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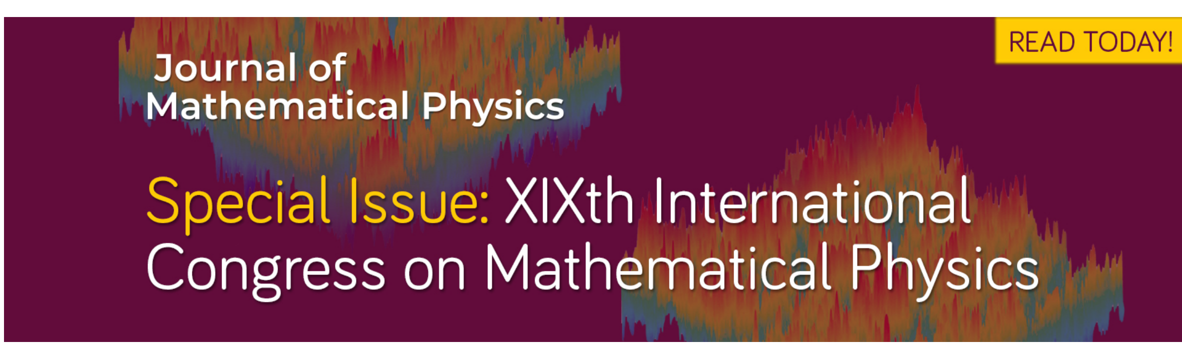

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ABSTRACT

In recent work, Wang and Tener defined a class of "extremal" vertex operator algebras (VOAs), consisting of those with at least two simple modules and conformal dimensions as large as possible for the central charge. In this article, we show that there are exactly 15 character vectors of extremal VOAs with two simple modules. All but one of the 15 character vectors are realized by a previously known VOA. The last character vector is realized by a new VOA with central charge 33.

I. INTRODUCTION

In 1988, a foundational article of Mathur, Mukhi, and Sen[1] pioneered an approach for the classification of rational chiral conformal field theories (CFTs). They studied the graded dimension functions, or characters, of CFTs by showing that they satisfied what are now called modular linear differential equations (MLDEs). The classification of CFTs by their characters, via MLDEs and other methods, has continued over the last two decades, and in recent years, there has been significant activity surrounding the classification of characters of chiral CFTs with two characters.

We study a mathematical version of this classification problem. We take vertex operator algebras (VOAs) as a mathematical model for chiral CFTs, and for a sufficiently nice ("strongly rational") vertex operator algebra V of central charge c, we consider its representation category $\mathcal{C} = \mathrm{Rep}(V)$, which is a modular tensor category (MTC).[2] One can recover the equivalence class of the central charge c mod 8 from $\mathcal{C}$, and motivated by this, we define an *admissible genus* to be a pair $(\mathcal{C}, c)$ consisting of a modular tensor category and a number c in the appropriate class mod 8 (cf. Ref. 3). It is natural to approach the problem of classification of VOAs and their characters by restricting to genera where both $\mathcal{C}$ and c are sufficiently small in an appropriate sense.

We will take the rank of an MTC (i.e., the number of simple objects) as our measure of its size. The smallest MTC is the trivial one, Vec, and a genus (Vec, c) is admissible when $c \equiv 0$ mod 8. There are a total of three VOAs in the genera $(\mathrm{Vec}, 8)$ and $(\mathrm{Vec}, 16)$, arising from even unimodular lattices of rank 8 and 16. The problem becomes interesting at $(\mathrm{Vec}, 24)$, where the classification of characters of VOAs can be read off from Schellekens's famous list[4] (see also Ref. 5). The classification of VOAs in $(\mathrm{Vec}, 24)$ is almost complete, except that the uniqueness of VOA(s) with the same character as the moonshine VOA has not been established. At $(\mathrm{Vec}, 32)$, the explicit classification problem of characters is already intractable, even just for the simplest examples coming from even unimodular lattices.

This difficulty propagates to the study of genera $(\mathcal{C}, c)$ with c large. Indeed, for any fixed $V \in (\mathcal{C}, c)$ one obtains from every $W \in (\mathrm{Vec}, 32)$ a new VOA $V \otimes W \in (\mathcal{C}, c + 32)$. Thus, if one wishes to consider classification problems for higher central charge and $\mathrm{rank}(\mathcal{C}) > 1$, it is necessary to restrict to a class of VOAs that excludes many VOAs of the form $V \otimes W$. There is a natural notion of "primeness" that one could consider in this context, but we will consider something slightly different.

The twist of a simple module $M \in \mathrm{Rep}(V)$ is given by $\theta_M = e^{2\pi i h}$, where h is the lowest conformal dimension of states in M. Thus for any (hypothetical) VOA $V \in (\mathcal{C}, c)$, we can recover the conformal dimensions of simple objects, mod 1. Moreover, there is an *a priori* bound[1,6] on the conformal dimensions,

$$\ell := \binom{n}{2} + \frac{nc}{4} - 6\sum_{j=0}^{n-1} h_j \geq 0, \tag{1.1}$$

where $V = M_0, \ldots, M_{n-1}$ are a complete list of simple V modules (assumed here to have linearly independent characters) and h_j is the lowest conformal dimension of M_j. Moreover, ℓ is an integer. A strongly rational VOA $V \in (\mathcal{C}, c)$ with $\mathrm{rank}(\mathcal{C}) > 1$ is called *extremal*[7] if $\sum h_j$ is as large as possible for c in light of (1.1). This is analogous to the extremality condition introduced by Höhn for holomorphic VOAs (i.e., VOAs V with $\mathrm{Rep}(V) = \mathrm{Vec}$).[8] Since the h_j are determined mod 1 by $\mathcal{C}$, extremality is equivalent to the condition $\ell < 6$.

The classification of (characters of) extremal non-holomorphic VOAs appears to be a tractable piece of the unrestricted general classification problem. In Ref. 7, it was demonstrated that if $\mathrm{rank}(\mathcal{C})$ is 2 or 3, then the characters of a corresponding VOA are determined by its genus, and a list of potential character vectors was obtained up to central charge 48. In this article, we give a complete classification of characters of extremal VOAs V with $\mathrm{rank}(\mathrm{Rep}(V)) = 2$, with no restriction on the central charge.

Main Theorem. *There are exactly 15 character vectors of strongly rational extremal (i.e., $\ell < 6$) VOAs with exactly two simple modules. These characters are listed in* Table V.

This theorem appears in the main body of the text as Theorem 3.13. Of the 15 character vectors in Table V, all but one are realized by the previously known VOAs. The remaining case, corresponding to the genus (Semion, 33), is realized by a new VOA constructed in Sec. IV.

This article fits into a cluster of activity regarding the classification of VOAs with two simple modules (or, more generally, two characters). Recently, Mason, Nagatomo, and Sakai[9] used MLDEs to establish a classification result for VOAs with two simple modules satisfying certain additional properties in the $\ell = 0$ regime. Our results extend the Mason–Nagatomo–Sakai classification to the case $\ell < 6$, although, in contrast, we only consider the problem of classifying characters of VOAs. The exceptional $c = 33$ character vector has $\ell = 4$ and thus did not appear in earlier classifications.

The Mason–Nagatomo–Sakai classification builds on the work of Franc and Mason,[10,11] which describes solutions to MLDEs in rank 2 in terms of hypergeometric series. In contrast, our approach is to use the general theory of vector-valued modular forms to derive an explicit recurrence between potential character vectors in the genus $(\mathcal{C}, c)$ and those in $(\mathcal{C}, c \pm 24)$. By studying the long-term behavior of this recurrence, we are able to obtain effective bounds on the possible central charges of extremal VOAs. While we only consider the case of VOAs with two simple modules in this article, our method does not rely on any special features of rank 2 MLDEs (such as a hypergeometric formula), and in future work, we hope to apply the same techniques in higher rank. For this reason, we avoid any explicit use of the hypergeometric series formulas.

Section III is an adaptation of the undergraduate thesis[12] of Grady, which obtained a classification of characters for extremal VOAs with two simple modules and which focused on the case $c, h \geq 0$. Not long after the thesis was published online, an article (Ref. 13) in the physics literature used MLDEs to obtain a classification similar to the one presented here, without having been aware of Ref. 12.

This article is organized as follows: In Sec. II, we review the classification of rank 2 modular tensor categories and modular data from the perspective of VOAs. In Sec. III A, we review the tools from Ref. 14, which we will use to describe the character vectors of VOAs. In Sec. III B, we derive a recurrence relation that describes how characteristic matrices change under the transformation $c \mapsto c \pm 24$. In Secs. III C and III D, we study the long-term behavior of this recurrence in the positive c and negative c situations, respectively, and in Sec. III E, we put these tools together to obtain our main theorem. Section IV provides a construction of an extremal VOA in the genus (Semion, 33). Finally, in the Appendix, we give tables of numerical data used in the proof of the main theorem, as well as all 15 extremal characters in rank 2.

II. RANK 2 MODULAR TENSOR CATEGORIES

In this article, we will consider VOAs that are simple, of CFT type, self-dual, and regular (or equivalently, rational and C_2-cofinite[15]). For brevity, we will use the term *strongly rational* to describe such VOAs. We refer the reader to Refs. 15 and 16 for background on the adjectives under consideration, but we will explain here the consequences that are relevant for our work.

A strongly rational VOA V possesses finitely many simple modules $V = M_0, M_1, \ldots, M_n$. We denote the category of V-modules by $\mathrm{Rep}(V)$ and write $\mathrm{rank}(\mathrm{Rep}(V))$ for the number of simple modules $n + 1$. We will assume throughout that every module M_j is self-dual, as it simplifies the exposition and is satisfied in the rank 2 case.

We are primarily interested in the characters of V,

$$\mathrm{ch}_j(\tau) = q^{-c/24}\sum_{n=0}^{\infty} \dim M_j(n + h_j)\, q^{n+h_j},$$

where as usual $q = e^{2\pi i \tau}$, c is the central charge of V, h_j is the smallest conformal dimension occurring in M_j, and $M_j(n + h_j)$ is the space of states of conformal dimension $n + h_i$. The foundational work of Zhu[17] demonstrated that the characters ch_j define holomorphic functions on the upper half-plane and that their span is invariant under the action of the modular group. Thus, if we set

$$\mathrm{ch}(\tau) = \begin{pmatrix} \mathrm{ch}_0 \\ \vdots \\ \mathrm{ch}_n \end{pmatrix},$$

there exists a representation $\rho_V : \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{GL}(n+1,\mathbb{C})$ such that

$$\mathrm{ch}(\gamma \cdot \tau) = \rho_V(\gamma)\mathrm{ch}(\tau) \tag{2.1}$$

for all $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ (recall that we assumed each M_i to be self-dual). Here, $\gamma \cdot \tau$ denotes the natural action of $\mathrm{PSL}(2,\mathbb{Z})$ on the upper half-plane.

By the work of Huang (Ref. 2, see also Ref. 18), $\mathrm{Rep}(V)$ is naturally a modular tensor category, and based on Huang's work, Dong–Lin–Ng[19] showed that Zhu's modular invariance is encoded by the S and T matrices of $\mathrm{Rep}(V)$ (see Ref. 20 for more details on the S and T matrices of a modular tensor category). Recall that the normalization of S is only canonical up to a sign and that for each choice of S, the normalization of T is only canonical up to a third root of unitary. By Theorem 3.10 of Ref. 19 (based on Ref. 2), we have that $\rho_V\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ coincides with a normalization of the categorical S matrix of $\mathrm{Rep}(V)$, and it is straightforward to check directly that

$$\rho_V\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{-2\pi i c/24}(\delta_{j,k}e^{2\pi i h_j})_{j,k}. \tag{2.2}$$

We now consider strongly rational VOAs V such that $\mathrm{rank}(\mathrm{Rep}(V)) = 2$. We will sometimes write h instead of h_1 for the non-trivial lowest conformal dimension, and similarly, we will sometimes write M instead of M_1. The classification of modular tensor categories of rank 2 was obtained in Ref. 21, and the complete list of normalized S matrices is

$$\pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & -1 \end{pmatrix} \qquad \text{and} \qquad \pm\frac{1}{\sqrt{2+\alpha}}\begin{pmatrix} 1 & \alpha \\ \alpha & -1 \end{pmatrix}, \tag{2.3}$$

where $\varepsilon^2 = 1$ and $\alpha^2 = 1 + \alpha$. This list coincides with the complete list of S matrices for two-dimensional congruence representations of the modular group earlier obtained by Mason (Ref. 6, Sec. 3).

Observe that

$$e^{2\pi i c/24}\,\mathrm{ch}_j(i) = \sum_{n=0}^{\infty} \dim M_j(n+h_j)\, e^{-2\pi(n+h_j)} > 0,$$

and thus the phase of $\mathrm{ch}_j(i)$ is independent of j (here, i is the imaginary unit). By (2.1), $\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathrm{ch}(i) = \mathrm{ch}(i)$, and thus $\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ fixes a vector all of whose entries have the same phase. This observation allows us to refine (2.3) and conclude that if $\mathrm{rank}(\mathrm{Rep}(V)) = 2$, then $\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ must be one of

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2+\varphi}}\begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{3-\varphi}}\begin{pmatrix} -1 & \varphi-1 \\ \varphi-1 & 1 \end{pmatrix}, \tag{2.4}$$

where we use positive square roots and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

By the classification of Ref. 21, there are exactly two modular tensor categories realizing each of (2.4) as a normalization of its S matrix, and these two are related by a reversal of the braiding. Fix one of these eight modular tensor categories $\mathcal{C}$ and its normalized S-matrix from (2.2). We wish to see how much information about a hypothetical V with $\mathrm{Rep}(V) = \mathcal{C}$ we may recover. By definition, the non-normalized T matrix of $\mathcal{C}$ is the diagonal matrix $e^{2\pi i h_j}\delta_{j,k}$, and thus the equivalence class of h mod 1 is determined by $\mathcal{C}$. Observe that if (S, T) are generators of a representation of $\mathrm{PSL}(2,\mathbb{Z})$ then $(S, \zeta T)$ again generate a representation only if $\zeta^3 = 1$, and thus from (2.2) we can see that c mod 8 is determined by $\mathcal{C}$ as well.

We summarize the eight cases in Table I. Each row corresponds to a modular tensor category, giving its normalized S matrix from (2.4), the equivalence classes of central charge and minimal conformal weight of a hypothetical VOA realization, as well as a familiar name for the category and a VOA realizing the category, where appropriate/known.

The genus of a strongly rational VOA V is the pair $(\mathrm{Rep}(V), c)$. In Ref. 7, Tener and Wang defined an *extremal* (non-holomorphic) VOA to be the one with $\mathrm{rank}(\mathrm{Rep}(V)) > 1$ such that the minimal conformal weights h_j were as large as possible in light of a certain *a priori* bound[1,6] (see Sec. 2.2 of Ref. 7 for more details). When $\mathrm{rank}(\mathrm{Rep}(V)) = 2$, then V is extremal when

$$0 \le 1 + \frac{c}{2} - 6h < 6. \tag{2.5}$$

The quantity $\ell := 1 + \frac{c}{2} - 6h$ is always a non-negative integer and has been used frequently in the study of VOAs (e.g., Refs. 1, 9, and 22, among many others).

TABLE I. The eight rank 2 modular tensor categories from the perspective of VOAs and an extremal realization where applicable.

No.	S	c mod 8	h mod 1	Name	Extremal realization
1	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	1	$\frac{1}{4}$	Semion	$A_{1,1}$ at $c = 1$
2	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	7	$\frac{3}{4}$	$\overline{\text{Semion}}$	$E_{7,1}$ at $c = 7$
3	$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	−3	$-\frac{3}{4}$	$\text{Semion}^\dagger$	None
4	$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	−5	$-\frac{1}{4}$	$\overline{\text{Semion}}^\dagger$	None
5	$\frac{1}{\sqrt{2+\varphi}}\begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$	$\frac{14}{5}$	$\frac{2}{5}$	Fib	$G_{2,1}$ at $c = \frac{14}{5}$
6	$\frac{1}{\sqrt{2+\varphi}}\begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$	$\frac{26}{5}$	$\frac{3}{5}$	$\overline{\text{Fib}}$	$F_{4,1}$ at $c = \frac{26}{5}$
7	$\frac{1}{\sqrt{3-\varphi}}\begin{pmatrix} -1 & \varphi - 1 \\ \varphi - 1 & 1 \end{pmatrix}$	$-\frac{22}{5}$	$-\frac{1}{5}$	Yang – Lee	Yang – Lee at $c = -\frac{22}{5}$
8	$\frac{1}{\sqrt{3-\varphi}}\begin{pmatrix} -1 & \varphi - 1 \\ \varphi - 1 & 1 \end{pmatrix}$	$-\frac{18}{5}$	$-\frac{4}{5}$	$\overline{\text{Yang – Lee}}$	None

The purpose of this article is to provide a list of all possible characters of extremal VOAs with $\text{rank}(\text{Rep}(V)) = 2$. Given a rank 2 modular tensor category $\mathcal{C}$ and central charge c in the appropriate class mod 8 (as in Table I), there is a unique rational number h_{ext} in the appropriate class mod 1 satisfying (2.5). When $\mathcal{C}$ is fixed, we will write $h_{ext}(c)$ to emphasize the dependence on c.

The pair $(\mathcal{C}, c)$ of a modular tensor category and appropriate choice of c is called an *admissible genus*.[3] For every admissible genus $(\mathcal{C}, c)$ described by Table I, there is a representation $\rho_c : \text{PSL}(2,\mathbb{Z}) \to U(2,\mathbb{C})$ whose S matrix is given by the entry of the table and whose T matrix is obtained by rescaling the categorical T matrix by $e^{-2\pi i c/24}$. These representations are simply a choice of normalization of the categorical S and T matrices, and their existence does not depend in any way on vertex operator algebras. However, they are defined in such a way that if there is a strongly rational VOA V with central charge c and $\text{Rep}(V) = \mathcal{C}$, then $\rho_V = \rho_c$.

III. CHARACTERS OF VOAs WITH TWO SIMPLE MODULES

A. Characters and vector-valued modular forms

We briefly recall the relevant theory of vector-valued modular forms, following Refs. 14 and 23. We refer the reader to these references, especially Ref. 14 (Sec. 2), for more details.

Let $\rho : \text{PSL}(2,\mathbb{Z}) \to \text{GL}(d,\mathbb{C})$ be an irreducible representation of the modular group, and assume that $\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonal with finite order. Let $\mathbb{X} : \mathbb{H} \to \mathbb{C}$ be a holomorphic function on the upper half-plane, which satisfies

$$\mathbb{X}(\gamma \cdot \tau) = \rho(\gamma)\mathbb{X}(\tau) \tag{3.1}$$

for all $\gamma \in \text{PSL}(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$. Choose a diagonal matrix Λ such that $\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{2\pi i \Lambda}$, called an *exponent matrix*. For any choice of exponent matrix, we may Fourier expand

$$q^{-\Lambda}\mathbb{X}(q) = \sum_{n \in \mathbb{Z}} \mathbb{X}[n] q^n \tag{3.2}$$

for coefficients $\mathbb{X}[n] \in \mathbb{C}^d$. Let $\mathcal{M}(\rho)$ denote the space of functions $\mathbb{X}$ satisfying (3.1) such that $\mathbb{X}[n] = 0$ for n sufficiently negative (observe that this does not depend on the choice of Λ).

Given a choice of exponent Λ, we define the principal part map

$$\mathcal{P}_\Lambda : \mathcal{M}(\rho) \to \operatorname{span}\{vq^{-n} : n > 0, v \in \mathbb{C}^d\}$$

by

$$\mathcal{P}_\Lambda \mathbb{X} = \sum_{n<0} \mathbb{X}[n]q^n,$$

where $\mathbb{X}[n]$ are as in (3.2).

An exponent matrix is called *bijective* if $\mathcal{P}_\Lambda$ is an isomorphism. For $\xi \in \{1,\ldots,d\}$, let $e_\xi \in \mathbb{C}^d$ be the corresponding standard basis vector. Given a choice of bijective exponent matrix, let $\mathbb{X}^{(\xi)} \in \mathcal{M}(\rho)$ be the function with $\mathcal{P}_\Lambda \mathbb{X}^{(\xi)} = q^{-1}e_\xi$. In this case, $\mathbb{X}^{(1)},\ldots,\mathbb{X}^{(d)}$ form a basis for $\mathcal{M}(\rho)$ as a free $\mathbb{C}[J]$-module, where

$$J = q^{-1} + 196884q + \cdots$$

is the J-invariant. The *fundamental matrix* Ξ is given by

$$\Xi = [\, \mathbb{X}^{(1)} \,|\, \cdots \,|\, \mathbb{X}^{(d)} \,].$$

The *characteristic matrix* χ is given by the constant terms of Ξ taken in the q-expansion [shifted by Λ as in (3.2)], that is,

$$\chi = [\, \mathbb{X}^{(1)}[0] \,|\, \cdots \,|\, \mathbb{X}^{(d)}[0] \,].$$

Now, fix as in Sec. II a modular tensor category $\mathcal{C}$ of rank 2 and a choice of real number c in the appropriate class mod 8. From these data, we specified a representation ρ_c of $\mathrm{PSL}(2,\mathbb{Z})$ with the property that if there exists a VOA V with central charge c and $\mathrm{Rep}(V) = \mathcal{C}$, then its character vector (ch_j) satisfies $(\mathrm{ch}_j) \in \mathcal{M}(\rho_c)$. The key observation of Ref. 7 is that

$$\Lambda(c) = \begin{pmatrix} 1 - \dfrac{c}{24} & 0 \\ 0 & h_{ext}(c) - \dfrac{c}{24} \end{pmatrix} =: \begin{pmatrix} \lambda_0(c) & 0 \\ 0 & \lambda_1(c) \end{pmatrix}$$

is a bijective exponent for ρ_c, where h_{ext} is the real number lying in the appropriate class mod 1 that satisfies (2.5). Thus, by the definition of fundamental matrix, we have the following theorem:

Theorem 3.1 (Ref. 7, Theorem 3.1). *Let $\mathcal{C}$ be a modular tensor category of rank 2, and let c be a real number in the appropriate class mod 8. If V is an extremal VOA with central charge c and* $\mathrm{rank}(\mathrm{Rep}(V)) = \mathcal{C}$*, then its character appears as the first column of the fundamental matrix corresponding to the bijective exponent $\Lambda(c)$.*

Let $\chi(c) = (\chi(c)_{ij})_{i,j=0}^{1}$ be the characteristic matrix taken with respect to $\Lambda(c)$. Thus, if V is an extremal VOA with central charge c and $\mathrm{rank}(\mathrm{Rep}(V)) = 2$, we have $\chi(c)_{00} = \dim V(1)$. We will determine the possible values of c for which there exists an extremal VOA by showing that for $|c|$ sufficiently large, one of $\chi(c)_{00}$ or $\chi(c)_{10}$ is not a non-negative integer.

B. General recurrence

They key idea[12] is to derive a recurrence relating the pair $(\chi(c+24), h_{ext}(c+24))$ to $(\chi(c), h_{ext}(c))$ and then study the long-term behavior of this recurrence. In fact, to handle the case $c \to +\infty$, one may derive a simple recurrence involving only the diagonal entries of χ (Ref. 12, Lemma 6.4). To handle the case $c \to -\infty$, we will use all of the entries of χ, and the relation will be slightly more complicated as a result.

Let $M^-_{2\times 2}$ be the set of 2×2 complex matrices whose bottom-left entry is non-zero, and let $M^+_{2\times 2}$ be the set of matrices whose top-right entry is non-zero. Define functions

$$f_\pm : M^\mp_{2\times 2} \times (\mathbb{R}\backslash\mathbb{Z}) \to M^\pm_{2\times 2} \times (\mathbb{R}\backslash\mathbb{Z})$$

by

$$f_+\left[\begin{pmatrix} x & y \\ z & w \end{pmatrix}, h\right] = \left[\begin{pmatrix} \dfrac{w + h(x-240)}{h+1} & \dfrac{1}{z} \\ \dfrac{(h+1)^2(h-2)yz - (x - w + 120(h-1))^2 + 746496(h+1)^2}{(h+2)(h+1)^2} z & \dfrac{x + h(w+240)}{h+1} \end{pmatrix}, h+2\right]$$

and

$$f_-\left[\begin{pmatrix} x & y \\ z & w \end{pmatrix}, h\right] = \left[\begin{pmatrix} \dfrac{-w + (h-2)(x+240)}{h-3} & \dfrac{h(h-3)^2yz + (x - w + 120(h-1))^2 - 746496(h-3)^2}{(h-4)(h-3)^2} y \\ \dfrac{1}{y} & \dfrac{-x + (h-2)(w-240)}{h-3} \end{pmatrix}, h-2\right].$$

By direct computation, one may check that these functions are invertible and $f_\pm^{-1} = f_\mp$. We will show that $f_\pm$ take characteristic matrices to characteristic matrices, but first we must check the following:

Lemma 3.2. Let χ be the fundamental matrix corresponding to a 2×2 bijective exponent Λ. Then, $\chi \in M^+_{2\times 2} \cap M^-_{2\times 2}$.

Proof. We show $\chi \in M^-_{2\times 2}$, and the other step is similar. Let $\mathbb{X} = \mathbb{X}^{(1)}$ be the first column of the fundamental matrix Ξ corresponding to $\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}$. If the bottom-left entry of χ were 0, this would imply that $\mathbb{X}$ was of the form

$$\mathbb{X} = \begin{pmatrix} q^{\lambda_0}(q^{-1} + \cdots) \\ q^{\lambda_1}(z_1 q + \cdots) \end{pmatrix} \tag{3.3}$$

for some $z_1 \in \mathbb{C}$. By Ref. 23, Theorem 4.1, $\Lambda' = \begin{pmatrix} \lambda_0 - 1 & 0 \\ 0 & \lambda_1 + 1 \end{pmatrix}$ is again a bijective exponent. However, examining (3.3), we see that $\mathcal{P}_{\Lambda'}\mathbb{X} = 0$, which is a contradiction. □

Lemma 3.3. Let $(\mathcal{C}, c)$ be an admissible genus from Table I, and let $\chi(c)$ denote the characteristic matrix of the representation ρ_c taken with respect to the bijective exponent $\Lambda(c)$. Then, $[\chi(c \pm 24), h_{ext}(c \pm 24)] = f_\pm[\chi(c), h_{ext}(c)]$.

Proof. It is clear from (2.5), which characterizes h_{ext}, that $h_{ext}(c \pm 24) = h_{ext}(c) \pm 2$, and by examining Table I, we see that $h_{ext}(c)$ is never an integer. Since $f_\pm^{-1} = f_\mp$, it suffices to show that $[\chi(c + 24), h_{ext}(c + 24)] = f_+[\chi(c), h_{ext}(c)]$.

Let $h = h_{ext}(c)$. By definition, we have

$$\Lambda := \Lambda(c) = \begin{pmatrix} 1 - \dfrac{c}{24} & 0 \\ 0 & h - \dfrac{c}{24} \end{pmatrix}$$

and

$$\Lambda_+ := \Lambda(c + 24) = \begin{pmatrix} -\dfrac{c}{24} & 0 \\ 0 & h + 1 - \dfrac{c}{24} \end{pmatrix} = \Lambda(c) + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

Let $\Xi := \Xi(c)$ and $\Xi^+ := \Xi(c + 24)$ be the fundamental matrices corresponding to the bijective exponents Λ and Λ_+, respectively, for the representation $\rho_c = \rho_{c+24}$. We may expand

$$\Xi = (\mathbb{X}^{(1)} \,|\, \mathbb{X}^{(2)}) = q^{\Lambda}\begin{pmatrix} q^{-1} + \sum_{n\geq 0} x_n q^n & \sum_{n\geq 0} y_n q^n \\ \sum_{n\geq 0} z_n q^n & q^{-1} + \sum_{n\geq 0} w_n q^n \end{pmatrix}$$

and

$$\Xi_+ = (\mathbb{X}_+^{(1)} \,|\, \mathbb{X}_+^{(2)}) = q^{\Lambda_+}\begin{pmatrix} q^{-1} + \sum_{n\geq 0} x^+_n q^n & \sum_{n\geq 0} y^+_n q^n \\ \sum_{n\geq 0} z^+_n q^n & q^{-1} + \sum_{n\geq 0} w^+_n q^n \end{pmatrix}.$$

Our goal is to show that

$$f_+\left[\begin{pmatrix} x_0 & y_0 \\ z_0 & w_0 \end{pmatrix}, h\right] = \left[\begin{pmatrix} x_0^+ & y_0^+ \\ z_0^+ & w_0^+ \end{pmatrix}, h + 2\right]. \tag{3.5}$$

To do this, we must obtain formulas for x_0^+, y_0^+, z_0^+, and w_0^+ in terms of x_0, y_0, z_0, and w_0, respectively.

Using (3.4), we have

$$\Xi_+ = q^{\Lambda}\begin{pmatrix} q^{-2} + x_0^+ q^{-1} + \cdots & y_0^+ q^{-1} + \cdots \\ z_0^+ q + \cdots & 1 + \cdots \end{pmatrix}. \tag{3.6}$$

Thus,

$$\mathcal{P}_\Lambda \mathbb{X}_+^{(1)} = \begin{pmatrix} q^{-2} + x_0^+ q^{-1} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{P}_\Lambda \mathbb{X}_+^{(2)} = \begin{pmatrix} y_0^+ q^{-1} \\ 0 \end{pmatrix}.$$

Since $\mathcal{P}_\Lambda$ is injective, we have

$$\mathbb{X}_+^{(2)} = y_0^+ \mathbb{X}^{(1)}. \tag{3.7}$$

By (3.6), this identity reads as

$$\begin{pmatrix} y_0^+ q^{-1} + \cdots \\ 1 + \cdots \end{pmatrix} = \begin{pmatrix} y_0^+ q^{-1} + \cdots \\ y_0^+ z_0 + \cdots \end{pmatrix},$$

and thus,

$$y_0^+ = \frac{1}{z_0}. \tag{3.8}$$

Note that $z_0 \neq 0$ by Lemma 3.2. This gives the formula for y_0^+, which corresponds to (3.5).

Substituting (3.8) into (3.7) yields $\mathbb{X}_+^{(2)} = \frac{1}{z_0}\mathbb{X}^{(1)}$ from which we conclude

$$w_0^+ = \frac{z_1}{z_0} \tag{3.9}$$

by considering the second q-coefficient in the second entry of $\mathbb{X}_+^{(2)}$. While this gives an expression for w_0^+ in terms of z variables, it is not the expression we are looking for due to the presence of the higher order coefficient z_1. We will derive an expression for z_1 in terms of lower order coefficients later in the proof (3.15).

For now, we continue on and find expressions for x_0^+ and z_0^+. By direct calculation,

$$\mathcal{P}_\Lambda\Big((J + x_0^+ - x_0)\mathbb{X}^{(1)} - z_0\mathbb{X}^{(2)}\Big) = \begin{pmatrix} q^{-2} + x_0^+ q^{-1} \\ 0 \end{pmatrix} = \mathcal{P}_\Lambda \mathbb{X}_+^{(1)},$$

where $J(q) = q^{-1} + 196884q + \cdots$. Since $\mathcal{P}_\Lambda$ is injective, we have

$$(J + x_0^+ - x_0)\mathbb{X}^{(1)} - z_0\mathbb{X}^{(2)} = \mathbb{X}_+^{(1)}. \tag{3.10}$$

We now multiply both sides of (3.10) by $q^{-\Lambda}$, expanding out the left-hand side and substituting the expression from (3.6) for the right-hand side, to obtain

$$\begin{pmatrix} q^{-2} + x_0^+ q^{-1} + \cdots \\ y_0 + y_1 q + \cdots \end{pmatrix} = \begin{pmatrix} q^{-2} + x_0^+ q^{-1} + \cdots \\ 0 + z_0^+ q + \cdots \end{pmatrix},$$

where

$$y_0 = (x_0^+ - w_0 - x_0)z_0 + z_1 \quad \text{and} \quad y_1 = 196884 z_0 - w_1 z_0 - x_0 z_1 + x_0^+ z_1 + z_2.$$

Thus, $y_0 = 0$ and $y_1 = z_0^+$. The former yields

$$x_0^+ = x_0 + w_0 - \frac{z_1}{z_0}. \tag{3.11}$$

Substituting (3.11) into the equation $z_0^+ = y_1$ and simplifying yield

$$z_0^+ = -\frac{z_1^2}{z_0} + z_1 w_0 + z_0(196884 - w_1) + z_2. \tag{3.12}$$

Our aim now is to replace the higher order coefficients w_1, z_1, and z_2 appearing in (3.9), (3.11), and (3.12) with expressions in terms of x_0, y_0, z_0, and w_0. To do this, we use the differential equation [Ref. 14, Eq. (2.14)]

$$\frac{1}{2\pi i}\frac{d\Xi}{d\tau} - \Xi(\tau)\mathcal{D}(\tau) = 0, \tag{3.13}$$

where

$$\mathcal{D}(\tau) = \frac{1}{\mathcal{E}(\tau)}[(J(\tau) - 240)(\Lambda - 1) + \chi + [\Lambda, \chi]]$$

for $\mathcal{E}(\tau) = q^{-1} - 240 - 141444q - \cdots$. Examining the coefficient of q in the bottom-right entry of (3.13) yields

$$w_1 = \frac{1}{2}\Big(w_0(w_0 + 240) - (h-2)y_0 z_0 + 338328(h - 1 - \frac{c}{24})\Big). \tag{3.14}$$

Similarly, examining the coefficients of q and q^2 in the bottom-left entry of (3.13) yields

$$z_1 = \frac{x_0 + h(w_0 + 240)}{h+1} z_0 \tag{3.15}$$

and

$$z_2 = \frac{x_0(z_1 + 240z_0) + z_0(hw_1 + 240hw_0 + 199044h - 338328\frac{c}{24})}{h+2}, \tag{3.16}$$

respectively.

We have now obtained expressions (3.14)–(3.16) for w_1, z_1, and z_2, respectively, in terms of lower order coefficients. We may substitute these formulas into expressions (3.9), (3.11), and (3.12) for w_0^+, x_0^+, and z_0^+, respectively, to obtain the formulas of (3.5). Combining with our earlier expression (3.8) for y_0^+ completes the proof. □

C. Recurrence for large positive c

We will show that for n sufficiently large, $\chi(c+24n)_{00} < 0$, and moreover, we will obtain an effective bound on such an n. We will do this by iterating f_+, although, in fact, a simpler function will suffice.

Lemma 3.4. Let $g : \mathbb{C}^2 \times (\mathbb{R}\backslash\mathbb{Z}) \to \mathbb{C}^2 \times (\mathbb{R}\backslash\mathbb{Z})$ be the function

$$g[x,w,h] = \left[\frac{w+h(x-240)}{h+1}, \frac{x+h(w+240)}{h+1}, h+2\right],$$

and let g^n denote its n-fold iterate. Then,

$$g^n[x,w,h] = \left[\frac{nw+(h+n-1)(x-240n)}{h+2n-1}, \frac{nx+(h+n-1)(w+240n)}{h+2n-1}, h+2n\right].$$

Proof. This follows by a straightforward induction. □

Lemma 3.5. Let $(\mathcal{C}, c)$ be an admissible genus from Table I, and let $\chi(c)$ denote the characteristic matrix of the representation ρ_c taken with respect to the bijective exponent $\Lambda(c)$. Suppose that $h_{ext}(c) > 0$. Then, $\chi(c+24n)_{00} < 0$ when

$$n > \frac{|M| + \sqrt{M^2 + 960|(h_{ext}(c)-1)\chi(c)_{00}|}}{480},$$

where $M = \chi(c)_{00} + \chi(c)_{11} - 240(h_{ext}(c)-1)$.

Proof. Set $a = \chi(c)_{00}$, $d = \chi(c)_{11}$, and $h = h_{ext}(c)$. By Lemmas 3.3 and 3.4, we have

$$\chi(c+24n)_{00} = \frac{nd+(h+n-1)(a-240n)}{h+2n-1}.$$

Since we assume $h > 0$, when $n \geq 0$, we have $h + 2n - 1 > 0$. Thus, $\chi(c+24n)_{00} < 0$ if and only if

$$0 > nd + (h+n-1)(a-240n) = -240n^2 + Mn + (h-1)\chi_{00}. \tag{3.17}$$

The right-hand side of (3.17) is a quadratic polynomial in n, which is concave down. Thus, (3.17) holds when n exceeds the largest real root of that quadratic (and it holds trivially if the quadratic has no real roots). The conclusion of the lemma now follows immediately from the quadratic formula. □

The purpose of Lemma 3.5 is to reduce the question of classifying extremal VOAs to a finite one. We apply it 24 times to obtain the following theorem:

Theorem 3.6. *For every rank 2 modular tensor category $\mathcal{C}$, there is an explicitly computable c_{max} such that there are no extremal VOAs in the genus $(\mathcal{C}, c)$ when $c > c_{max}$. The values are given in Table II, and the numbering of categories is the same as in Table I.*

Proof. Let us first take $\mathcal{C}$ to be the Semion MTC. In this case, $c \equiv 1 \bmod 8$. We consider first the case $c \equiv 1 \bmod 24$. For $c = 1$, we can compute the characteristic matrix $\chi(1) = \begin{pmatrix} 3 & 26752 \\ 2 & -247 \end{pmatrix}$ using, for example, the method of Ref. 7 (based on Ref. 14) or the method of hypergeometric series.[11] We can compute $h_{ext}(1) = \frac{1}{4}$ from the definition of h_{ext} and the fact that $h \equiv \frac{1}{4} \bmod 1$. Applying Lemma 3.5 with these data, we see that $\chi(1+24n) < 0$ when $n > 0.298\ldots$. Thus, if $n_{max} = 0$, we have $\chi(1+24n) < 0$ when $n > n_{max}$. By Theorem 3.1, there are no extremal VOAs in the genera $(\mathcal{C}, 1+24n)$ when $n > n_{max}$.

We can repeat the above exercise for the values $c = 9$ and $c = 17$ and three times again for each row of Table I. The resulting characteristic matrices, h_{ext}, and n_{max} are given in Table VI. For each category $\mathcal{C}$, the value c_{max} in Table II is the maximum of the three values of $c + 24n_{max}$ corresponding to the three possible classes of c mod 24. □

D. Recurrence for very negative c

We will show that for n sufficiently large, we have $|\chi(c-24n)_{10}| < 1$. Since $\chi(c-24n)_{10} \neq 0$ by Lemma 3.2, this will guarantee that $\chi(c-24n)_{10}$ is not an integer. As with the case of very positive c, we will avoid finding an explicit expression for $f^n_\pm[\chi(c), h_{ext}(c)]$. Instead, we extract the following pieces of the data that will be easier to work with. Let $\alpha(c) = \chi(c)_{00} - \chi(c)_{11}$ and $\beta(c) = \chi(c)_{10}\chi(c)_{01}$.

TABLE II. c_{mx} values for rank 2 modular tensor categories.

No.	$\mathcal{C}$	c_{max}
1	Semion	57
2	$\overline{\text{Semion}}$	39
3	Semion†	67
4	$\overline{\text{Semion}}^\dagger$	37
5	Fib	$\frac{174}{5}$
6	$\overline{\text{Fib}}$	$\frac{186}{5}$
7	Yang-Lee	$\frac{338}{5}$
8	$\overline{\text{Yang – Lee}}$	$\frac{222}{5}$

The utility of studying $\beta(c)$ is the following:

Lemma 3.7. Let $(\mathcal{C}, c)$ *be an admissible genus from* Table I, *and suppose that* $|\chi(c)_{10}| \leq 1$ *and* $|\beta(c - 24n)| > 1$ *for all* $n \geq 1$. *Then,* $|\chi(c - 24n)_{10}| < 1$ *for all* $n \geq 0$.

Proof. By Lemma 3.3, we have $\chi(c-24)_{10} = \chi(c)_{01}^{-1}$. Thus,

$$\beta(c) = \chi(c)_{10}\chi(c)_{01} = \frac{\chi(c)_{10}}{\chi(c-24)_{10}}.$$

Thus, if we know that $|\chi(c)_{10}| \leq 1$ and $|\beta(c-24)| > 1$, we can conclude that $|\chi(c-24)|_{10} < 1$. We repeat this argument n times to complete the proof. □

To see how $\beta(c)$ depends on c, we introduce the function

$$k : \mathbb{C}^2 \times (\mathbb{R}\backslash\mathbb{Z}) \to \mathbb{C}^2 \times (\mathbb{R}\backslash\mathbb{Z}),$$

given by

$$k[\alpha, \beta, h] = \left[\frac{\alpha(h-1) + 480(h-2)}{h-3}, \frac{(h-3)^2(h\beta - 746496) + (\alpha + 120(h-1))^2}{(h-4)(h-3)^2}, h-2\right].$$

This function was chosen so that

Lemma 3.8. Let $(\mathcal{C}, c)$ *be an admissible genus from* Table I. *Then,*

$$[\alpha(c-24), \beta(c-24), h_{ext}(c-24)] = k[\alpha(c), \beta(c), h_{ext}(c)].$$

Proof. This follows by direct algebraic manipulation applied to Lemma 3.3 and the formula for f_-. □

It is now an algebra exercise to determine the long-term behavior of $\alpha(c-24n)$ and $\beta(c-24n)$.

Lemma 3.9. The n-fold iterate of k *is given by*

$$\begin{aligned} &k^n[\alpha, \beta, h] = \\ &\left[\frac{\alpha(h-1) + 480n(h-n-1)}{h-2n-1},\right. \\ &\quad \frac{n(h-n-1)(\alpha + 120(h-1))^2}{(h-2n)(h-2n-1)^2(h-2n-2)} + \frac{h(h-2)\beta - 746496n(h-n-1)}{(h-2n)(h-2n-2)}, \\ &\quad \left. h-2n\right]. \end{aligned}$$

Proof. The formula may be verified by a straightforward induction using the definition of k. □

We carefully examine the expression obtained in Lemma 3.9 to obtain a criterion to bound $|\beta(c+24n)| > 1$.

Lemma 3.10. Let $(\mathcal{C}, c)$ be an admissible genus from Table I, and suppose that $h_{ext}(c) < 0$ and $\beta(c) > 1$. Then, $|\beta(c - 24n)| > 1$ whenever

$$n > \frac{|\alpha(c) - 120(1 - h_{ext}(c))|(1 - h_{ext}(c))}{860}. \tag{3.18}$$

Proof. Let $\beta_n = \beta(c - 24n)$, which by Lemmas 3.8 and 3.9 is given by the formula

$$\beta_n = \frac{n(h-n-1)(\alpha + 120(h-1))^2}{(h-2n)(h-2n-1)^2(h-2n-2)} + \frac{h(h-2)\beta - 746496n(h-n-1)}{(h-2n)(h-2n-2)},$$

where $\alpha = \alpha(c)$ and $h = h_{ext}(c) < 0$.

We can write $\beta_n = \frac{p(n)}{q(n)}$ for $q(n) = (h-2n)(h-2n-1)^2(h-2n-2)$, and p is a certain polynomial of n. To show $|\beta_n| > 1$, it suffices to show that $|p(n)| > |q(n)|$. Since $q(n) > 0$ by inspection when $n \geq 1$, it suffices to show that $p(n) > q(n)$ or that $r(n) := p(n) - q(n) > 0$. Through straightforward manipulation of the formula for β_n, we have $r(n) = r_1(n) + r_2(n)$, where

$$\begin{aligned} r_1(n) =& 2985968n^4 + 5971936(1-h)n^3 + \\ &+ (3732456(1-h)^2 + 4)n^2 + (746488(1-h)^3 + 4(1-h))n + \\ &+ 4(-h)n(2-h)(1-h+n) + (-h)(2-h)(\beta - 1)(1-h+2n)^2, \\ r_2(n) =& -(1-h+n)n(\alpha - 120(1-h))^2. \end{aligned}$$

Since $h < 0$ and $\beta > 1$, every term of $r_1(n)$ is positive. Thus, to show $r(n) > 0$, it suffices to find a term of $r_1(n)$ that controls $r_2(n)$.

To show

$$2985968n^4 + r_2(n) > 0,$$

it suffices to show

$$2985968n^4 > (1-h+n)^2(\alpha - 120(1-h))^2.$$

This will follow from the simple estimate Lemma 3.11 with $A = 2\,985\,968$, $B = (\alpha - 120(1-h))^2$, and $C = 1 - h$, provided

$$2985968n^2 > 2(\alpha - 120(1-h))^2(1 + (1-h)^2).$$

This would follow from

$$2985968n^2 > 4(\alpha - 120(1-h))^2(1-h)^2$$

or equivalently

$$(2985968)^{\frac{1}{2}}n > 2|\alpha - 120(1-h)|(1-h).$$

This is an immediate consequence of our assumption (3.18). □

We used the following simple observation in the Proof of Lemma 3.10:

Lemma 3.11. Let A, B, C, and n be positive real numbers with $n \geq 1$. Then, if

$$An^2 > 2B(1 + C^2),$$

it follows that

$$An^4 > B(n + C)^2.$$

Proof. It suffices to show $An^4 > 2B(n^2 + C^2)$ or equivalently $(An^2 - 2B)n^2 > 2BC^2$. Instead, we may show $An^2 - 2B > 2BC^2$ since $n \geq 1$ and $An^2 > 2B$. This follows immediately from our hypothesis. □

We now apply Lemma 3.10 in 24 cases to obtain a lower bound on the central charge of extremal VOAs.

Theorem 3.12. *For every rank 2 modular tensor category $\mathcal{C}$, there is an explicitly computable c_{min} such that there are no extremal VOAs in the genus $(\mathcal{C}, c)$ when $c < c_{min}$. The values are given in Table III. The numbering of categories is the same as Table I.*

Proof. As in the Proof of Theorem 3.6, we will work through the necessary computation when $\mathcal{C}$ = Semion and obtain a bound that holds for $c \equiv 1 \bmod 24$. Since $h_{ext}(1) > 0$, we must instead consider $c = -23$ in order to apply Lemma 3.10. We compute $h_{ext}(-23) = -\frac{7}{4}$ using (2.5), and we compute

TABLE III. c_{min} values for rank 2 modular tensor categories.

No.	$\mathcal{C}$	c_{min}
1	Semion	-23
2	$\overline{\text{Semion}}$	-17
3	$\text{Semion}^{\dagger}$	-13
4	$\overline{\text{Semion}^{\dagger}}$	-19
5	Fib	$-\frac{106}{5}$
6	$\overline{\text{Fib}}$	$-\frac{94}{5}$
7	Yang-Lee	$-\frac{62}{5}$
8	$\overline{\text{Yang-Lee}}$	$-\frac{98}{5}$

$$\chi(-23) = \begin{pmatrix} \frac{713}{11} & \frac{57264144384}{11} \\ \frac{1}{26752} & -\frac{3397}{11} \end{pmatrix},$$

and from there, $\alpha(-23) = \frac{4110}{11}$ and $\beta(-23) = \frac{23546112}{121}$. Thus, by Lemma 3.10, we have $|\beta(-23-24n)| > 1$ when $n > 0.13\ldots$. Taking $n_{max} = 0$, we have $|\beta(-23-24n)| > 1$ when $n > n_{max}$. As $|\chi(-23-24n_{max})_{10}| < 1$, we conclude that $|\chi(-23-24n)_{10}| < 1$ for all $n > n_{max}$, and thus, by Theorem 3.1, there cannot be a VOA in the genus $(\mathcal{C}, c)$ when $c < -23$ and $c \equiv 1 \bmod 24$. We repeat this argument for the other two equivalence classes of c mod 24, and the value c_{min} from Table III is the minimum of the allowed values.

We apply the above procedure to each of the eight modular categories appearing in Table I. The data from each of the cases are given in Table VII. □

E. Main result

Combining Theorems 3.6 and 3.12, we obtain for every rank 2 modular tensor category $\mathcal{C}$, a pair of numbers c_{min} and c_{max} such that if V is an extremal VOA in the genus $(\mathcal{C}, c)$, then $c_{min} \le c \le c_{max}$. We can now compute the characteristic matrix of every remaining pair $(\mathcal{C}, c)$ (e.g., by Lemma 3.3) and throw away any for which the first column does not consist of positive integers. We are left with 15 possibilities, all but one of which are realized by VOAs that have previously been studied. The remaining character vector is realized by a VOA constructed in Sec. IV. We summarize the result in Table IV.

Theorem 3.13. *Let V be a strongly rational extremal VOA with two simple modules. Then it lies in one of the genera specified in Table IV (and its character vector is given in Table V).*

TABLE IV. Genera of extremal VOAs corresponding to rank 2 modular tensor categories.

$\mathcal{C}$	C	Realization	h_{ext}	ℓ
Semion	1	$A_{1,1}$	$\frac{1}{4}$	0
Semion	9	$A_{1,1} \otimes E_{8,1}$	$\frac{1}{4}$	4
Semion	17	[22]	$\frac{5}{4}$	2
Semion	33	Sec. 4	$\frac{9}{4}$	4
$\overline{\text{Semion}}$	7	$E_{7,1}$	$\frac{3}{4}$	0
$\overline{\text{Semion}}$	15	$E_{7,1} \otimes E_{8,1}$	$\frac{3}{4}$	4
$\overline{\text{Semion}}$	23	[22]	$\frac{7}{4}$	2
Fib	$\frac{14}{5}$	$G_{2,1}$	$\frac{2}{5}$	0
Fib	$\frac{54}{5}$	$G_{2,1} \otimes E_{8,1}$	$\frac{2}{5}$	4
Fib	$\frac{94}{5}$	[22]	$\frac{7}{5}$	2
$\overline{\text{Fib}}$	$\frac{26}{5}$	$F_{4,1}$	$\frac{3}{5}$	0
$\overline{\text{Fib}}$	$\frac{66}{5}$	$F_{4,1} \otimes E_{8,1}$	$\frac{3}{5}$	4
$\overline{\text{Fib}}$	$\frac{106}{5}$	[22]	$\frac{8}{5}$	2
Yang-Lee	$-\frac{22}{5}$	Yang-Lee	$-\frac{1}{5}$	0
Yang-Lee	$\frac{18}{5}$	Y-L $\otimes E_{8,1}$	$-\frac{1}{5}$	4

One of the main purposes of establishing classification results such as Theorem 3.13 is to find interesting new examples, such as the VOA in the genus (Semion, 33) constructed in Sec. IV. The most interesting genera for which no VOA realizations are known are (DHaag, $8n$), where DHaag is the double of the Haagerup fusion category. Evans and Gannon computed possible character vectors for potential "Haagerup VOAs" in the cases $n = 1, 2,$ and 3[24] and used these characters to suggest strategies for constructing them. Subsequently, Gannon analyzed the possible Lie algebra structures on the weight 1 vectors of Haagerup VOAs, as well as the structure of their cosets. Despite all of the circumstantial evidence, however, no construction has been found for a Haagerup VOA.

The success in constructing a (Semion, 33) VOA may be regarded as further evidence of the fruitfulness of the Evans–Gannon approach to the Haagerup VOA. While the Semion category is quite a bit simpler than DHaag, the central charge $c = 33$ is, in practice, quite large compared to $c = 8, 16,$ or 24, which is a source of added difficulty.

In the case of the (Semion, 33) VOA, the subVOA generated by weight 1 vectors is of type $A_{1,1}$. The coset of this affine VOA is of independent interest, and we record here its character vector,

$$q^{-32/24}\begin{pmatrix} 1 + 0q + 69616q^2 + 34668544q^3 + \cdots \\ q^{9/4}(426192 + 121366368q + \cdots) \\ q^{7/4}(10245 + 11330970q + \cdots) \\ q^2(69888 + 34664448q + \cdots) \end{pmatrix},$$

as well as the character vector of its holomorphic extension (the twisted orbifold of the rank 32 Barnes–Wall lattice),

$$q^{-32/24}(1 + 0q + 139504q^2 + 69332992q^3 + \cdots).$$

IV. CONSTRUCTION OF THE EXTREMAL $c = 33$ EXAMPLE

The goal of this section is to prove the following theorem:

Theorem 4.1. *There exists an extremal VOA in the genus* (Semion, 33).

The key step in the construction will be the following:

Theorem 4.2. *There exists a $c = 32$ holomorphic framed VOA V and its involution $\theta \in \mathrm{Aut}(V)$ satisfying the following conditions:*

(1) $V(1) = 0$.
(2) *The unique irreducible θ-twisted V-module W has top weight $7/4$.*

We first give a Proof of Theorem 4.1 using Theorem 4.2.

Proof of Theorem 4.1. Suppose we have V, θ, and W as in Theorem 4.2. Let $V^{\pm} = \{a \in V \mid \theta a = \pm a\}$ be the eigenspace decompositions and $W = W^+ \oplus W^-$ be the irreducible decomposition as a V^+-module. Note that V^+ is a strongly rational VOA by Ref. 25 (see also Ref. 26). We assign the labeling $W^{\pm}$ such that the conformal weight of W^+ is $7/4$. It turns out that the conformal weight of W^- is equal to $9/4$ so that the conformal weights of V^+, V^-, W^+, and W^- are $0, 2, 7/4$, and $9/4$, respectively. Since V is holomorphic, V^+ has exactly four irreducible modules $V^{\pm}$ and $W^{\pm}$, and all of them are simple currents. The fusion algebra of V^+ is isomorphic to the group algebra associated with $(\mathbb{Z}/2\mathbb{Z})^2$.

We consider a tensor product $V^+ \otimes L_{\hat{\mathfrak{sl}}_2}(1,0)$ and its $\mathbb{Z}/2\mathbb{Z}$-graded simple current extension

$$U = V^+ \otimes L_{\hat{\mathfrak{sl}}_2}(1,0) \oplus W^+ \otimes L_{\hat{\mathfrak{sl}}_2}(1,1),$$

which is strongly rational by Refs. 27 and 28 (see also Ref. 29, Theorem 4.13). Then, the weight 1 subspace of U is three-dimensional. It is easy to see that U has exactly two irreducible untwisted modules, U and

$$M = V^- \otimes L_{\hat{\mathfrak{sl}}_2}(1,1) \oplus W^- \otimes L_{\hat{\mathfrak{sl}}_2}(1,0),$$

whose conformal weight is $2 + 1/4 = 9/4 + 0 = 9/4$. Thus, U is an extremal VOA with central charge 33 and two simple modules, as desired. □

We will now prove Theorem 4.2 based on the theory of framed VOAs. We will use the same notation and terminology for the framed VOAs as in Ref. 30. In particular, the product of codewords is defined by

$$\alpha \cdot \beta = (\alpha_1\beta_1, \ldots, \alpha_n\beta_n) \in \mathbb{Z}_2^n$$

for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n$. We denote by $\mathbf{1}_n$ the codeword $(1^n) \in \mathbb{Z}_2^n$.

Let V be a framed VOA with a Virasoro frame $F = \langle e^1, \dots, e^n \rangle \cong L(1/2, 0)^{\otimes n}$. Let (C, D) be the structure codes with respect to F and $V = \oplus_{\alpha \in D} V^\alpha$ be the corresponding 1/16-word decomposition. For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$, we define

$$\sigma_\beta := \prod_{i \in \mathrm{supp}(\beta)} (-1)^{2o(e^i)}, \quad \tau_\beta := \prod_{i \in \mathrm{supp}(\beta)} (-1)^{16 o(e^i)}, \tag{4.1}$$

where $o(a)$ denotes the grade preserving operator of $a \in V$. It follows from the fusion rules of $L(1/2, 0)$-modules that $\sigma_\beta \in \mathrm{Aut}(V^0)$ and $\tau_\beta \in \mathrm{Aut}(V)$ (cf. Ref. 31). The maps $\sigma : \mathbb{Z}_2^n \to \mathrm{Aut}(V^0)$ and $\tau : \mathbb{Z}_2^n \to \mathrm{Aut}(V)$ are group homomorphisms such that $\ker \sigma = C^\perp$ and $\ker \tau = D^\perp$. Therefore, we have the following exact sequences:

$$\begin{aligned} 0 \to C^\perp \to \mathbb{Z}_2^n \xrightarrow{\sigma} \mathrm{Im}\, \sigma \to 0, \\ 0 \to D^\perp \to \mathbb{Z}_2^n \xrightarrow{\tau} \mathrm{Im}\, \tau \to 0. \end{aligned} \tag{4.2}$$

We define the point-wise frame stabilizer [denoted by $\mathrm{Stab}_V^{\mathrm{pt}}(F)$ in Ref. 30] by

$$\mathrm{Aut}_F(V) := \{ g \in \mathrm{Aut}(V) \mid g(e^i) = e^i \text{ for } 1 \le i \le n \}. \tag{4.3}$$

The structure of the point-wise frame stabilizer is determined in Ref. 30 as follows:

Theorem 4.3 (Ref. 30). *Let V be a framed VOA with structure codes (C, D).*

(1) *$\mathrm{Im}\, \tau \cong \mathbb{Z}_2^n / D^\perp$ is a central subgroup of $\mathrm{Aut}_F(V)$.*
(2) *For $\theta \in \mathrm{Aut}_F(V)$, there exists $\xi \in \mathbb{Z}_2^n$ and $\eta \in \mathbb{Z}_2^n$ such that $\theta|_{V^0} = \sigma_\xi$ and $\theta^2 = \tau_\eta$. In particular, $\theta^4 = 1$.*
(3) *For $\xi \in \mathbb{Z}_2^n$, there exists $\theta \in \mathrm{Aut}_F(V)$ such that $\theta|_{V^0} = \sigma_\xi$ if and only if $\xi \cdot D \subset C$, and in this case, the order $|\theta| = 2$ if and only if $\xi \cdot D$ is a doubly even subcode of C and otherwise $|\theta| = 4$.*
(4) *Let $P = \{ \xi \in \mathbb{Z}_2^n \mid \xi \cdot D \subset C \}$. Then, $C^\perp \subset P$, and we have an exact sequence*

$$1 \to \mathbb{Z}_2^n / D^\perp \to \mathrm{Aut}_F(V) \to P / C^\perp \to 1.$$

We review a construction of the twisted modules from Ref. 30. Suppose we have a codeword $\xi \in P$ such that $\xi \cdot D$ is a doubly even subcode of C. Let $\theta \in \mathrm{Aut}_F(V)$ be an involution such that $\theta|_{V^0} = \sigma_\xi$. Note that such a θ_ξ is not unique and only determined modulo $\mathrm{Im}\tau$ by (4) of Theorem 4.3. The fixed point subalgebra $V^+ = V^{\langle \theta \rangle}$ is a framed VOA with structure codes (C^0, D), where $C^0 = C \cap \langle \xi \rangle^\perp$. We denote its 1/16-word decomposition by $V^+ = \oplus_{\alpha \in D} V^{+,\alpha}$. Let X be an irreducible F-module isomorphic to $L(^1/_2, ^1/_{16})^{\otimes \mathrm{wt}(\xi)} \otimes L(^1/_2, 0)^{\otimes n - \mathrm{wt}(\xi)}$ whose 1/16-word is ξ. There exists an irreducible $V^{+,0}$-module Y that contains X as an F-submodule (cf. Refs. 30 and 32). Since $V = \oplus_{\alpha \in D} V^\alpha$ is a $\mathbb{Z}_2 \oplus D$-graded simple current extension of $V^{+,0}$, there exists $\tau_\eta \in \mathrm{Im}\, \tau$ such that the fusion product

$$W = V \boxtimes_{V^{+,0}} Y = \bigoplus_{\alpha \in D} V^\alpha \boxtimes_{V^{+,0}} Y \tag{4.4}$$

has a unique structure of an irreducible $\theta\tau_\eta$-twisted V-module. Since θ and $\theta\tau_\eta$ define the same automorphism σ_ξ on the subalgebra V^0 of V, by replacing θ by $\theta\tau_\eta$ if necessary, we may regard W as an irreducible θ-twisted V-module. Each summand $V^\alpha \boxtimes_{V^{+,0}} Y$, $\alpha \in D$, of W has the 1/16-word $\alpha + \xi$ so that its top weight is at least $\mathrm{wt}(\xi + \alpha)/16$. Summarizing, we have the following theorem:

Theorem 4.4. *Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA with structure codes (C, D) with respect to a frame $F \cong L(^1/_2, 0)^{\otimes n}$. Let $\xi \in P$ be a codeword such that $\xi \cdot D$ is a doubly even subcode of C. Let X be the irreducible F-module isomorphic to $L(^1/_2, ^1/_{16})^{\otimes \mathrm{wt}(\xi)} \otimes L(^1/_2, 0)^{\otimes n - \mathrm{wt}(\xi)}$ such that its 1/16-word is ξ. Then there exists an involutive automorphism $\theta \in \mathrm{Aut}(V)$ and an irreducible θ-twisted V-module W such that $\theta|_{V^0} = \sigma_\xi$ and W contains X as an irreducible F-submodule. The top weight of W is at least $\min\{\mathrm{wt}(\xi + \alpha)/16 \mid \alpha \in D\}$.*

Recall the Reed–Muller (RM) codes. The first order Reed–Muller code RM(1, 4) of length 2^4 is defined by the following generating matrix:

$$\begin{bmatrix} 1111 & 1111 & 1111 & 1111 \\ 1111 & 1111 & 0000 & 0000 \\ 1111 & 0000 & 1111 & 0000 \\ 1100 & 1100 & 1100 & 1100 \\ 1010 & 1010 & 1010 & 1010 \end{bmatrix}. \tag{4.5}$$

The dual code of RM(1, 4) is the second order Reed–Muller code $\mathrm{RM}(2, 4) = \mathrm{RM}(1, 4) \cdot \mathrm{RM}(1, 4)$. The first order Reed–Muller code RM(1, 6) of length 2^6 is defined by

$$\mathrm{RM}(1, 6) = \mathrm{Span}_{\mathbb{Z}_2} \{ (\alpha, \alpha, \alpha, \alpha),\ (0^{16} 1^{16} 0^{16} 1^{16}),\ (0^{32} 1^{32}) \mid \alpha \in \mathrm{RM}(1, 4) \}. \tag{4.6}$$

The weight enumerator of RM(1, 6) is $x^{64} + 126x^{32} + 1$. The dual code of RM(1, 6) is the fourth order Reed–Muller code RM(4, 6) of length 2^6. It follows from the MacWilliams identity that the minimum weight of RM(4, 6) is 4. It is easy to see that

$$\begin{array}{r}(\alpha,\beta,\gamma,\delta)\in \mathrm{RM}(4,6)\\ (\alpha,\beta,\gamma,\delta\in \mathbb{Z}_2^{16})\end{array} \Leftrightarrow \begin{array}{l}\alpha+\beta+\gamma+\delta\in \mathrm{RM}(2,4)=\mathrm{RM}(1,4)^{\perp},\\ \mathrm{wt}(\alpha)\equiv \mathrm{wt}(\beta)\equiv \mathrm{wt}(\gamma)\equiv \mathrm{wt}(\delta) \bmod 2.\end{array} \tag{4.7}$$

Since RM(1, 6) is triply even, there exists a $c = 32$ holomorphic framed VOA with structure codes (RM(4, 6), RM(1, 6)) by Remark 6 and Theorem 10 of Ref. 30. Since the minimum weights of RM(1, 6) and RM(4, 6) are 32 and 4, respectively, the weight 1 subspace of such a VOA is trivial. It follows from Theorem 3.13 of Ref. 33 that such a framed VOA is uniquely determined by its structure codes. On the other hand, it is shown in Ref. 34 that the $\mathbb{Z}_2$-orbifold construction $\tilde{V}_{\mathrm{BW}_{32}}$ of the lattice VOA associated with the Barnes–Wall lattice BW_{32} of rank 32 is a $c = 32$ holomorphic framed VOA with structure codes (RM(4, 6), RM(1, 6)). The VOA $\tilde{V}_{\mathrm{BW}_{32}}$ is studied in Refs. 34 and 35, and it has a finite automorphism group of the shape $2^{27}.E_6(2)$.

Theorem 4.5. *The $\mathbb{Z}_2$-orbifolding $\tilde{V}_{\mathrm{BW}_{32}}$ is the unique $c = 32$ holomorphic framed VOA with structure codes* (RM(4, 6), RM(1, 6)).

We will prove Theorem 4.2 by using $\tilde{V}_{\mathrm{BW}_{32}}$. Let $\alpha^1 = (1^{16})$, $\alpha^2 = (1^8 0^8)$, $\alpha^3 = (1^4 0^4 1^4 0^4)$, $\alpha^4 = (\{1^2 0^2\}^4)$, and $\alpha^5 = (\{10\}^8)$ be the basis of RM(1, 4) in (4.5). Then, by (4.6), the Reed–Muller code RM(1, 6) has a basis

$$\gamma^i = (\alpha^i,\alpha^i,\alpha^i,\alpha^i),\quad 1\le i\le 5,\quad \gamma^6 = (0^{16}1^{16}0^{16}1^{16}),\quad \gamma^7 = (0^{32}1^{32}). \tag{4.8}$$

Lemma 4.6. Let $v^1, v^2, v^3, v^4 \in \mathbb{Z}_2^{16}$, and let $\xi = (v^1, v^2, v^3, v^4) \in \mathbb{Z}_2^{64}$. Then, $\xi\cdot$ RM(1, 6) *is a subcode of* RM(4, 6) *if and only if the following conditions are satisfied:*

(i) $v^1 + v^2 + v^3 + v^4 \in \mathrm{RM}(1, 4)$,
(ii) $v^i + v^j \in \mathrm{RM}(1, 4)^\perp$ *for* $1 \le i < j \le 4$, *and*
(iii) v^1, v^2, v^3, v^4 *are even.*
Moreover, $\xi\cdot$ RM(1, 6) *is a doubly even subcode of* RM(4, 6) *if and only if it further satisfies the following condition:*
(iv) $\xi\cdot\gamma^i$, $1 \le i \le 5$, *are doubly even.*

Proof. First, we prove that $\xi\cdot\mathrm{RM}(1, 6)$ is a subcode of $\mathrm{RM}(4, 6) = \mathrm{RM}(1, 6)^\perp$ if and only if ξ satisfies conditions (i)–(iii). Let α, $\beta \in \mathrm{RM}(1, 6)$. We have $(\xi\cdot\alpha|\beta) = (\xi|\alpha\cdot\beta)$ so that $\xi\cdot\mathrm{RM}(1, 6)$ is a subcode of $\mathrm{RM}(1, 6)^\perp$ if and only if $(\xi|\gamma^i\cdot\gamma^j) = 0$ for $1 \le i \le j \le 7$. For $1 \le i, j \le 5$, we have

$$\begin{aligned}(\xi|\gamma^i\cdot\gamma^j) &= ((v^1,v^2,v^3,v^4)|(\alpha^i\cdot\alpha^j,\alpha^i\cdot\alpha^j,\alpha^i\cdot\alpha^j,\alpha^i\cdot\alpha^j)\\ &= (v^1+v^2+v^3+v^4|\alpha^i\cdot\alpha^j)\end{aligned}$$

so that $(\xi|\gamma^i\cdot\gamma^j) = 0$ for $1 \le i, j \le 5$ if and only if $v^1 + v^2 + v^3 + v^4 \in (\mathrm{RM}(1, 4)\cdot\mathrm{RM}(1, 4))^\perp = \mathrm{RM}(2, 4)^\perp = \mathrm{RM}(1, 4)$. Thus, we obtain condition (i). Similarly, from $(\xi|\gamma^i\cdot\gamma^j) = 0$ for $1 \le i \le 5$ and $j = 6$ and 7, we obtain $v^2 + v^4, v^3 + v^4 \in \mathrm{RM}(1, 4)^\perp$. Since $\mathrm{RM}(1, 4) \subset \mathrm{RM}(2, 4)$, it follows that $v^i + v^j \in \mathrm{RM}(1, 4)^\perp$ for $1 \le i < j \le 4$, and we obtain condition (ii). In addition, from $(\xi|\gamma^i\cdot\gamma^j) = 0$ for $6 \le i, j \le 7$, we obtain condition (iii). Thus, $\xi\cdot\mathrm{RM}(1, 6)$ is a subcode of RM(4, 6) if and only if ξ satisfies conditions (i)–(iii).

Now suppose $\xi\cdot\mathrm{RM}(1, 6)$ is a subcode of RM(4, 6). As we have discussed, this is equivalent to that $(\xi\cdot\gamma^i|\xi\cdot\gamma^j) = (\xi|\gamma^i\cdot\gamma^j) = 0$ for $1 \le i, j \le 7$ so that $\xi\cdot\mathrm{RM}(1, 6)$ is self-orthogonal. Since a sum of mutually orthogonal doubly even codewords is again doubly even, $\xi\cdot\mathrm{RM}(1, 6)$ is doubly even if and only if all seven vectors $\xi\cdot\gamma^i$, $1 \le i \le 7$, are doubly even. It follows from condition (iii) that $\xi\cdot\gamma^6$ and $\xi\cdot\gamma^7$ are doubly even. Therefore, $\xi\cdot\mathrm{RM}(1, 6)$ is doubly even if and only if ξ satisfies condition (iv). □

Lemma 4.7. Let α be a weight 6 codeword of RM(2, 4). *Then, the codeword*

$$\xi = (\alpha,\alpha,\alpha,\alpha^c) \in \mathbb{Z}_2^{64}$$

satisfies conditions (i)–(iv) *in Lemma 4.6, where $\alpha^c = \mathbf{1}_{16} + \alpha$. The weight enumerator of the coset $\xi + \mathrm{RM}(1, 6)$ is $64x^{28} + 64x^{36}$.*

Proof of Theorem 4.2. Let $V = \oplus_{\alpha\in\mathrm{RM}(1,6)} V^\alpha$ be a holomorphic framed VOA with structure codes (RM(4, 6), RM(1, 6)). Then, $V \cong \tilde{V}_{\mathrm{BW}_{32}}$ by Theorem 4.5. Let

$$\alpha = (0110\,1100\,1010\,0000) \in \mathrm{RM}(2, 4),$$

and set

$$\xi = (\alpha,\alpha,\alpha,\alpha^c) \in \mathbb{Z}_2^{64}. \tag{4.9}$$

Let X be an irreducible $L(1/2, 0)^{\otimes 64}$-module isomorphic to $L(^1/_2, ^1/_{16})^{\otimes 28} \otimes L(^1/_2, 0)^{\otimes 36}$ such that its 1/16-word is ξ. By Theorem 4.4 and Lemmas 4.6 and 4.7, there is an involution $\theta \in \mathrm{Aut}(V)$ such that $\theta|_{V^0} = \sigma_\xi$ and the irreducible θ-twisted V-module W has top weight $28/16 = 7/4$. □

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APPENDIX: DATA

As in Ref. 7, we can compute potential character vectors for each of the allowed genera of Theorem 3.13, and they are given in Table V. The realizations labeled with Ref. 22 arise as cosets of affine VOAs.

Table VI (which was used in the Proof of Theorem 3.6) has 24 rows, in groups of 3. Each group corresponds to a modular tensor category from Table I, and each row within the group selects a representative of an equivalence class of admissible c mod 24. Table VI provides the characteristic matrix (computed as in Ref. 7), the extremal conformal dimension $h_{ext}(c)$ computed from the definition (2.5), and a value n_{max} such that $\chi(c+24n)_{00} < 0$ when $n > n_{max}$, as computed using Lemma 3.5.

TABLE V. Characters of strongly rational extremal VOAs with two simple modules.

$\mathcal{C}$	C	Realization	Character
Semion	1	$A_{1,1}$	$q^{-1/24}\begin{pmatrix} 1+3q+4q^2+\cdots \\ q^{\frac{1}{4}}(2+2q+6q^2+\cdots) \end{pmatrix}$
Semion	9	$A_{1,1}\otimes E_{8,1}$	$q^{-9/24}\begin{pmatrix} 1+251q+4872q^2+\cdots \\ q^{\frac{1}{4}}(2+498q+8750q^2+\cdots) \end{pmatrix}$
Semion	17	22	$q^{-17/24}\begin{pmatrix} 1+323q+60860q^2+\cdots \\ q^{\frac{5}{4}}(1632+162656q+4681120q^2+\cdots) \end{pmatrix}$
Semion	33	Sec. 4	$q^{-33/24}\begin{pmatrix} 1+3q+86004q^2+\cdots \\ q^{\frac{9}{4}}(565760+192053760q+\cdots) \end{pmatrix}$
$\overline{\text{Semion}}$	7	$E_{7,1}$	$q^{-7/24}\begin{pmatrix} 1+133q+1673q^2+\cdots \\ q^{\frac{3}{4}}(56+968q+7504q^2+\cdots) \end{pmatrix}$
$\overline{\text{Semion}}$	15	$E_{7,1}\otimes E_{8,1}$	$q^{-15/24}\begin{pmatrix} 1+381q+38781q^2+\cdots \\ q^{\frac{3}{4}}(56+14856q+478512q^2+\cdots) \end{pmatrix}$
$\overline{\text{Semion}}$	23	22	$q^{-23/24}\begin{pmatrix} 1+69q+131905q^2+\cdots \\ q^{\frac{7}{4}}(32384+4418944q+189846784q^2+\cdots) \end{pmatrix}$
Fib	$\frac{14}{5}$	$G_{2,1}$	$q^{-7/60}\begin{pmatrix} 1+14q+42q^2+\cdots \\ q^{\frac{2}{5}}(7+34q+119q^2+\cdots) \end{pmatrix}$
Fib	$\frac{54}{5}$	$G_{2,1}\otimes E_{8,1}$	$q^{-27/60}\begin{pmatrix} 1+262q+7638q^2+\cdots \\ q^{\frac{2}{5}}(7+1770q+37419q^2+\cdots) \end{pmatrix}$
Fib	$\frac{94}{5}$	22	$q^{-47/60}\begin{pmatrix} 1+188q+62087q^2+\cdots \\ q^{\frac{7}{5}}(4794+532134q+17518686q^2+\cdots) \end{pmatrix}$
$\overline{\text{Fib}}$	$\frac{26}{5}$	$F_{4,1}$	$q^{-13/60}\begin{pmatrix} 1+52q+377q^2+\cdots \\ q^{\frac{3}{5}}(26+299q+1702q^2+\cdots) \end{pmatrix}$
$\overline{\text{Fib}}$	$\frac{66}{5}$	$F_{4,1}\otimes E_{8,1}$	$q^{-33/60}\begin{pmatrix} 1+300q+17397q^2+\cdots \\ q^{\frac{3}{5}}(26+6747q+183078q^2+\cdots) \end{pmatrix}$
$\overline{\text{Fib}}$	$\frac{106}{5}$	22	$q^{-33/60}\begin{pmatrix} 1+106q+84429q^2+\cdots \\ q^{\frac{8}{5}}(15847+1991846q+76895739q^2+\cdots) \end{pmatrix}$
Yang-Lee	$-\frac{22}{5}$	Yang-Lee	$q^{11/60}\begin{pmatrix} 1+0q+q^2+\cdots \\ q^{-\frac{1}{5}}(1+q+q^2+\cdots) \end{pmatrix}$
Yang-Lee	$\frac{18}{5}$	$Y-L\otimes E_{8,1}$	$q^{-3/20}\begin{pmatrix} 1+248q+4125q^2+\cdots \\ q^{-\frac{1}{5}}(1+249q+4373q^2+\cdots) \end{pmatrix}$

TABLE VI. Values of n_{max} computed from Lemma 3.5.

No.	$\mathcal{C}$	C	$\chi(c)$	$h_{ext}(c)$	n_{max}
1	Semion	1	$\begin{pmatrix} 3 & 26752 \\ 2 & -247 \end{pmatrix}$	$\frac{1}{4}$	0
1	Semion	9	$\begin{pmatrix} 251 & 26752 \\ 2 & 1 \end{pmatrix}$	$\frac{1}{4}$	2
1	Semion	17	$\begin{pmatrix} 323 & 88 \\ 1632 & -319 \end{pmatrix}$	$\frac{5}{4}$	0
2	$\overline{\text{Semion}}$	7	$\begin{pmatrix} 133 & 1248 \\ 56 & -377 \end{pmatrix}$	$\frac{3}{4}$	0
2	$\overline{\text{Semion}}$	15	$\begin{pmatrix} 381 & 1248 \\ 56 & -129 \end{pmatrix}$	$\frac{3}{4}$	1
2	$\overline{\text{Semion}}$	23	$\begin{pmatrix} 69 & 10 \\ 32384 & -65 \end{pmatrix}$	$\frac{7}{4}$	0
3	$\text{Semion}^\dagger$	11	$\begin{pmatrix} -319 & 1632 \\ 88 & 323 \end{pmatrix}$	$\frac{3}{4}$	0
3	$\text{Semion}^\dagger$	19	$\begin{pmatrix} -247 & 2 \\ 26752 & 3 \end{pmatrix}$	$\frac{7}{4}$	2
3	$\text{Semion}^\dagger$	27	$\begin{pmatrix} 1 & 2 \\ 26752 & 251 \end{pmatrix}$	$\frac{7}{4}$	0
4	$\overline{\text{Semion}}^\dagger$	5	$\begin{pmatrix} -65 & 32384 \\ 10 & 69 \end{pmatrix}$	$\frac{1}{4}$	0
4	$\overline{\text{Semion}}^\dagger$	13	$\begin{pmatrix} -377 & 56 \\ 1248 & 133 \end{pmatrix}$	$\frac{5}{4}$	1
4	$\overline{\text{Semion}}^\dagger$	21	$\begin{pmatrix} -129 & 56 \\ 1248 & 381 \end{pmatrix}$	$\frac{5}{4}$	0
5	Fib	$\frac{14}{5}$	$\begin{pmatrix} 14 & 12857 \\ 7 & -258 \end{pmatrix}$	$\frac{2}{5}$	0
5	Fib	$\frac{54}{5}$	$\begin{pmatrix} 262 & 12857 \\ 7 & -10 \end{pmatrix}$	$\frac{2}{5}$	1
5	Fib	$\frac{94}{5}$	$\begin{pmatrix} 188 & 46 \\ 4794 & -184 \end{pmatrix}$	$\frac{7}{5}$	0
6	$\overline{\text{Fib}}$	$\frac{26}{5}$	$\begin{pmatrix} 52 & 3774 \\ 26 & -296 \end{pmatrix}$	$\frac{3}{5}$	0
6	$\overline{\text{Fib}}$	$\frac{66}{5}$	$\begin{pmatrix} 300 & 3774 \\ 26 & -48 \end{pmatrix}$	$\frac{3}{5}$	1
6	$\overline{\text{Fib}}$	$\frac{106}{5}$	$\begin{pmatrix} 106 & 17 \\ 15847 & -102 \end{pmatrix}$	$\frac{8}{5}$	0
7	Yang – Lee	$\frac{58}{5}$	$\begin{pmatrix} -406 & 902 \\ 87 & 410 \end{pmatrix}$	$\frac{4}{5}$	0
7	Yang – Lee	$\frac{98}{5}$	$\begin{pmatrix} -245 & 1 \\ 26999 & 1 \end{pmatrix}$	$\frac{9}{5}$	2
7	Yang – Lee	$\frac{138}{5}$	$\begin{pmatrix} 3 & 1 \\ 26999 & 249 \end{pmatrix}$	$\frac{9}{5}$	0
8	$\overline{\text{Yang – Lee}}$	$\frac{22}{5}$	$\begin{pmatrix} -55 & 32509 \\ 11 & 59 \end{pmatrix}$	$\frac{1}{5}$	1
8	$\overline{\text{Yang – Lee}}$	$\frac{62}{5}$	$\begin{pmatrix} -434 & 57 \\ 682 & 190 \end{pmatrix}$	$\frac{6}{5}$	1
8	$\overline{\text{Yang – Lee}}$	$\frac{102}{5}$	$\begin{pmatrix} -186 & 57 \\ 682 & 438 \end{pmatrix}$	$\frac{6}{5}$	1

TABLE VII. Values of n_{max} computed from Lemma 3.10.

No.	$\mathcal{C}$	c	$\chi(c)$	$h_{ext}(c)$	$\alpha(c)$	$\beta(c)$	n_{max}	$\chi_{10}(c)$
1	Semion	-7	$\begin{pmatrix} 59 & 13424640 \\ \frac{1}{88} & -55 \end{pmatrix}$	$-\frac{3}{4}$	114	$\frac{1678080}{11}$	0	$\frac{1}{88}$
1	Semion	-15	$\begin{pmatrix} \frac{3441}{11} & \frac{57264144384}{11} \\ \frac{1}{26752} & -\frac{669}{11} \end{pmatrix}$	$-\frac{7}{4}$	$\frac{4110}{11}$	$\frac{23546112}{121}$	0	$\frac{1}{26752}$
1	Semion	-23	$\begin{pmatrix} \frac{713}{11} & \frac{57264144384}{11} \\ \frac{1}{26752} & -\frac{3397}{11} \end{pmatrix}$	$-\frac{7}{4}$	$\frac{4110}{11}$	$\frac{23546112}{121}$	0	$\frac{1}{26752}$
2	$\overline{\text{Semion}}$	-1	$\begin{pmatrix} \frac{49}{5} & \frac{3281408}{5} \\ \frac{1}{10} & \frac{-29}{5} \end{pmatrix}$	$\frac{-1}{4}$	$\frac{78}{5}$	$\frac{1640704}{25}$	0	$\frac{1}{10}$
2	$\overline{\text{Semion}}$	-9	$\begin{pmatrix} \frac{863}{3} & \frac{747151360}{3} \\ \frac{1}{1248} & \frac{-107}{3} \end{pmatrix}$	$\frac{-5}{4}$	$\frac{970}{3}$	$\frac{23348480}{117}$	0	$\frac{1}{1248}$
2	$\overline{\text{Semion}}$	-17	$\begin{pmatrix} \frac{119}{3} & \frac{747151360}{3} \\ \frac{1}{1248} & \frac{-851}{3} \end{pmatrix}$	$\frac{-5}{4}$	$\frac{970}{3}$	$\frac{23348480}{117}$	0	$\frac{1}{1248}$
3	Semion†	3	$\begin{pmatrix} 249 & 565760 \\ \frac{1}{2} & 3 \end{pmatrix}$	$\frac{-1}{4}$	246	282 880	0	$\frac{1}{2}$
3	Semion†	-5	$\begin{pmatrix} 1 & 565760 \\ \frac{1}{2} & -245 \end{pmatrix}$	$\frac{-1}{4}$	246	282 880	0	$\frac{1}{2}$
3	Semion†	-13	$\begin{pmatrix} \frac{299}{3} & \frac{827924480}{3} \\ \frac{1}{1632} & \frac{-287}{3} \end{pmatrix}$	$\frac{-5}{4}$	$\frac{586}{3}$	$\frac{1521920}{9}$	0	$\frac{1}{1632}$
4	$\overline{\text{Semion}}^{\dagger}$	-3	$\begin{pmatrix} \frac{1857}{7} & \frac{83232768}{7} \\ \frac{1}{56} & \frac{-93}{7} \end{pmatrix}$	$\frac{-3}{4}$	$\frac{1950}{7}$	$\frac{10404096}{49}$	0	$\frac{1}{56}$
4	$\overline{\text{Semion}}^{\dagger}$	-11	$\begin{pmatrix} \frac{121}{7} & \frac{827924480}{3} \\ \frac{1}{1632} & \frac{-1829}{7} \end{pmatrix}$	$\frac{-3}{4}$	$\frac{1950}{7}$	$\frac{10404096}{49}$	0	$\frac{1}{56}$
4	$\overline{\text{Semion}}^{\dagger}$	-19	$\begin{pmatrix} \frac{1501}{11} & \frac{62591041536}{11} \\ \frac{1}{32384} & \frac{-1457}{11} \end{pmatrix}$	$\frac{-7}{4}$	$\frac{2958}{11}$	$\frac{21260544}{121}$	0	$\frac{1}{32384}$
5	Fib	$\frac{-26}{5}$	$\begin{pmatrix} \frac{91}{2} & \frac{13051833}{2} \\ \frac{1}{46} & \frac{-83}{2} \end{pmatrix}$	$\frac{-8}{5}$	87	$\frac{567471}{4}$	0	$\frac{1}{46}$
5	Fib	$\frac{-66}{5}$	$\begin{pmatrix} \frac{3966}{13} & \frac{32712244109}{13} \\ \frac{1}{12857} & \frac{-690}{13} \end{pmatrix}$	$\frac{-8}{5}$	$\frac{4656}{13}$	$\frac{33076081}{169}$	0	$\frac{1}{12857}$
5	Fib	$\frac{-106}{5}$	$\begin{pmatrix} \frac{742}{13} & \frac{32712244109}{13} \\ \frac{1}{12857} & \frac{-3914}{13} \end{pmatrix}$	$\frac{-8}{5}$	$\frac{4656}{13}$	$\frac{33076081}{169}$	0	$\frac{1}{12857}$

TABLE VII. *(Continued.)*

No.	$\mathcal{C}$	c	$\chi(c)$	$h_{ext}(c)$	$\alpha(c)$	$\beta(c)$	n_{max}	$\chi_{10}(c)$
6	$\overline{\text{Fib}}$	$\frac{-14}{5}$	$\begin{pmatrix} 26 & 1951158 \\ \frac{1}{17} & -22 \end{pmatrix}$	$\frac{-2}{5}$	48	114 774	0	$\frac{1}{17}$
6	$\overline{\text{Fib}}$	$\frac{-54}{5}$	$\begin{pmatrix} 295 & 745916226 \\ \frac{1}{3774} & -43 \end{pmatrix}$	$\frac{-7}{5}$	338	$\frac{3359983}{17}$	0	$\frac{1}{3774}$
6	$\overline{\text{Fib}}$	$\frac{-94}{5}$	$\begin{pmatrix} 47 & 745916226 \\ \frac{1}{3774} & -291 \end{pmatrix}$	$\frac{-7}{5}$	338	$\frac{3359983}{17}$	0	$\frac{1}{3774}$
7	Yang – Lee	$\frac{18}{5}$	$\begin{pmatrix} 248 & 310124 \\ 1 & 4 \end{pmatrix}$	$\frac{-1}{5}$	244	310 124	0	1
7	Yang – Lee	$\frac{-22}{5}$	$\begin{pmatrix} 0 & 310124 \\ 1 & -244 \end{pmatrix}$	$\frac{-1}{5}$	244	310 124	0	1
7	Yang – Lee	$\frac{-62}{5}$	$\begin{pmatrix} \frac{1054}{11} & \frac{1667924403}{11} \\ \frac{1}{902} & \frac{-1010}{11} \end{pmatrix}$	$\frac{-6}{5}$	$\frac{2064}{11}$	$\frac{40681083}{242}$	0	$\frac{1}{902}$
8	$\overline{\text{Yang – Lee}}$	$\frac{-18}{5}$	$\begin{pmatrix} \frac{802}{3} & \frac{35954954}{3} \\ \frac{1}{57} & \frac{-46}{3} \end{pmatrix}$	$\frac{-4}{5}$	$\frac{848}{3}$	$\frac{1892366}{9}$	0	$\frac{1}{57}$
8	$\overline{\text{Yang – Lee}}$	$\frac{-58}{5}$	$\begin{pmatrix} \frac{58}{3} & \frac{35954954}{3} \\ \frac{1}{57} & \frac{-790}{3} \end{pmatrix}$	$\frac{-4}{5}$	$\frac{848}{3}$	$\frac{1892366}{9}$	0	$\frac{1}{57}$
8	$\overline{\text{Yang – Lee}}$	$\frac{-98}{5}$	$\begin{pmatrix} 140 & 5726299516 \\ \frac{1}{323509} & -136 \end{pmatrix}$	$\frac{-9}{5}$	276	$\frac{3346756}{19}$	0	$\frac{1}{323509}$

Table VII (which was used in the Proof of Theorem 3.12) again has 24 rows, in the same groups of 3. Each group corresponds to a modular tensor category from Table I, and each row within the group selects a representative of an equivalence class of admissible c mod 24. Table VII provides the characteristic matrix (computed as in Ref. 7), the extremal conformal dimension $h_{ext}(c)$ computed from the definition (2.5), and $\beta(c)$ and $\alpha(c)$ computed directly from the characteristic matrix. Using these data, we apply Lemma 3.10 to obtain a value n_{max} such that $|\beta(c-24n)| > 1$ when $n > n_{max}$. In fact, $n_{max} = 0$ in all cases.

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