the definition of parallelism and the fact that $C(X, \delta_1)$ does not cross any hyperplanes of the form $H(\alpha_1, \delta_1')$. Since F must be contained in a single twist flat and does not cross any twist hyperplane of the form $H(\alpha_1, \delta_1')$, it follows that F must be contained in a combinatorial switch hyperplane of the form $C(X', \delta_1')$ where $X' \in \mathcal{P}(\{\alpha_2, \alpha_3\})^{(0)}$ and δ_1' is a dual to $X' - \{\alpha_1, \alpha_2\}$. Via the cubical isometric embedding described in Lemma 2.2, F must actually be an entire combinatorial hyperplane $C(X', \delta_1')$, i.e. a combinatorial switch hyperplane corresponding to an edge in $\mathcal{P}(\{\alpha_2, \alpha_3\})$.

The two families of hyperplanes that have non-empty intersection with $C(X, \delta_1)$ are twist hyperplanes of the form $H(\alpha_2, \delta_2)$ and $H(\alpha_3, \delta_3)$, where δ_2 and δ_3 are any duals to α_2 and α_3 respectively. So in fact the parallelism class of $C(X, \delta_1)$ contains all switch combintorial hyperplanes corresponding to edges in $p(H(\alpha_2, \delta_2) \cap H(\alpha_3, \delta_3)) = \mathcal{P}(\{\alpha_2, \alpha_3\})$. □

Let $\mathrm{Stab}(\alpha_2, \alpha_3)$ denote the subgroup of $\mathcal{H}_2$ that fixes both α_2 and α_3, without interchanging them. For the following corollary, note that the subgroup $\langle T_{\alpha_2}, T_{\alpha_3}\rangle$ is contained in the center of $\mathrm{Stab}(\alpha_2, \alpha_3)$, so it is a normal subgroup, (see for example [FM12, Fact 3.8]). Furthermore, notice that $\langle T_{\alpha_2}, T_{\alpha_3}\rangle$ is in the kernel of the group action because every element of this group fixes the cut system $\{\alpha_2, \alpha_3\}$.

Corollary 3.6. *The group $\mathrm{Stab}(\alpha_2, \alpha_3)/\langle T_{\alpha_2}, T_{\alpha_3}\rangle$ acts simply transitively on $[C(X, \delta_1)]$ where X and δ_1 are as in Lemma 3.5. Consequently, the set $[C(X, \delta_1)]$ has infinite cardinality.*

Proof. By Lemma 3.5, it is clear that $\mathrm{Stab}(\alpha_2, \alpha_3)$ acts transitively on $[C(X, \delta_1)]$. Additionally, since $\langle T_{\alpha_2}, T_{\alpha_3} \rangle$ is in the kernel of the group action, $\mathrm{Stab}(\alpha_2, \alpha_3)/\langle T_{\alpha_2}, T_{\alpha_3} \rangle$ also acts transitively on $[C(X, \delta_1)]$. It remains to show that the action of $\mathrm{Stab}(\alpha_2, \alpha_3)/\langle T_{\alpha_2}, T_{\alpha_3} \rangle$ is free.

Let $\phi, \psi \in \mathrm{Stab}(\alpha_2, \alpha_3)$ such that $\phi \cdot C(X, \delta_1) = \psi \cdot C(X, \delta_1)$. This means that $\psi^{-1}\phi$ must fix $C(X, \delta_1)$. In order to fix $C(X, \delta_1)$, $\psi^{-1}\phi$ is allowed to alter the duals to α_2 and α_3 by Dehn twists, but any curve disjoint from $\alpha_2 \cup \alpha_3$ must be fixed. It then follows that $\psi^{-1}\phi \in \langle T_{\alpha_2}, T_{\alpha_3} \rangle$. Hence, the action of $\mathrm{Stab}(\alpha_2, \alpha_3)/\langle T_{\alpha_2}, T_{\alpha_3} \rangle$ is free as well.

The group $\mathrm{Stab}(\alpha_2, \alpha_3)$ has infinite cardinality, (it contains Dehn twists along α_1, for example), and the group $\langle T_{\alpha_2}, T_{\alpha_3} \rangle$ has infinite index in $\mathrm{Stab}(\alpha_2, \alpha_3)$, (consider cosets of powers of Dehn twists along α_1). It follows that $\mathrm{Stab}(\alpha_2, \alpha_3)/\langle T_{\alpha_2}, T_{\alpha_3} \rangle$ has infinite cardinality, and since $\mathrm{Stab}(\alpha_2, \alpha_3)/\langle T_{\alpha_2}, T_{\alpha_3} \rangle$ acts simply transitively on $[C(X, \delta_1)]$, the cardinality of $[C(X, \delta_1)]$ is infinite. □

3.4. **Edges in the contact graph.** Recall that if there is an edge between two hyperplanes H_1 and H_2 in a contact graph. then either H_1 and H_2 cross or they osculate. In this section we make some observations regarding what it looks like for different types of hyperplanes in $\mathcal{M}$ to cross or osculate.

Let us start by considering two switch hyperplanes that are connected by an edge in the contact graph. Each switch hyperplane is contained in its own switch bridge, so no two switch hyperplanes can cross each other. In order for two switch hyperplanes to osculate, they must be adjacent to a common twist flat, so we can name our hyperplanes $H(X, X')$ and $H(X, X'')$. If $H(X, X')$ and $H(X, X'')$ correspond to switching the same curve, then $H(X, X')^{\pm}$ and $H(X, X'')^{\pm}$ will be parallel and disjoint, so the hyperplanes cannot osculate. If on the other hand $H(X, X')$ and $H(X, X'')$ correspond to switching different curves, then the two hyperplanes will osculate. An example of osculating switch hyperplanes can be seen in Figure 6.

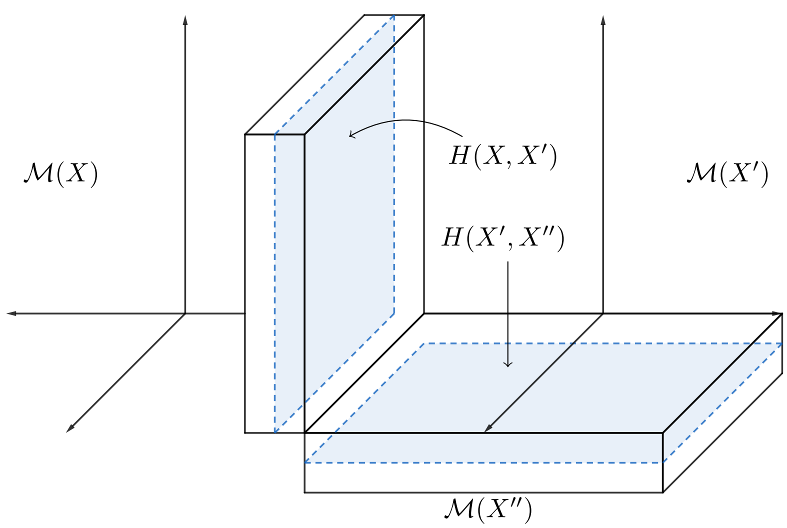


FIGURE 6. An example of two switch hyperplanes that osculate, and are thus connected by an edge in $\mathcal{CM}$.

Consider now two twist hyperplanes $H(\alpha, \delta)$ and $H(\alpha', \delta')$ that are connected by an edge in the contact graph. These hyperplanes can either cross or osculate. In the case that the hyperplanes cross, the image of their intersection under p will be $p(H(\alpha,\delta) \cap H(\alpha',\delta')) = \mathcal{P}(\{\alpha,\alpha'\})$. In the case that $H(\alpha,\delta)$ and $H(\alpha',\delta')$ osculate, the two hyperplanes are actually parallel to one another. In fact, $\alpha = \alpha'$, and one of the combinatorial hyperplanes associated to $H(\alpha,\delta)$ must be one of the combinatorial hyperplanes associated to $H(\alpha',\delta')$. See figure 7 for examples of these cases.

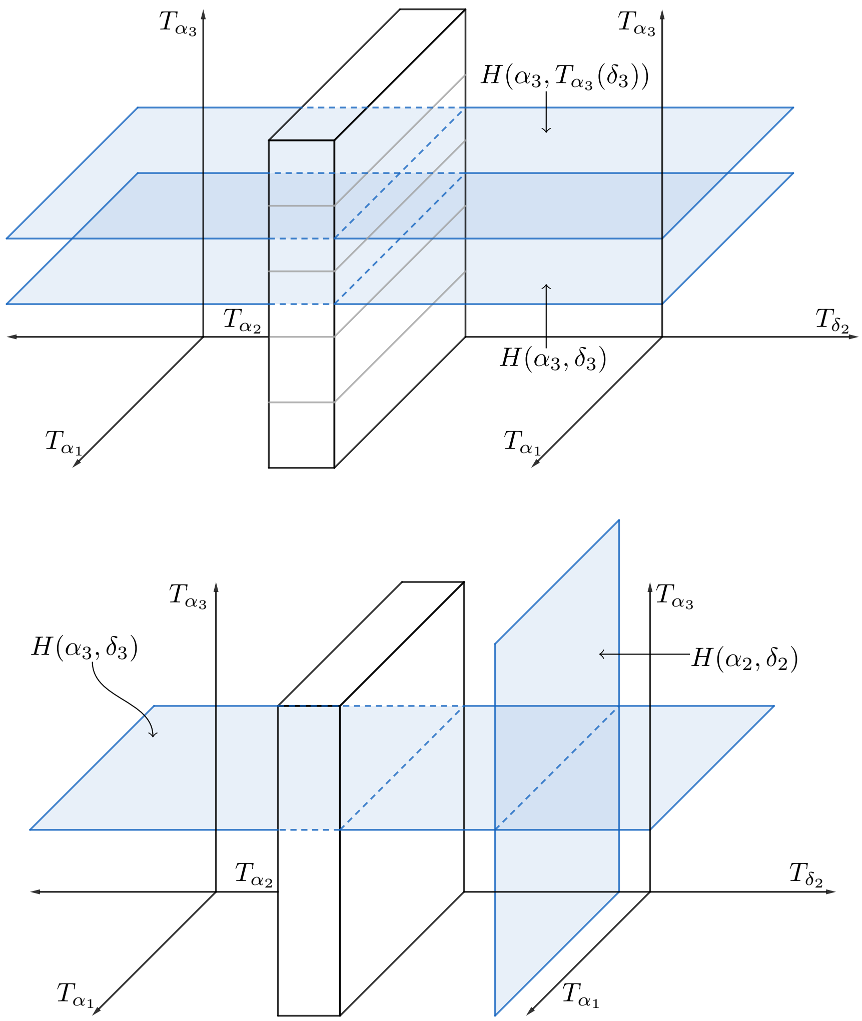


FIGURE 7. Examples of twist hyperplanes connected by an edge in $\mathcal{CM}$. Top: two twist hyperplanes that osculate. Bottom: two twist hyperplanes that cross.

Lastly, let us consider when a switch hyperplane $H(X,X')$ is connected by an edge in the contact graph to a twist hyperplane $H(\alpha,\delta)$. Again, it is possible for the two hyperplane to either cross or osculate. In the case that they osculate, the switch hyperplane $H(X,X')$ will actually be parallel into $H(\alpha,\delta)$. In fact, one of the combinatorial hyperplanes associated to $H(X,X')$ will be contained in one of the combinatorial hyperplanes associated to $H(\alpha,\delta)$. Figure 8 illustrates these two cases.

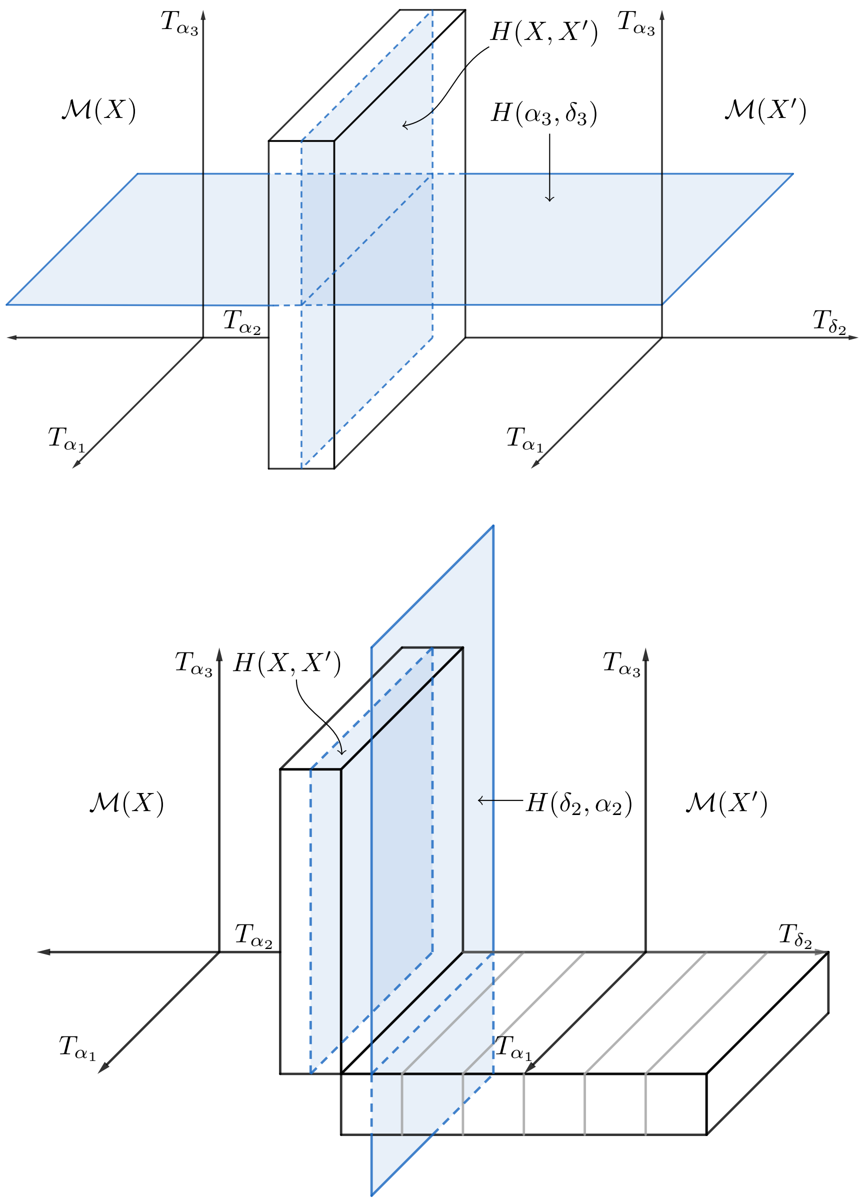


FIGURE 8. Examples of a twist and switch hyperplane connected by an edge in $\mathcal{CM}$. Top: twist and switch hyperplanes crossing. Bottom: twist and switch hyperplanes osculating.

3.5. **Intersections of combinatorial hyperplanes.** The construction of a factor system for $\mathcal{M}$ involves understanding the non-empty intersections of combinatorial hyperplanes. In this section, we classify the subcomplexes that are obtained by taking the non-empty intersection of two combinatorial hyperplanes.

Suppose F_1 and F_2 are two combinatorial hyperplanes that have non-empty intersection. Based on what type of hyperplanes F_1 and F_2 are, we get three different cases.

(1) Let $F_1 = C(X, \delta)$ and $F_2 = C(X, \delta')$ be two distinct combinatorial switch hyperplanes. Recall that every vertex in $C(X, \delta)$ must contain δ as a dual curve, and every vertex in $C(X, \delta')$ must contain δ' as a dual curve. The intersection $F_1 \cap F_2$ of the two combinatorial hyperplanes is then an isometrically embedded copy of $\mathbb{R}$ contained in $\mathcal{M}(X)^{(1)}$ such that every vertex contains both δ and δ' as dual curves. We will denote such a line by $l(X, \delta, \delta')$.
(2) Let $F_1 = C(\alpha, \delta)$ and $F_2 = C(\alpha', \delta')$ be two distinct combinatorial twist hyperplanes, where δ, δ' are duals to α, α' in the same twist flat $\mathcal{M}(X)$. Then the intersection $F_1 \cap F_2$ will be a tree $t(\alpha, \alpha', \delta, \delta')$ such that every vertex contains both α and α' as pants curves, and such that the intersection with $\mathcal{M}(X)$ has δ, δ' as duals to α, α'. Any non-empty intersection of $t(\alpha, \alpha', \delta, \delta')$ with a twist flat will be a copy of $\mathbb{R}$, (in particular a line of the type described in (1)), and these copies of $\mathbb{R}$ will be connected across switch bridges to other copies of $\mathbb{R}$ via intervals $[-\frac{1}{2}, \frac{1}{2}]$. In terms of the map p, $p|_{t(\alpha,\alpha',\delta,\delta')}$ maps onto $\mathcal{P}(\{\alpha, \alpha'\})$ and the fiber of a vertex $X \in \mathcal{P}(\{\alpha, \alpha'\})^{(0)}$ is the line $l(X, \delta, \delta')$. In this way we can think of $t(\alpha, \alpha', \delta, \delta')$ as a kind of "blow-up" of the tree $\mathcal{P}(\{\alpha, \alpha'\})$.
(3) Now $F_1 = C(\alpha, \delta)$ be a combinatorial twist hyperplane and let $F_2 = C(X, \delta')$ be a combinatorial switch hyperplane. It is possible that $C(X, \delta') \subset C(\alpha, \delta)$, in which case the intersection is just $C(X, \delta')$. Otherwise, the intersection of the two will be an isometrically embedded copy of $\mathbb{R}$ as in case (1) above, (since $C(X, \delta') \subset \mathcal{M}(X)$ and the intersection of $C(\alpha, \delta)$ with $\mathcal{M}(X)$ is the combinatorial switch hyperplane $C(X, \delta)$).

4. Constructing a Factor System with Unbounded Products

In this section, we show that $\mathcal{M}$ contains a factor system, and that consequently $\mathcal{H}_2$ is an HHG. Furthermore, we show that the factor system has unbounded products.

4.1. **Characterizing the hyperclosure.** Let $\mathfrak{F}$ be the hyperclosure of $\mathcal{M}$. In this section, we will use Lemma 2.7 to determine exactly what subcomplexes are contained in $\mathfrak{F}$. In particular, we will show that $\mathfrak{F}$ is equal to the set $\mathfrak{F}'$ consisting of the set of subcomplexes of the following types:

(1) the whole space $\mathcal{M}$,
(2) subcomplexes $F_1 \cap F_2$ such that F_1 and F_2 are (not necessarily distinct) combinatorial hyperplanes of $\mathcal{M}$, and
(3) 0-cubes of $\mathcal{M}$.

Proposition 4.1. *$\mathfrak{F} = \mathfrak{F}'$.*

One direction of the containment is straight forward. Along the way to proving it, we describe all of the elements contained in $\mathfrak{F}'$.

Lemma 4.2. *If $F \in \mathfrak{F}'$, then F falls into one of the following categories:*

(1) $F = \mathcal{M}$,
(2) $F = C(X, \delta)$ *is a combinatorial switch hyperplane,*
(3) $F = C(\alpha, \delta)$ *is a combinatorial twist hyperplane,*
(4) $F = l(X, \delta, \delta')$ *is the intersection of two combinatorial switch hyperplanes,*
(5) $F = t(\alpha, \alpha', \delta, \delta')$ *is the intersection of two combinatorial twist hyperplanes, or*
(6) F *is a* 0*-cube of* $\mathcal{M}$*.*

Furthermore, $\mathfrak{F}' \subset \mathfrak{F}$*.*

Proof. The characterization of $F \in \mathfrak{F}'$ follows from the definition of $\mathfrak{F}'$ and the analysis in Section 3.5.

For the containment $\mathfrak{F}' \subset \mathfrak{F}$, notice that subcomplexes of types (1)-(3) are in $\mathfrak{F}$ because they satisfy properties (1) and (2) of Definition 2.6, (so they are contained in every set $\mathfrak{G}$ as in Definition 2.6). Additionally, by Lemma 2.4, subcomplexes of types (4) and (5) are projections of combinatorial hyperplanes, so they are contained in $\mathfrak{F}$ via properties (2) and (3) of Definition 2.6. Lastly, any 0-cube (X, Δ), where $\Delta = \{\delta_1, \delta_2, \delta_3\}$, can be viewed as the intersection of the three combinatorial hyperplanes $\{C(X, \delta_i)\}_{i=1}^3$. Hence, by Lemma 2.4,

$$(X, \Delta) = C(X, \delta_1) \cap C(X, \delta_2) \cap C(X, \delta_3) = \mathfrak{g}_{C(X,\delta_1)}(\mathfrak{g}_{C(X,\delta_2)}(C(X, \delta_3))) \in \mathfrak{F}.$$

Therefore, $\mathfrak{F}' \subset \mathfrak{F}$. □

To continue towards proving Proposition 4.1, we prove the following lemma, which allows us to factor projections of subcomplexes of $\mathcal{M}$ through projections to combinatorial switch hyperplanes. Note that this is a version of Lemma 2.1 from [HS20] that is specific to our context.

Lemma 4.3. *Suppose $\{C_i\}_{i=1}^n$ is a collection of combinatorial switch hyperplanes separating two subcomplexes $F_1, F_2 \subset \mathcal{M}$. Suppose the C_i are ordered by distance to F_1, with C_n being closest to F_1 and C_1 being furthest from F_1. Then $\mathfrak{g}_{F_1}(F_2)$ is parallel to $\mathfrak{g}_{F_1}(\mathfrak{g}_{C_n}(\cdots \mathfrak{g}_{C_1}(F_2) \cdots))$.*

Proof. We prove this inductively, and use Lemma 2.3, which says that the hyperplanes crossing $\mathfrak{g}_{F_1}(F_2)$ are exactly those crossing both F_1 and F_2.

Suppose for the base case that there is a combinatorial switch hyperplane C separating F_1 and F_2. If H is a hyperplane crossing both F_1 and F_2, (and therefore also $\mathfrak{g}_{F_1}(F_2)$ by Lemma 2.3), then H must cross C as well. This is because F_1 and F_2 are separated by C, and hence cannot cross any of the same twist flats, so any hyperplane crossing F_1 and F_2 must be a twist hyperplane crossing C as well. By Lemma 2.3, it follows that H crosses $\mathfrak{g}_C(F_2)$, and one more application of this lemma to F_1 and $\mathfrak{g}_C(F_2)$ implies H crosses $\mathfrak{g}_{F_1}(\mathfrak{g}_C(F_2))$. So if a hyperplane H crosses $\mathfrak{g}_{F_1}(F_2)$, it must also cross $\mathfrak{g}_{F_1}(\mathfrak{g}_C(F_2))$.

Now suppose instead that H is a hyperplane that crosses $\mathfrak{g}_{F_1}(\mathfrak{g}_C(F_2))$. By applying Lemma 2.3 several times, we get that H must cross F_1, C, and F_2. Since H crosses both F_1 and F_2, applying Lemma 2.3 one last time tells us that H crosses $\mathfrak{g}_{F_1}(F_2)$. We have now shown that any hyperplane H crosses $\mathfrak{g}_{F_1}(F_2)$ if and only if H crosses $\mathfrak{g}_{F_1}(\mathfrak{g}_C(F_2))$, and hence the two subcomplexes are parallel.

For the inductive step, assume that when F_1 and F_2 are separated by $n-1$ ordered combinatorial switch hyperplanes $\{C_i\}_{i=1}^{n-1}$, then $\mathfrak{g}_{F_1}(F_2)$ is parallel to $\mathfrak{g}_{F_1}(\mathfrak{g}_{C_{n-1}}(\cdots \mathfrak{g}_{C_1}(F_2) \cdots))$. Now suppose F_1 and F_2 are separated by at least n hyperplanes, and that $\{C_i\}_{i=1}^n$ is a collection of combinatorial switch hyperplanes separating the two subcomplexes, ordered so that C_n is closest to F_1 and C_1 is closest to F_2. Then $\mathfrak{g}_{C_1}(F_2)$ is a subcomplex that is separated from F_1 by $n-1$ hyperplanes. Applying the inductive hypothesis tells us that $\mathfrak{g}_{F_1}(\mathfrak{g}_{C_n}(\cdots \mathfrak{g}_{C_1}(F_2) \cdots))$ is parallel to $\mathfrak{g}_{F_1}(\mathfrak{g}_{C_1}(F_2))$. By the same reasoning as the base case, $\mathfrak{g}_{F_1}(\mathfrak{g}_{C_1}(F_2))$ is parallel to $\mathfrak{g}_{F_1}(F_2)$. Since parallelism is an equivalence relation, it follows that $\mathfrak{g}_{F_1}(F_2)$ is parallel to $\mathfrak{g}_{F_1}(\mathfrak{g}_{C_n}(\cdots \mathfrak{g}_{C_1}(F_2) \cdots))$. □

We also show that $\mathfrak{F}'$ is closed under parallelism.

Lemma 4.4. *If $F \in \mathfrak{F}'$ and $F' \subset \mathcal{M}$ is a convex subcomplex that is parallel to F, then $F' \in \mathfrak{F}'$.*

Proof. We will prove this lemma in four cases.

Case 1: $\mathcal{M}$ is contained in its own parallelism class since it is the only convex subcomplex of $\mathcal{M}$ that intersects every hyperplane. Additionally, all 0-cubes are parallel to one another since they are the only convex subcomplexes that do not intersect any hyperplane. Hence, if $F = \mathcal{M}$ or F is a 0-cube, then $F' \in \mathfrak{F}'$.

Case 2: Suppose F is a combinatorial hyperplane. By Lemmas 3.3 and 3.5, F' must also be a combinatorial hyperplane. Hence, $F' \in \mathfrak{F}'$.

Case 3: Suppose $F = l(X, \delta_1, \delta_2)$, where $X = \{\alpha_1, \alpha_2, \alpha_3\}$ and δ_1, δ_2 are duals to α_1, α_2, respectively. Because F is contained entirely within a single twist flat, it does not intersect any switch hyperplanes. This means that F' also does not intersect any switch hyperplanes and must therefore be contained in a single twist flat. The only hyperplanes that F does intersect are twist hyperplanes of the form $H(\alpha_3, T_{\alpha_3}^{n_3}(\delta_3))$, where δ_3 is a dual to α_3 in $\mathcal{M}(X)$ and $n_3 \in \mathbb{Z}$. Additionally, any twist hyperplane that crosses a hyperplane $H(\alpha_3, T_{\alpha_3}^{n_3}(\delta_3))$ does not cross F. In particular, any twist hyperplanes $H(\alpha_1', \delta_1')$ and $H(\alpha_2', \delta_2')$ such that $\{\alpha_1', \alpha_2', \alpha_3\} \in \mathcal{P}(\alpha_3)^{(0)}$ do not cross F. Lemma 2.2 tells us that $F' \cong F \cong \mathbb{R}$. Combining the above facts implies that F' is a line $l(X', \delta_1', \delta_2')$, where $X' \in \mathcal{P}(\alpha_3)^{(0)}$ and δ_1', δ_2' are duals to the curves in $X' - \{\alpha_3\}$. Hence, $F' \in \mathfrak{F}'$.

Case 4: Suppose $F = t(\alpha_1, \alpha_2, \delta_1, \delta_2)$, and let $X = \{\alpha_1, \alpha_2, \alpha_3\}$, (so F has non-empty intersection with $\mathcal{M}(X)$). First, we know that F intersects switch hyperplanes that correspond to edges in $p(H(\alpha_1, \delta_1) \cap H(\alpha_2, \delta_2)) = \mathcal{P}(\{\alpha_1, \alpha_2\})$, meaning that F' must also intersect each of these switch hyperplanes. In particular, this means $p(F') = \mathcal{P}(\{\alpha_1, \alpha_2\})$. Additionally, F does not intersect the twist hyperplanes $H(\alpha_1, T_{\alpha_1}^{n_1}(\delta_1))$ and $H(\alpha_2, T_{\alpha_2}^{n_2}(\delta_2))$ where $n_1, n_2 \in \mathbb{Z}$. Lemma 2.2 tells us that $F' \cong F$. Combining the above facts implies that F' must be a tree $t(\alpha_1, \alpha_2, T_{\alpha_1}^{n_1}(\delta_1), T_{\alpha_2}^{n_2}(\delta_2))$. Thus, $F' \in \mathfrak{F}'$. $\square$

We are now equipped to prove Proposition 4.1.

Proof of Proposition 4.1. First, recall via Lemma 4.2 that $\mathfrak{F}' \subset \mathfrak{F}$. Thus, it remains to show that $\mathfrak{F} \subset \mathfrak{F}'$.

To do this, we will explicitly determine the convex subcomplexes contained in $\mathfrak{F}$, and will see that they are indeed elements of $\mathfrak{F}'$. To this end, we will use the characterization of the hyperclosure given by Lemma 2.7, i.e. $\mathfrak{F} = \cup_{n \geq 0} \mathfrak{F}_n$, where $\mathfrak{F}_0 = \{\mathcal{M}\}$ and $\mathfrak{F}_n$ for $n \geq 1$ is the set of convex subcomplexes of $\mathcal{M}$ that can be written in the form $\mathfrak{g}_C(F)$ for some $C \in \mathfrak{C}$ and $F \in \mathfrak{F}_{n-1}$.

We will start by showing that projections $\mathfrak{g}_C(F)$, where C and F are not separated by any combinatorial switch hyperplanes, are all contained in $\mathfrak{F}'$. Then we will show that for any n, any projection of a subcomplex $F \in \mathfrak{F}_n$ onto a combinatorial hyperplane can be decomposed into projections between subcomplexes that are not separated by any combinatorial switch hyperplanes.

Case 1: Let $\mathfrak{F}_0' = \mathfrak{F}_0$, and let $\mathfrak{F}_n'$ be the set of convex subcomplexes that can be written as $\mathfrak{g}_C(F_{n-1})$ such that $C \in \mathfrak{C}$, $F_{n-1} \in \mathfrak{F}_{n-1}'$, and C and F_{n-1} are not separated by a switch hyperplane. We will show first that $\cup_{n \geq 0} \mathfrak{F}_n' \subset \mathfrak{F}'$.

For $n = 1$, we have $\mathfrak{F}_1 = \mathfrak{F}_1' = \mathfrak{C}$ since no combinatorial hyperplane is separated from $\mathcal{M}$ by any other combinatorial hyperplane.

Suppose $F_2 \in \mathfrak{F}_2'$, meaning that $F_2 = \mathfrak{g}_C(F_1)$ for some $C \in \mathfrak{C}$ and some $F_1 \in \mathfrak{F}_1' = \mathfrak{C}$, where F_1 and C are not separated by any combinatorial switch hyperplanes. Because F_1 and C are not separated by a switch hyperplane, both F_1 and C have non-empty intersection with some twist flat $\mathcal{M}(X)$. Futher, because F_1 and C are combinatorial hyperplanes that intersect a common twist flat, one of the following must hold:

(i) F_1 and C are parallel;
(ii) one of F_1, C is parallel into the other; or
(ii) F_1 and C have non-empty intersection.

(It cannot be that F_1 and C are disjoint and neither is parallel into the other because $F_1 \cap \mathcal{M}(X)$ and $C \cap \mathcal{M}(X)$ must both be combinatorial switch hyperplanes contained in $\mathcal{M}(X) \cong \mathbb{R}^3$). Thus, F_2 is determined by one of the following cases.

(a) If F_1 and C are parallel, then by Lemmas 2.2, 3.3, and 3.5, $F_2 = \mathfrak{g}_C(F_1)$ is a combinatorial hyperplane.
(b) If F_1 is parallel into C, then $\mathfrak{g}_C(F_1)$ is the subcomplex of C that is parallel to F_1. Again by Lemmas 2.2, 3.3, and 3.5, $F_2 = \mathfrak{g}_C(F_1)$ is a combinatorial hyperplane.
(c) If C is parallel into F_1, we know that $\mathfrak{g}_{F_1}(C)$ is the subcomplex of F_1 that is parallel to C. Then Lemma 2.4 implies $\mathfrak{g}_C(F_1)$ is also parallel to $\mathfrak{g}_{F_1}(C)$. Again by Lemmas 2.2, 3.3, and 3.5, $F_2 = \mathfrak{g}_C(F_1)$ is a combinatorial hyperplane.
(d) If F_1 and C have non-empty intersection, then Lemma 2.4 tells us that $\mathfrak{g}_C(F_1) = F_1 \cap C$, which is the intersection of two combinatorial hyperplanes. Specifically, by the analysis in Section 3.5, $F_2 = \mathfrak{g}_C(F_1)$ is either a combinatorial switch hyperplane, a line $l(X, \delta_1, \delta_2)$, or a tree $t(\alpha_1, \alpha_2, \delta_1, \delta_2)$.

In each of these cases, F_2 is a convex subcomplex found in $\mathfrak{F}'$. Thus, $\mathfrak{F}'_2 \subset \mathfrak{F}'$.

Now suppose $F_3 \in \mathfrak{F}'_3$, meaning that $F_3 = \mathfrak{g}_C(F_2)$ for some $C \in \mathfrak{C}$ and some $F_2 \in \mathfrak{F}'_2$ where F_2 and C are not separated by any combinatorial switch hyperplanes. Because $F_2 \in \mathfrak{F}'_2$, we know that F_2 is either a combinatorial hyperplane, a tree, or a line. This means F_3 is determined by one of the following cases.

(a) If F_2 is a combinatorial hyperplane, then $F_2 \in \mathfrak{F}'_1$ and $F_3 = \mathfrak{g}_C(F_2) \in \mathfrak{F}'_2$. We already know that F_3 will be a combinatorial hyperplane, a tree, or a line.
(b) Suppose $F_2 = l(X, \delta_1, \delta_2)$. Since we have assumed that C and F_2 are not separated by a switch hyperplane, they must both intersect the twist flat $\mathcal{M}(X)$. This means that either F_2 is parallel into C, or F_2 intersects C in a single point. In the first case, $F_3 = \mathfrak{g}_C(F_2)$ will be the line contained in C that is parallel to F_2, and in the second case $F_3 = \mathfrak{g}_C(F_2)$ is the single 0-cube of intersection.
(c) Suppose $F_2 = t(\alpha_1, \alpha_2, \delta_1, \delta_2)$. Then the classification of $\mathfrak{g}_C(F_2)$ depends on whether C is a combinatorial switch or twist hyperplane.
    (i) Suppose C is a combinatorial switch hyperplane, and say it is contained in the twist flat $\mathcal{M}(X)$, where $\alpha_1, \alpha_2 \in X$. Then $\mathfrak{g}_C(F_2) \subset C \subset \mathcal{M}(X)$. The intersection of F_2 with $\mathcal{M}(X)$ is the line $l(X, \delta_1, \delta_2)$. If C and F_2 have non-empty intersection, then $\mathfrak{g}_C(F_2) = C \cap F_2$ will be a single 0-cube or the line $l(X, \delta_1, \delta_2)$. If instead C and F_2 are disjoint, then $\mathfrak{g}_C(F_2)$ will be the line contained in C that is parallel to F_2. This can be seen via Lemma 2.3 and the fact that the only hyperplanes intersecting both C and F_2 will be the collection of twist hyperplanes $H(\alpha_3, \delta_3)$, where $\alpha_3 \in X - \{\alpha_1, \alpha_2\}$.
    (ii) Suppose C is a combinatorial twist hyperplane. To determine the projection $\mathfrak{g}_C(F_2)$ we must understand how the two combinatorial twist hyperplanes $C(\alpha_1, \delta_1)$ and $C(\alpha_2, \delta_2)$ relate to C. Because F_2 and C intersect a common twist flat, it must be that C intersects one or both of $C(\alpha_1, \delta_1)$ and $C(\alpha_2, \delta_2)$. If C intersects both $C(\alpha_1, \delta_1)$ and $C(\alpha_2, \delta_2)$, then either their intersection is a single 0-cube, (meaning $\mathfrak{g}_C(F_2)$ is that 0-cube), or the intersection is the entire tree, meaning $\mathfrak{g}_C(F_2)$ is the entire tree. If C intersects only one of $C(\alpha_1, \delta_1)$ and $C(\alpha_2, \delta_2)$, say

to $C(\alpha_1, \delta_1)$, then C must be parallel to $C(\alpha_1, \delta_1)$, and F_2 must be parallel into C. This means that $\mathfrak{g}_C(F_2)$ will be the tree contained in C that is parallel to F_2.

Thus, the convex subcomplexes contained in $\mathfrak{F}_3'$ are either combinatorial hyperplanes, trees, lines, or single 0-cubes, meaning $\mathfrak{F}_3' \subset \mathfrak{F}'$. Additionally, we see that $\mathfrak{F}_3' = \mathfrak{F}_2' \cup \{0-\text{cubes}\}$. By this description, we can see that projections of elements of $\mathfrak{F}_3'$ to combinatorial hyperplanes are still elements of $\mathfrak{F}_3'$. This means that for $n \geq 3$, $\mathfrak{F}_n' = \mathfrak{F}_3' \subset \mathfrak{F}'$. Thus, $\cup_{n\geq 0}\mathfrak{F}_n' \subset \mathfrak{F}'$.

Case 2: For the general case, we will show that if $F \in \mathfrak{F}_n$, then $F \in \cup_{n\geq 0}\mathfrak{F}_n' \subset \mathfrak{F}'$. We proceed by induction.

Recall that $\mathfrak{F}_0 = \mathfrak{F}_0'$ and $\mathfrak{F}_1 = \mathfrak{F}_1'$. Clearly, for $n = 0$ or 1, $\mathfrak{F}_n \subset \mathfrak{F}'$.

For the inductive step, suppose that for $k \geq 2$, if $F_{k-1} \in \mathfrak{F}_{k-1}$, then $F_{k-1} \in \cup_{n\geq 0}\mathfrak{F}_n'$. Now suppose that $F_k \in \mathfrak{F}_k$. This means $F_k = \mathfrak{g}_C(F_{k-1})$ for some combinatorial hyperplane C and some $F_{k-1} \in \mathfrak{F}_{k-1}$. Suppose that $\{C_i\}_{i=1}^m$ are *all* of the combinatorial switch hyperplanes separating C from F_{k-1}, where the C_i are ordered such that C_1 is closest to F_{k-1} and C_m is furthest from F_{k-1}. Consider the subcomplex $\mathfrak{g}_C(\mathfrak{g}_{C_m}(\cdots \mathfrak{g}_{C_1}(F_{k-1})\cdots))$. By assumption, the subcomplexes F_{k-1} and C_1 are not separated by any combinatorial switch hyperplanes. The inductive hypothesis then tells us that $\mathfrak{g}_{C_1}(F_{k-1}) \in \cup_{n\geq 0}\mathfrak{F}_n' \subset \mathfrak{F}'$. Furthermore, because C_i and C_{i+1} are not separated by any combinatorial switch hyperplanes, $\mathfrak{g}_C(\mathfrak{g}_{C_m}(\cdots \mathfrak{g}_{C_1}(F_{k-1})\cdots)) \in \mathfrak{F}_{k+m}'$. By Lemma 4.3, we know that $\mathfrak{g}_C(\mathfrak{g}_{C_m}(\cdots \mathfrak{g}_{C_1}(F_{k-1})\cdots))$ is parallel to $F_k = \mathfrak{g}_C(F_{k-1})$, so Lemma 4.4 implies $F_k \in \cup_{n\geq 0}\mathfrak{F}_n' \subset \mathfrak{F}'$. This proves that $\mathfrak{F} \subset \mathfrak{F}'$.

Thus, we have shown that $\mathfrak{F} = \mathfrak{F}'$. □

Corollary 4.5. *Let $\mathfrak{M}$ be the closure of $\mathcal{M}$. Then $\mathfrak{M} = \mathfrak{F}$.*

Proof. Clearly $\mathfrak{F}' \subset \mathfrak{M} \subset \mathfrak{F}$. Then Proposition 4.1 implies $\mathfrak{F}' = \mathfrak{M} = \mathfrak{F}$. □

4.2. $\mathcal{H}_2$ **is an HHG with unbounded products.** With the classification of the subcomplexes of $\mathfrak{F}$ in hand, we can now prove that $\mathcal{H}_2$ is an HHG with unbounded products. We start by using our characterization of $\mathfrak{F}$ given by Proposition 4.1 to prove that $\mathfrak{F}$ is a factor system.

Lemma 4.6. *$\mathfrak{F}$ is a factor system for $\mathcal{M}$.*

Proof. By the definition of the hyperclosure, all that must be proven is that property (2) of Definition 2.5 is satisfied, i.e. that $\mathfrak{F}$ has finite multiplicity. We will use the classification $\mathfrak{F} = \mathfrak{F}'$ afforded by Proposition 4.1 to show that the finite multiplicity property holds with $N = 14$.

Let (X, Δ) be a vertex in $\mathcal{M}$, where $X = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\Delta = \{\delta_1, \delta_2, \delta_3\}$. Of course $(X, \Delta) \in \mathcal{M}$ and $(X, \Delta) \in (X, \Delta)$. Additionally, (X, Δ) is in three combinatorial switch hyperplanes corresponding to each of the three dual curves: $C(X, \delta_1)$, $C(X, \delta_2)$, and $C(X, \delta_3)$. Similarly, (X, Δ) will be contained in three combinatorial twist hyperplanes corresponding to the three pants curves and their duals: $C(\alpha_1, \delta_1)$, $C(\alpha_2, \delta_2)$, and $C(\alpha_3, \delta_3)$. The vertex is also contained in several lines and trees. Particularly, (X, Δ) is contained in the lines $l(X, \delta_1, \delta_2)$, $l(X, \delta_1, \delta_3)$, and $l(X, \delta_2, \delta_3)$, as well as the trees $t(\alpha_1, \alpha_2, \delta_1, \delta_2)$, $t(\alpha_1, \alpha_3, \delta_1, \delta_3)$, and $t(\alpha_2, \alpha_3, \delta_2, \delta_3)$. These are all the types of subcomplexes in $\mathfrak{F}$ that contain (X, Δ), for a total of 14 subcomplexes. Thus, property (2) is satisfied, and we have shown that $\mathfrak{F}$ is a factor system for $\mathcal{M}$. □

The existence of a factor system for $\mathcal{M}$ leads to the following corollary.

Corollary 4.7. *$(\mathcal{H}_2, \mathfrak{S})$ is a hierarchically hyperbolic group, where $\mathfrak{S}$ is a subset of $\mathfrak{F}$ containing a single element from each parallelism class in $\mathfrak{F}$, (excluding single 0-cubes), and our associated set of δ-hyperbolic spaces is the set of factored contact graphs $\{\hat{\mathcal{C}}F : F \in \mathfrak{S}\}$.*

Proof. Via [BHS17, Remark 13.2] and Lemma 4.6, we can conclude that $\mathcal{M}$ is hierarchically hyperbolic, with domains $\mathfrak{S}$ and δ-hyperbolic spaces as described in the corollary. We exclude 0-cubes so that nesting and orthogonality are mutually exclusive.

In [HH18, Proposition 6.7], Hamenstädt and Hensel prove that $\mathcal{H}_2$ acts properly, cocompactly, and by isometries on $\mathcal{M}$. Since $\mathcal{M}$ is an HHS, it follows that $\mathcal{H}_2$ is an HHG with the same domains and associated δ-hyperbolic spaces. □

Lastly, in order to be able to apply Theorem 2.8 to $(\mathcal{H}_2, \mathfrak{S})$, we prove that $(\mathcal{H}_2, \mathfrak{S})$ has unbounded products.

Lemma 4.8. *$(\mathcal{H}_2, \mathfrak{S})$ has unbounded products.*

Proof. Every subcomplex $F \in \mathfrak{S} - \{\mathcal{M}\}$ has infinite diameter, so we must show that the corresponding factors E_F, as defined in Lemma 2.2, also have infinite diameter.

If F is a combinatorial hyperplane, then by Corollaries 3.4 and 3.6, there are infinitely many elements in the parallelism class of F. Each unique subcomplex in $[F]$ intersects E_F in a unique 0-cube, (by Lemma 2.2), and hence, E_F must have infinite diameter.

Suppose instead that $F \in \mathfrak{S} - \{\mathcal{M}\}$ is arbitrary. By the characterization of $\mathfrak{F} \subset \mathfrak{S}$ given in Proposition 4.1, we know that F is contained in some combinatorial hyperplane C. Consider the cubical isometric embedding $C \times E_C \to \mathcal{X}$ given by Lemma 2.2. If $F' \in [F]$ such that there is some $C' \in [C]$ with $F' \subset C'$, then there is a 0-cube $e \in E_C$ such that $F \times \{e\} \to \mathcal{X}$ factors as $F \times \{e\} \xrightarrow{id} F' \hookrightarrow F$. Thus, $E_C \subset E_F$. Since $\operatorname{diam}(E_C) = \infty$, it follows that $\operatorname{diam}(E_F) = \infty$.

Thus, $(\mathcal{M}, \mathfrak{S})$ has unbounded products. □

5. The Factored Contact Graph is Quasi-isometric to the disk graph

The last piece necessary to prove the main theorems is to prove that the disk graph $\mathcal{D}(V_2)$ is coarsely $\mathcal{H}_2$-equivariantly quasi-isometric to the factored contact graph $\hat{\mathcal{C}}\mathcal{M}$. In this section we prove this claim, and then finally prove the main theorems.

Let $\mathcal{ND}(V_g)$ be the *non-separating disk graph*, i.e. the induced subgraph of $\mathcal{D}(V_g)$ whose vertices correspond to the non-separating meridians on ∂V_g. Since the vertices in the model $\mathcal{M}$ include only non-separating meridians, it will be easier to work with $\mathcal{ND}(V_2)$ rather than $\mathcal{D}(V_2)$ when constructing a quasi-isometry to $\hat{\mathcal{C}}\mathcal{M}$. The following proposition allows us to make this simplification.

Proposition 5.1. *For $g \geq 2$, the non-separating disk graph $\mathcal{ND}(V_g)$ isometrically embeds as a $\frac{3}{2}$-dense subgraph of the disk graph $\mathcal{D}(V_g)$.*

Proof. Because $\mathcal{ND}(V_g)$ is a subgraph of $\mathcal{D}(V_g)$, the inclusion is 1-Lipschitz.

Suppose now that $\gamma : [0, n] \to \mathcal{D}(V_g)$ is a geodesic in the the disk graph such that $\gamma(0)$ and $\gamma(n)$ are non-separating meridians. Suppose for some $0 < i < n$ that $\gamma(i)$ is a separating meridian. The complement $V_g - \gamma(i)$ consists of two spotted handlebodies Y_1 and Y_2, each with genus at least 1. The meridians $\gamma(i-1)$ and $\gamma(i+1)$ must intersect since they are distance two apart, but both must be disjoint from $\gamma(i)$. This means that $\gamma(i-1) \cup \gamma(i+1)$ is contained in say Y_1. Since Y_2 is a spotted handlebody of genus at least one, it must contain at least one non-separating meridian δ. The meridian δ is disjoint from $\gamma(i-1) \cup \gamma(i+1)$, so we can replace $\gamma(i)$ with δ. In this way, we can replace each separating meridian in γ with a non-separating meridian, and thus the distance between $\gamma(0)$ and $\gamma(n)$ in the non-separating disk graph is at most the distance between them in the disk graph. This gives us the lower bound.

Lastly, by the above we see that any separating meridian is always disjoint from at least one non-separating meridian, so every vertex of $\mathcal{D}(V_g)$ is distance 1 from a vertex in $\mathcal{ND}(V_g)$. Then any point on any edge in $\mathcal{D}(V_g)$ is at most distance $\frac{3}{2}$ from some point in $\mathcal{ND}(V_g)$. Thus, $\mathcal{ND}(V_g)$ is a $\frac{3}{2}$-dense subgraph of $\mathcal{D}(V_g)$. □

Proposition 5.2. *The non-separating disk graph $\mathcal{ND}(V_2)$ isometrically embeds as a $\frac{3}{2}$-dense subgraph of the contact graph $\mathcal{CM}$.*

Proof. We define first a map $\iota : \mathcal{ND}(V_2)^{(0)} \to \mathcal{CM}^{(0)}$ as $\iota(\alpha) = H(\alpha, \delta)$, where $H(\alpha, \delta)$ is any twist hyperplane such that every vertex in $N(H(\alpha, \delta))$ contains α as a pants curve. This map is injective because given any two non-separating meridians α and β, if $H(\alpha, \delta) = H(\beta, \delta')$, then indeed $\alpha = \beta$.

Next we assume α and β are two distinct, non-separating meridians connected by an edge in $\mathcal{ND}(V_2)$, and show that $\iota(\alpha)$ and $\iota(\beta)$ will also be connected by an edge. The complement $V_2 - \{\alpha \cup \beta\}$ is a genus 0 handlebody with four spots, two spots corresponding to α, and two spots corresponding to β. Then any meridian η on $V_2 - \{\alpha \cup \beta\}$ separating the two spots corresponding to α and separating the two spots corresponding to β will be a non-separating meridian on V_2 that is disjoint from both α and β. Together, $X = \{\alpha, \beta, \eta\}$ is a non-separating pants decomposition on V_2. Furthermore, $\iota(\alpha) = H(\alpha, \delta)$ and $\iota(\beta) = H(\beta, \delta')$ are two twist hyperplanes that cross one another in the twist flat $\mathcal{M}(X)$. This then means that in $\mathcal{CM}$, the vertices corresponding to $H(\alpha, \delta)$ and $H(\beta, \delta')$ are connected by an edge.

Because ι is injective and sends disjoint meridians to hyperplanes that cross one another, (i.e. sends edges to edges), it follows that the map ι extends to a simplicial embedding $\iota : \mathcal{ND}(V_2) \to \mathcal{CM}$, which is thus 1-Lipschitz.

Suppose now that $\gamma : [0, n] \to \mathcal{CM}$ is a geodesic parametrized by arc length such that $\gamma(0)$ and $\gamma(n)$ are in the image of ι. We will show that we can use γ to produce a new geodesic consisting entirely of twist hyperplanes in the image of ι. The first step is to show that starting from one end of γ, we can replace any switch hyperplane in γ with a twist hyperplane. Then we must show that we can choose the twist hyperplanes to be in the image of ι. If $n = 0$ or 1, then we are already done, so assume $n \geq 2$.

Fix i such that $0 < i < n$. Suppose $\gamma(i)$ corresponds to a switch hyperplane and $\gamma(i-1)$ corresponds to a twist hyperplane. Then either

(1) $\gamma(i+1)$ is a twist hyperplane, or
(2) $\gamma(i+1)$ is a switch hyperplane.

In either case, both $N(\gamma(i-1))$ and $N(\gamma(i+1))$ must have non-empty intersection with $N(\gamma(i))$, but must be disjoint from one another. Recall also that each edge in $\mathcal{CM}$ corresponds to the two hyperplanes either crossing or osculating.

For case (1), suppose $\gamma(i+1)$ is a twist hyperplane. It cannot be the case that one of $\gamma(i-1)$ or $\gamma(i+1)$ osculates with $\gamma(i)$ and the other crosses it because then $\gamma(i-1)$ and $\gamma(i+1)$ would intersect one another (see Figure 9). More specifically, if $\gamma(i-1)$ osculates with $\gamma(i)$ and $\gamma(i+1)$ crosses $\gamma(i)$, then as discussed in Section 3.4, $\gamma(i)$ will be parallel into $\gamma(i-1)$, and by the definition of parallel into, $\gamma(i+1)$ must cross $\gamma(i-1)$. This means we have only two subcases:

(1a) both $\gamma(i-1)$ and $\gamma(i+1)$ osculate with $\gamma(i)$, or

(1b) both $\gamma(i-1)$ and $\gamma(i+1)$ cross $\gamma(i)$. Notice that since $N(\gamma(i+1))$ and $N(\gamma(i-1))$ must be disjoint, $\gamma(i-1)$ and $\gamma(i+1)$ must actually be parallel to one another.

These two subcases are illustrated in Figure 10. In case (1a), we can replace $\gamma(i)$ with any twist hyperplane that crosses $\gamma(i)$, as this hyperplane must also cross $\gamma(i-1)$ and $\gamma(i+1)$. This follows from the fact that $\gamma(i)$ must be parallel into both $\gamma(i-1)$ and $\gamma(i+1)$, (see Section 3.4). In case (1b), we can replace $\gamma(i)$ with any twist hyperplane that osculates with $\gamma(i)$, as this hyperplane must cross $\gamma(i-1)$ and $\gamma(i+1)$. Again, this follows from the fact that $\gamma(i)$ is parallel into any twist hyperplane with which it osculates.

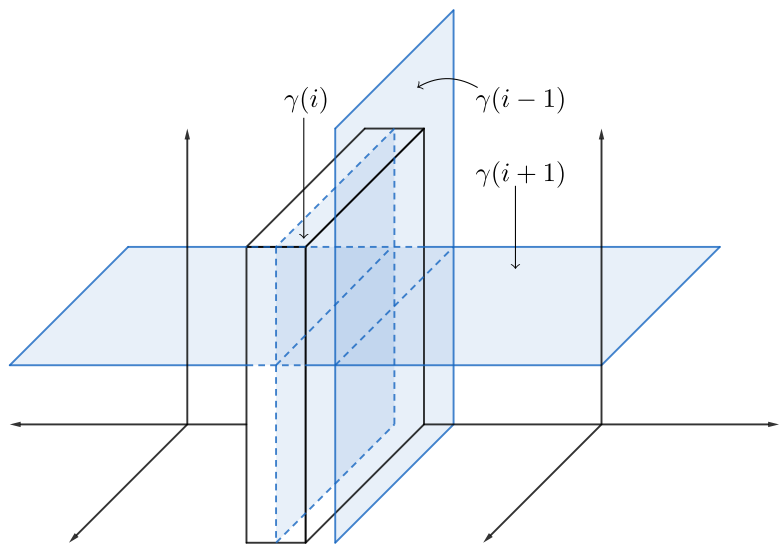


FIGURE 9. Ruling out a subcase of case (1).

In case (2), where $\gamma(i+1)$ is a switch hyperplane, $\gamma(i)$ and $\gamma(i+1)$ must osculate because no two switch hyperplanes can cross, (again see Section 3.4). This presents us with two subcases:

(2a) $\gamma(i-1)$ osculates with $\gamma(i)$, or
(2b) $\gamma(i-1)$ crosses $\gamma(i)$.

Figure 11 illustrates these two cases. In case (2a), we can replace $\gamma(i)$ with a twist hyperplane that crosses both $\gamma(i)$ and $\gamma(i+1)$, as this hyperplane must also cross $\gamma(i-1)$. Again we are using the fact that a switch hyperplane is parallel into any twist hyperplane with which it osculates. In case (2b), we can also replace $\gamma(i)$ with a twist hyperplane that crosses both $\gamma(i)$ and $\gamma(i+1)$ as such a hyperplane must also cross $\gamma(i-1)$. This is because $\gamma(i+1)$ must be parallel into $\gamma(i-1)$.

By starting with one end of γ, say $\gamma(0)$, we can use the above arguments to replace any vertices in γ corresponding to switch hyperplanes with vertices corresponding to twist hyperplanes. Let γ' be the altered geodesic. Next we show that we can γ' so that every vertes corresponds to a twist hyperplane in the image of ι.

Suppose $\gamma'(i) = H(\alpha, \delta)$ is a twist hyperplane that is not in the image of ι. If $\gamma'(i-1)$ and $\gamma'(i+1)$ both cross $\gamma'(i)$, then because $\iota(\alpha)$ is parallel to $\gamma(i)$, (by Lemma 3.3), it must also cross $\gamma'(i-1)$ and $\gamma'(i+1)$; we can replace $\gamma'(i)$ with $\iota(\alpha)$. If instead $\gamma'(i-1)$ and $\gamma'(i+1)$ both osculate with $\gamma'(i)$, then by the discussion in Section 3.4, all three hyperplanes

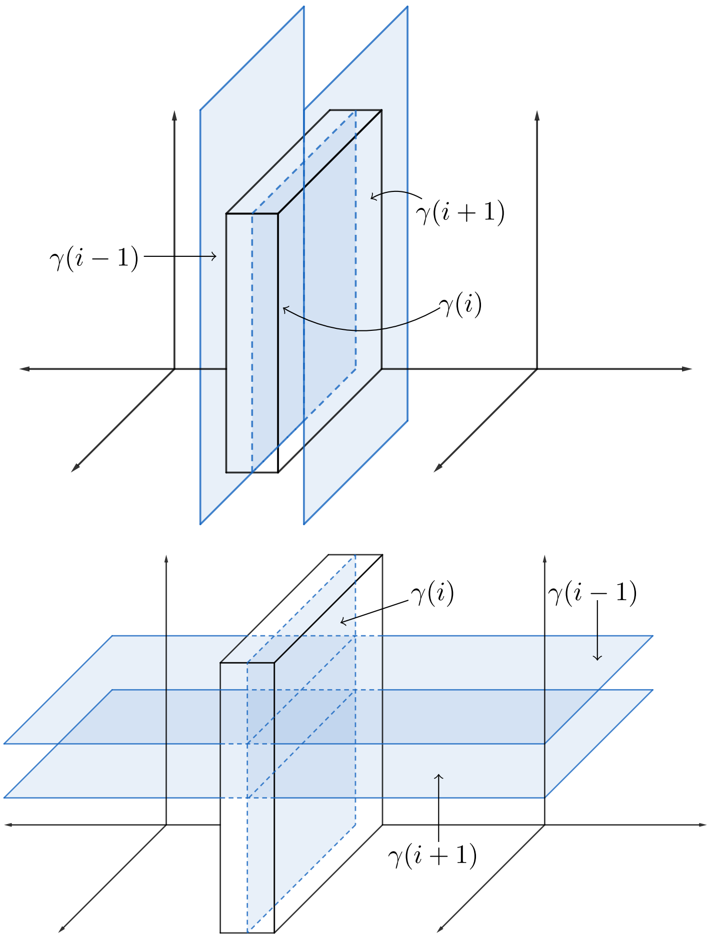


FIGURE 10. Cases (1a) and (1b), respectively.

are parallel, and there must be some twist hyperplane $H(\beta, \eta)$ that crosses all three of them; we can replace $\gamma'(i)$ with $\iota(\beta)$. Lastly, it cannot be the case that $\gamma'(i-1)$ osculates with $\gamma'(i)$ and that $\gamma'(i+1)$ crosses $\gamma'(i)$ because $\gamma'(i+1)$ would then also cross $\gamma'(i-1)$.

We can thus replace γ' with a geodesic consisting entirely of twist hyperplanes in the image of ι. This means that the length of a geodesic in $\mathcal{ND}(V_2)$ connecting $\gamma(0)$ and $\gamma(n)$ is at most as long as γ', (and so at most as long as γ), and hence we attain our lower bound.

It remains to show that ι is $\frac{3}{2}$-dense. Let $H(\alpha, \delta)$ be a twist hyperplane not in the image of ι. For any twist hyperplane $H(\alpha', \delta')$ that crosses $H(\alpha, \delta)$, parallelism implies $H(\alpha, \delta)$ must also cross $\iota(\alpha')$. Thus, any twist hyperplane $H(\alpha, \delta)$ not in the image of ι is a distance 1 away from the image of ι. Consider now a switch hyperplane $H(X, X')$. This hyperplane crosses all twist hyperplanes that have non-empty intersection with $\mathcal{M}(X)$ and $\mathcal{M}(X')$, and at least one of these twist hyperplanes will be in the image of ι. Hence, $H(X, X')$ will also be a distance 1 from the image of ι. Thus, any vertex in $\mathcal{CM}$ will be a distance 1 from some point in $\mathcal{ND}(V_2)$, and any point on an edge in $\mathcal{CM}$ will be at most distance $\frac{3}{2}$ from a point in $\mathcal{ND}(V_2)$. Thus, $\mathcal{ND}(V_2)$ is a $\frac{3}{2}$-dense subgraph of $\mathcal{CM}$. □

Propositions 5.1 and 5.2 gives us the following.

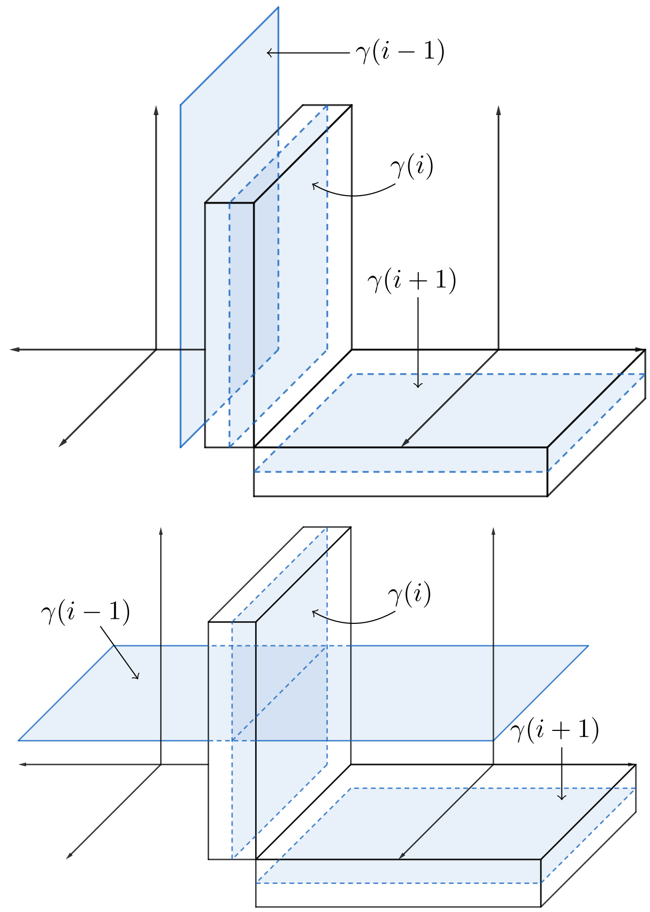


FIGURE 11. Cases (2a) and (2b), respectively.

Corollary 5.3. *The factored contact graph $\hat{\mathcal{C}}\mathcal{M}$ is coarsely $\mathcal{H}_2$-equivariantly quasi-isometric to the disk graph $\mathcal{D}(V_2)$.*

Proof. Recall from Corollary 4.5 that our factor system for $\mathcal{M}$ is exactly the minimal factor system as described in [BHS17]. [BHS17, Remark 8.18] tells us that for minimal factor systems, the factored contact graph $\hat{\mathcal{C}}\mathcal{M}$ is quasi-isometric to the contact graph $\mathcal{C}\mathcal{M}$ via the inclusion. Proposition 5.1 tells us that the inclusion $\mathcal{N}\mathcal{D}(V_2) \hookrightarrow \mathcal{D}(V_2)$ is a quasi-isometry. By constructing a quasi-inverse r for this inclusion, and composing this r with ι from Proposition 5.2 and the inclusion $\mathcal{C}\mathcal{M} \hookrightarrow \hat{\mathcal{C}}\mathcal{M}$, we have a quasi-isometry $\phi : \mathcal{D}(V_2) \to \hat{\mathcal{C}}\mathcal{M}$.

To see that ϕ is coarsely $\mathcal{H}_2$-equivariant, first note that clearly the inclusions $\mathcal{N}\mathcal{D}(V_2) \hookrightarrow \mathcal{D}(V_2)$ and $\mathcal{C}\mathcal{M} \hookrightarrow \hat{\mathcal{C}}\mathcal{M}$ are $\mathcal{H}_2$-equivariant, and so the quasi-inverse r will be coarsely $\mathcal{H}_2$-equivariant. Furthermore, ι is coarsely $\mathcal{H}_2$-equivariant because for any $\alpha \in \mathcal{N}\mathcal{D}(V_2)^{(0)}$ and $g \in \mathcal{H}_2$, the twist hyperplanes $g \cdot \iota(\alpha)$ and $\iota(g \cdot \alpha)$ will be parallel, meaning their distance in $\mathcal{C}\mathcal{M}$ will be at most two. $\square$

It is now straightforward to prove Theorems 1.2 and 1.1.

Proof of Theorem 1.2. Corollary 4.7 tells us that $(\mathcal{H}_2, \mathfrak{S})$ is an HHG with maximal δ-hyperbolic space $\hat{\mathcal{CM}}$. Corollary 5.3 tells us that $\hat{\mathcal{CM}}$ is coarsely equivariantly quasi-isometric to $\mathcal{D}(V_2)$. $\square$

Proof of Theorem 1.1. Theorem 1.2 tells us that $(\mathcal{H}_2, \mathfrak{S})$ is an HHG with maximal δ-hyperbolic space coarsely $\mathcal{H}_2$-equivariantly quasi-isometric to $\mathcal{D}(V_2)$. Lemma 4.8 tells us that $(\mathcal{H}_2, \mathfrak{S})$ has unbounded products, which allows us to apply Theorem 2.8. Because $\mathcal{D}(V_2)$ is coarsely $\mathcal{H}_2$-equivariantly quasi-isometric to $\hat{\mathcal{CM}}$, (the maximal δ-hyperbolic space in our HHS), then if the orbit map of a subgroup into $\hat{\mathcal{CM}}$ is a quasi-isometric embedding, so too is the orbit map into $\mathcal{D}(V_2)$. $\square$

6. Discussion of Higher Genus Cases

The HHG techniques used in the genus two case are not applicable to higher genus. This is because for genus $g \geq 3$, Hamenstädt and Hensel prove in [HH18] that $\mathcal{H}_g$ has exponential Dehn function. Consequently, $\mathcal{H}_g$ cannot be an HHG (see [BHS19, Corollary 7.5]). In fact, Proposition 6.1 below shows that the analogue to Theorem 1.1 is false. At a minimum, an application of [ADT17, Theorem 1.6] tells us that if $H \leq \mathcal{H}_g \leq MCG(\partial V_g)$ is a stable subgroup of $MCG(\partial V_g)$, then H is a stable subgroup of $\mathcal{H}_g$ as well. For instance, purely pseudo-Anosov subgroups of $H \leq \mathcal{H}_g \leq MCG(\partial V_g)$ are stable in $MCG(\partial V_g)$ so they will be stable in $\mathcal{H}_g$ [BBKL20].

While pseudo-Anosov mapping classes are the only elements that act loxodromically on the curve graph, there are reducible elements in the handlebody group that act loxodromically on the disk graph. It is from such mapping classes that we find a counterexample to the higher genus analogue of Theorem 1.1. Specifically, we prove the following.

Proposition 6.1. *For $g \geq 3$, there exists an element $\Phi \in \mathcal{H}_g$ such that the orbit map $\langle\Phi\rangle \to \mathcal{D}(V_g)$ is a quasi-isometric embedding but such that $\langle\Phi\rangle$ is not stable in $\mathcal{H}_g$.*

For the remainder of this section, we will construct such a Φ and prove Proposition 6.1.

6.1. **Constructing Φ.** Let S_0^{g+1}, for $g \geq 3$, be a sphere with $g+1$ boundary components. Let δ_1 and δ_2 be two of the boundary components. Glue δ_1 and δ_2 together so that the resulting surface S_1^{g-1} is a torus with $g-1$ boundary components. Say that $\alpha \subset S_1^{g-1}$ is the curve corresponding to δ_1 and δ_2. Let N be a regular neighborhood of α and let $S = \overline{S_1^{g-1} - N}$, which is homeomorphic to S_0^{g+1}. Choose some reducible $\phi \in MCG(S_1^{g-1})$ that is the identity on N and is pseudo-Anosov on S. Now let $V_g = S_1^{g-1} \times I$ where $I = [-1, 1]$; V_g is a genus g handlebody. We define

$$\Phi = \phi \times id \in MCG(V_g) \cong \mathcal{H}_g.$$

We will show that Φ satisfies the properties described in Proposition 6.1.

6.2. **Φ is loxodromic.** We say that an element $g \in G$ acting on a hyperbolic G-space X is *loxodromic* if the orbit map $\mathbb{Z} \to X$ given by $n \mapsto g^n \cdot x$ for some (any) $x \in X$ is a quasi-isometric embedding. Considering the orbit map of the entire group $G \to X$, being loxodromic easily implies $\langle g\rangle$ quasi-isometrically embedding in G.

To see that Φ is loxodromic, we use the idea of witnesses, (previously called holes due to Masur and Schleimer [MS13]). A *witness* for the disk graph $\mathcal{D}(V_g)$ is a essential subsurface $\Sigma \subset \partial V_g$ such that every representative of every meridian on V_g has non-empty intersection with Σ. Masur and Schleimer show that distances in the disk graph can be estimated using distances in the curve graphs of witnesses. Specifically, for large enough A, there is a constant B such that the following holds:

$$d_{\mathcal{D}(V_g)}(\alpha,\beta) =_B \sum_{X \text{ witness}} [d_{\mathcal{C}(X)}(\pi_X(\alpha),\pi_X(\beta))]_A.$$

Here $=_B$ indicates equality up to additive and multiplicative errors, $[x]_A$ is x if $x \geq A$ and is 0 otherwise, and π_X indicates the subsurface projection to X. For more details about witnesses and the distance formula, see [MS13].

Using the distance formula and witnesses, we can prove the following lemma.

Lemma 6.2. Φ *is loxodromic.*

Proof. The upper bound follows from the fact that orbit maps of finitely generated groups are Lipschitz.

For the lower bound, recall that $S = \overline{S_1^{g-1} - N}$ and let

$$S_i = S \times \{i\} \subset S_1^{g-1} \times \{i\} \subset \partial V_g$$

for $i \in \{-1, 1\}$. By the construction of V_g, S_1 must be a witnesses for V_g. To see this, notice that the inclusions $S_{-1} \cup (N \times \{-1\}) \hookrightarrow V_g$ and $N \times \{1\} \hookrightarrow V_g$ are π_1-injective, implying $S_{-1} \cup (N \times \{-1\})$ and $N \times \{1\}$ are incompressible in V_g. It follows that no meridian is contained in $S_{-1} \cup (N \times \{\pm 1\})$. Further, no meridian is contained in any component of $\partial S_1^{g-1} \times I$. Hence, S_1 must be a witness.

Because $\Phi|_{S_1}$ is a pseudo-Anosov, then for any $\beta \in \mathcal{D}(V_g)^{(0)}$, the distance $d_{\mathcal{C}(S_1)}(\Phi^n \cdot \beta, \beta)$ must be growing linearly in n. Since S_1 is a witness for V_g, the distance formula tells us that $d_{\mathcal{D}(V_g)}(\Phi^n \cdot \beta, \beta)$ must also be growing linearly. □

6.3. $\langle \Phi \rangle$ **is not stable.** In order to prove $\langle \Phi \rangle$ is not stable, we will show that $\langle \Phi \rangle \subset \mathcal{H}_g$ is contained in a quasi-isometrically embedded copy of $\mathbb{Z}^2 \subset \mathcal{H}_g$. To this end, let $A_\alpha \subset V_g$ be the properly embedded annulus bounded by $\alpha \times \{-1\}$ and $\alpha \times \{1\}$, where α is as in Section 6.1. Let Ψ be the annulus twist about A_α, i.e. $\Psi = T_{\alpha \times \{1\}} T^{-1}_{\alpha \times \{-1\}} \in \mathcal{H}_g$.

Lemma 6.3. $\langle \Phi \rangle$ *is not stable in* $\mathcal{H}_g$.

Proof. The map Ψ commutes with Φ, so $\langle \Phi, \Psi \rangle \cong \mathbb{Z}^2$. Furthermore, by appealing to the Masur-Minsky distance formula for $MCG(\partial V_g)$, (see [MM00]), we find that $\langle \Phi, \Psi \rangle \hookrightarrow MCG(\partial V_g)$ is a quasi-isometric embedding. Since the inclusion $\mathcal{H}_g \hookrightarrow MCG(\partial V_g)$ is Lipschitz, the inclusion $\langle \Phi, \Psi \rangle \hookrightarrow \mathcal{H}_g$ must be a quasi-isometric embedding. Finally, because $\langle \Phi \rangle$ is contained in a quasi-isometrically embedded copy of $\mathbb{Z}^2 \subset \mathcal{H}_g$, it cannot be stable. □

Lemmas 6.2 and 6.3 give us Proposition 6.1.

6.4. **Consequence for acylindricity.** One consequence of Proposition 6.1 is that the action of $\mathcal{H}_g$ for $g \geq 3$ on $\mathcal{D}(V_g)$ cannot be acylindrical. To see this, we know via [DGO17, Corollary 2.9] and [Sis16, Theorem 1] that if G is a group acting acylindrically on a hyperbolic space X, then any infinite order, loxodromic element $g \in G$ must be stable, (i.e. $\langle g \rangle$ is stable in G). Propostion 6.1 provides us with an infinite order element acting loxodromically on $\mathcal{D}(V_g)$ that is not stable.

We should point out that this does not mean that $\mathcal{H}_g$ is not acylindrically hyperbolic. In fact, one can see that $\mathcal{H}_g$ is acylindrically hyperbolic via the fact that the action of the mapping class group on the curve graph is acylindrical [Bow08]. Since $\mathcal{H}_g \leq MCG(\partial V_g)$, the action of $\mathcal{H}_g$ on the curve graph must also be acylindrical.

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