Evaluation of Harmonic Sums with Integrals

Vivek Kaushik and Daniele Ritelli

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Abstract

We consider the sums $S(k) = \sum_{n=0}^{\infty} \frac{(-1)^{nk}}{(2n+1)^k}$ and $\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ with k being a positive integer. We evaluate these sums with multiple integration, a modern technique. First, we start with three different double integrals that have been previously used in the literature to show $S(2) = \pi^2/8$, which implies Euler's identity $\zeta(2) = \pi^2/6$. Then, we generalize each integral in order to find the considered sums. The k dimensional analogue of the first integral is the density function of the quotient of k independent, nonnegative Cauchy random variables. In seeking this function, we encounter a special logarithmic integral that we can directly relate to $S(k)$. The k dimensional analogue of the second integral, upon a change of variables, is the volume of a convex polytope, which can be expressed as a probability involving certain pairwise sums of k independent uniform random variables. We use combinatorial arguments to find the volume, which in turn gives new closed formulas for $S(k)$ and $\zeta(2k)$. The k dimensional analogue of the last integral, upon another change of variables, is an integral of the joint density function of k Cauchy random variables over a hyperbolic polytope. This integral can be expressed as a probability involving certain pairwise products of these random variables, and it is equal to the probability from the second generalization. Thus, we specifically highlight the similarities in the combinatorial arguments between the second and third generalizations.

1 Introduction

One of the most celebrated problems in Classical Analysis is the *Basel Problem*, which is to evaluate the infinite series

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{1.1}$$

Pietro Mengoli initially posed this problem in 1644, and Euler [8] was the first to give the correct answer $\pi^2/6$. Since Euler's time, however, more solutions to the problem have appeared in the literature. For example, Kalman [11] records ones with techniques from different areas such as Fourier Analysis and complex variables.

In this article, we generalize solutions involving multiple integration to the closely related problem of finding the sums:

$$S(k) = \sum_{n=0}^{\infty} \frac{(-1)^{nk}}{(2n+1)^k}, \quad k \in \mathbb{N}, \tag{1.2}$$

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}, \quad k \in \mathbb{N}, \tag{1.3}$$

with the latter sum being the Riemann Zeta Function evaluated at the positive even integers. Apostol [1] pioneered this approach on $\zeta(2)$. He evaluates

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\ dx\ dy \tag{1.4}$$

in two ways: first, by converting the integrand into a geometric series and exchanging summation and integration to obtain $\zeta(2)$, and then using the linear change of variables

$$x = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}}$$

to obtain $\pi^2/6$ upon evaluating arctangent integrals.

We focus on three different double integrals as our starting point:

$$D_1 = \int_0^\infty \int_0^\infty \frac{y}{(x^2y^2+1)(y^2+1)}\ dx\ dy, \tag{1.5}$$

whose variations appear in [3, 10, 14, 15, 16],

$$D_2 = \int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\ dx\ dy, \tag{1.6}$$

which appears in [2, 7], and

$$D_3 = \int_0^1 \int_0^1 \frac{1}{\sqrt{xy}(1-xy)}\ dx\ dy, \tag{1.7}$$

which appears in [12, p. 9]. Each integral is used to show

$$S(2) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \tag{1.8}$$

from which we can algebraically obtain $\zeta(2) = \pi^2/6$.

First, D_1 evaluates to $\pi^2/4$, but upon reversing the order of integration, it becomes

$$\int_0^\infty \frac{\ln(x)}{x^2-1}\ dx, \tag{1.9}$$

which we show is $2S(2)$. We generalize D_1, considering the density function of the quotient of k Cauchy random variables expressed as a multiple integral. Our approach differs from that of Bourgade, Fujita and Yor [3], who also use these same random variables. We evaluate this multiple integral by repeatedly reversing the order of integration and using partial fractions several times. The result is we obtain a logarithmic integral generalizing (1.9), which we can relate directly back to $S(k)$. We then modify our approach to find special closed forms to the bilateral alternating series

$$S(k,a) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{nk}}{(an+1)^k}, \quad a > 1, \quad \frac{1}{a} \notin \mathbb{N}, \quad k \in \mathbb{N}, \tag{1.10}$$

of which the special case $k = 2$ is examined in [5].

Next, D_2, similar to (1.4), can be evaluated in two ways. First, we convert the integrand into a geometric series and exchange summation and integration to obtain $S(2)$. Then, using Calabi's trigonometric change of variables [2] (the hyperbolic version provided in [6, 13, 17, 18])

$$x = \frac{\sin(u)}{\cos(v)}, \quad y = \frac{\sin(v)}{\cos(u)}, \tag{1.11}$$

whose Jacobian determinant is $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 - x^2y^2$, we see D_2 is the area of a right isosceles triangle with the shorter legs each of length $\pi/2$. Hence, $D_2 = \pi^2/8$. The generalized version of this approach leads us to a convex polytope

$$\Delta^k = \{(u_1, \ldots, u_k) \in \mathbb{R}^k : u_i + u_{i+1} < 1,\ u_i > 0, 1 \le i \le k\}, \tag{1.12}$$

in which we use cyclical indexing mod k : $u_{k+1} := u_1$. The volume of Δ^k, which is equal to $(2/\pi)^k S(k)$, has already been computed in three different ways: Beukers, Calabi, and Kolk [2] dissect Δ^k into pairwise disjoint congruent pyramids, Elkies [7] and Silagadze [18] both perform spectral analysis on its characteristic function, and Stanley [20] uses properties of alternating permutations. We give another approach to the volume computation, viewing it as the probability that k independent uniform random variables on $(0, 1)$ have cyclically pairwise consecutive sums less than 1. We use combinatorial arguments to evaluate the probability. Our approach leads to new and interesting closed formulas of $S(k)$ and $\zeta(2k)$ that do not invoke the traditional Bernoulli and Eulerian numbers.

Finally, upon a substitution $x = \sqrt{u}, y = \sqrt{v}$ to D_2, we see that $D_3 = 4S(2)$. On the other hand, Zagier's and Kontsevich's change of variables [12, p. 9]

$$x = \frac{\xi^2(\eta^2+1)}{\xi^2+1}, \quad y = \frac{\eta^2(\xi^2+1)}{\eta^2+1}, \tag{1.13}$$

which has Jacobian Determinant $\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = \frac{4\sqrt{xy}(1-xy)}{(\xi^2+1)(\eta^2+1)}$, transforms D_3 into

$$\iint_{\substack{\xi\eta<1,\\ \xi,\eta>0}} \frac{4}{(\xi^2+1)(\eta^2+1)}\, d\eta\, d\xi,$$

whose value is $\pi^2/2$. We generalize this approach, which leads us to integrating the joint density function of k independent, nonnegative Cauchy random variables over a hyperbolic polytope:

$$\mathbb{H}^k = \{(\xi_1, \ldots, \xi_k) \in \mathbb{R}^k : \xi_i\xi_{i+1} < 1,\ \xi_i > 0, 1 \le i \le k\}, \tag{1.14}$$

in which we use cyclical indexing mod k : $\xi_{k+1} := \xi_1$. This is the same as the probability that k independent, nonnegative Cauchy random variables have cyclically pairwise consecutive products less than 1. Combinatorial analysis of this probability leads to the exact same closed formulas from the second approach, but we highlight the similarities between this analysis and that of our second approach. Hence, this approach, along with the second, induces two equivalent probabilistic viewpoints of $S(k)$ and $\zeta(2k)$.

Finally, it is worth noting that $\zeta(2k)$ can be obtained from $S(2k)$ by observing

$$\zeta(2k) = \frac{1}{2^{2k}}\zeta(2k) + S(2k),$$

which implies

$$\zeta(2k) = \frac{2^{2k}}{2^{2k}-1}S(2k). \tag{1.15}$$

1.1 Some Concepts From Probability Theory

We briefly recall some notions from probability theory that we use throughout this article.

Let $A_1, \ldots, A_k$ be k events. We denote $\Pr(A_1, \ldots, A_k)$ to be the probability that $A_1, \ldots, A_k$ occur at the same time.

Let $X_1, \ldots, X_k$ be k continuous, nonnegative random variables. Their *joint density function* $f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$ is a nonnegative function such that

$$\int_0^\infty \cdots \int_0^\infty f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)\ dx_1 \ldots dx_k = 1, \tag{1.16}$$

and for all a_1, b_1, $\ldots$, a_k, $b_k \geq 0$, we have

$$\Pr\left(a_1 \leq X_1 \leq b_1, \ldots, a_k \leq X_k \leq b_k\right) = \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)\ dx_1 \ldots dx_k. \tag{1.17}$$

The *joint cumulative distribution function* of $X_1, \ldots, X_k$ is the function

$$F_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \Pr(X_1 \leq x_1, \ldots, X_k \leq x_k). \tag{1.18}$$

In the case there is one random variable X_1, we simply refer to $f_{X_1}(x_1)$ as its *density function* and $F_{X_1}(x_1)$ as its *cumulative distribution function*. It follows from the Fundamental Theorem of Calculus that

$$\frac{\partial^k}{\partial x_1 \ldots \partial x_k} F_{X_1,\ldots,X_k} = f_{X_1,\ldots,X_k}, \tag{1.19}$$

and in the case of only one variable X_1, we have $F'_{X_1} = f_{X_1}$.

If $X_1, \ldots, X_k$ are *independent*, then we have

$$F_{X_1,\ldots,X_k}(x_1, \ldots x_k) = F_{X_1}(x_1) \ldots F_{X_k}(x_k), \tag{1.20}$$

and from (1.19), we have

$$f_{X_1,\ldots,X_k}(x_1, \ldots x_k) = f_{X_1}(x_1) \ldots f_{X_k}(x_k). \tag{1.21}$$

Let X and Y be independent, nonnegative random variables with density functions $f_X(x)$ and $f_Y(y)$, respectively. Their *product* $Z = XY$ is a new random variable with density function

$$f_Z(z) = \int_0^\infty \frac{1}{x} f_Y\left(\frac{z}{x}\right) f_X(x)\ dx. \tag{1.22}$$

If $X \neq 0$, their *quotient* $T = Y/X$ has density function

$$f_T(t) = \int_0^\infty x f_Y(tx) f_X(x)\ dx. \tag{1.23}$$

Random variable operations are studied in Springer's book [19].

Finally, the nonnegative random variable X is said to be *Cauchy* if

$$f_X(x) = \frac{2}{\pi} \frac{1}{x^2 + 1}, \tag{1.24}$$

and X is said to be *uniform* on (a, b) with $a < b$ if

$$f_X(x) = \frac{1}{b - a}. \tag{1.25}$$

2 Cauchy Random Variables

We first give a slightly modified version of the solution to the Basel Problem given by Pace [15] (see also [3]), which we will then generalize. Using the results of our generalization, we highlight two specific applications, in which we show $S(3) = \pi^3/32$ and $S(4) = \pi^4/96$. The latter result gives a new proof of $\zeta(4) = \pi^4/90$ with multiple integration. Finally, we extend our generalization to find $S(k, a)$ and provide closed formulas for the cases $k = 2$ and 3.

2.1 Luigi Pace's Solution to the Basel Problem

Let X_1 and X_2 be independent, nonnegative Cauchy random variables. Define

$$Z = X_1/X_2.$$

We compute the density function $f_Z(z)$. By (1.23) and (1.24), we have

$$f_Z(z) = \frac{4}{\pi^2} \int_0^\infty \frac{x_2}{(z^2x_2^2 + 1)(x_2^2 + 1)}\, dx_2. \tag{2.1}$$

To evaluate (2.1), we use the partial fractions identity

$$\frac{x}{(x^2y^2 + 1)(x^2 + 1)} = \frac{xy^2}{(x^2y^2 + 1)(y^2 - 1)} - \frac{x}{(x^2 + 1)(y^2 - 1)}. \tag{2.2}$$

Hence,

$$f_Z(z) = \frac{4}{\pi^2} \frac{\ln(z)}{z^2 - 1}.$$

Using (1.16) and then rearranging terms, we have

$$\int_0^\infty \frac{\ln(z)}{z^2 - 1}\, dz = \frac{\pi^2}{4}. \tag{2.3}$$

Following the argument of [16], we write

$$\begin{aligned} \int_0^\infty \frac{\ln(z)}{z^2 - 1} &= \int_0^1 \frac{\ln(z)}{z^2 - 1}\, dz + \int_1^\infty \frac{\ln(z)}{z^2 - 1}\, dz && (2.4) \\ &= 2\int_0^1 \frac{\ln(z)}{z^2 - 1}\, dz, && (2.5) \end{aligned}$$

in which (2.5) follows from a substitution $z = 1/u$ to the second term on the right hand side of (2.4). We expand the integrand of (2.5) into a geometric series

$$\frac{\ln(z)}{z^2 - 1} = -\sum_{n=0}^\infty z^{2n} \ln(z). \tag{2.6}$$

Putting the right hand side of (2.6) in place of the integrand in (2.5), we have (2.5) is equal to

$$\begin{aligned} 2\int_0^1 -\sum_{n=0}^\infty z^{2n} \ln(z)\, dz &= -2\sum_{n=0}^\infty \int_0^1 z^{2n} \ln(z)\, dz && (2.7) \\ &= 2\sum_{n=0}^\infty \frac{1}{(2n + 1)^2}, && (2.8) \end{aligned}$$

by which (2.7) follows by the Monotone Convergence Theorem (see [4, p. 95-96]), and (2.8) follows upon integration by parts. Hence, we have

$$S(2) = \frac{\pi^2}{8}.$$

Finally, using (1.15), we obtain

$$\zeta(2) = \frac{4}{3}S(2) = \frac{\pi^2}{6}.$$

Thus, the Basel Problem is solved.

2.2 Generalization for $S(k)$

We now give the generalization of Pace's solution, as well as all the solutions presented in [10, 14, 15, 16].

Let $X_1, \ldots, X_k$ be k independent, nonnegative Cauchy random variables with $k \geq 2$. Define the quotient

$$Z_k = X_1/\ldots/X_k.$$

We seek the density function $f_{Z_k}(z)$.

Theorem 2.2.1. *The density function $f_{Z_k}(z)$ is*

$$f_{Z_k}(z) = \frac{2^k}{\pi^k}\int_0^\infty \cdots \int_0^\infty \frac{x_2\ldots x_k}{(z^2x_2^2\ldots x_k^2+1)(x_2^2+1)\ldots(x_k^2+1)}\,dx_2\ldots dx_k. \tag{2.9}$$

Proof. The case $k = 2$ was already computed in (2.1).

Let the statement hold for $k = m$ with $m > 2$. We show it must also hold for $k = m + 1$. Applying (1.23) to $Z_{m+1} = Z_m/X_{m+1}$, we have

$$\begin{aligned} f_{Z_{m+1}}(z) &= \frac{2}{\pi}\int_0^\infty x_{m+1} f_{Z_m}(zx_{m+1}) f_{X_{m+1}}(x_{m+1})\,dx_{m+1} \\ &= \frac{2^{m+1}}{\pi^{m+1}}\int_0^\infty \cdots \int_0^\infty \frac{x_2\ldots x_m x_{m+1}}{(z^2x_2^2\ldots x_{m+1}^2+1)(x_2^2+1)\ldots(x_{m+1}^2+1)}\,dx_2\ldots dx_{m+1}, \quad (2.10) \end{aligned}$$

in which (2.10) is the result of the inductive hypothesis on $f_{Z_m}(z)$.

Note that the integrand of (2.10) is nonnegative. Thus, Tonelli's Theorem allows us to reverse the order of integration on the integral

$$\int_0^\infty f_{Z_{m+1}}(z)\,dz. \tag{2.11}$$

Integrating (2.11) with respect to z first, and then with respect to each of the other m variables, we see (2.11) is equal to 1, satisfying (1.16). ◻

Remark. *Theorem 2.2.1 still holds if Z_k is alternatively formulated by the quotient of X_1 and the product $X_2\ldots X_k$. In this case, we would need to use* (1.22) *and* (1.23).

The crux of our generalization is evaluating (2.9), which involves reversing the order of integration several times and using a generalized partial fractions identity

$$\frac{x}{(x^2y^2-(-1)^s)(x^2+1)} = \frac{xy^2}{(x^2y^2-(-1)^s)(y^2-(-1)^s)} - \frac{x}{(x^2+1)(y^2-(-1)^s)}, \quad s \in \mathbb{N}. \tag{2.12}$$

Along the way, we encounter integrals of the form

$$\int_0^\infty \frac{\ln^k(z)}{z^2-(-1)^k}\ dz, \tag{2.13}$$

which vanish, according to Gradshteyn and Rhyzik table entries [9, Section 4.271: 7 and 9]. The end result is that we arrive at a logarithmic integral of the form

$$J_k = \int_0^\infty \frac{\ln^{k-1}(z)}{z^2-(-1)^k}\ dz. \tag{2.14}$$

By splitting the region of integration in the same way as in (2.4) and making a substitution $z = 1/u$ to the resulting second term, we find that

$$J_k = 2\int_0^1 \frac{\ln^{k-1}(z)}{z^2-(-1)^k}\ dz. \tag{2.15}$$

From entries [9, Section 4.271: 6 and 10], we have

$$J_k = \begin{cases} \dfrac{2^{2k}-2^k}{k}\left(\dfrac{\pi}{2}\right)^k |B_k| & k \text{ even} \\ \left(\dfrac{\pi}{2}\right)^k |E_{k-1}| & k \text{ odd} \end{cases}. \tag{2.16}$$

Here, B_m and E_m denote the Bernoulli number and Eulerian number of order m, respectively. These numbers satisfy

$$\frac{x}{e^x-1} = \sum_{m=0}^\infty \frac{B_m}{m!}x^m, \qquad \frac{1}{\cosh(x)} = \sum_{m=0}^\infty \frac{E_m}{m!}x^m.$$

It is worth noting B_m and E_m can be alternatively defined through the generating functions of $\tan(x)$ and $\sec(x)$, respectively (see [20]). Finally, we convert the integrand in (2.15) into a geometric series

$$\frac{\ln^{k-1}(z)}{z^2-(-1)^k} = -\sum_{n=0}^\infty \ln^{k-1}(z)(-1)^{nk}z^{2n}. \tag{2.17}$$

We put the series in (2.17) in place of the original integrand in (2.15), then exchange summation and integration as permitted by the Monotone Convergence Theorem, and finally use repeated integration by parts to obtain the result

$$S(k) = \frac{J_k}{2(k-1)!}. \tag{2.18}$$

2.3 Evaluating $S(3)$ and $S(4)$

We use our density function results to evaluate $S(3)$ and $S(4)$, with the latter giving $\zeta(4) = \pi^4/90$.

Theorem 2.3.1. *The value of* $S(3)$ *is*

$$S(3) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

Proof. Let X_1, X_2, and X_3 be independent, nonnegative Cauchy random variables, and let Z_3 be their quotient. Then, by Theorem 2.2.1, we have the density function

$$f_{Z_3}(z) = \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \frac{x_2 x_3}{(z^2 x_2^2 x_3^2 + 1)(x_2^2 + 1)(x_3^2 + 1)}\ dx_2\ dx_3.$$

Using the partial fractions identity in (2.12), we have

$$\begin{aligned} f_{Z_3}(z) &= \frac{8}{\pi^3} \int_0^\infty \frac{\ln(z x_3)}{(z^2 x_3^2 - 1)(x_3^2 + 1)}\ dx_3 \\ &= \frac{8}{\pi^3} \int_0^\infty \frac{x_3 \ln(z)}{(z^2 x_3^2 - 1)(x_3^2 + 1)} + \frac{8}{\pi^3} \int_0^\infty \frac{x_3 \ln(x_3)}{(z^2 x_3^2 - 1)(x_3^2 + 1)} dx_3. \end{aligned} \tag{2.19}$$

Using (2.12) again on the first term of (2.19), we see

$$f_{Z_3}(z) = \frac{8}{\pi^3} \frac{\ln^2(z)}{z^2 + 1} + \frac{8}{\pi^3} \int_0^\infty \frac{x_3 \ln(x_3)}{(z^2 x_3^2 - 1)(x_3^2 + 1)}\ dx_3. \tag{2.20}$$

Now, by (1.16), we have

$$1 = \frac{8}{\pi^3} J_3 + \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \frac{x_3 \ln(x_3)}{(z^2 x_3^2 - 1)(x_3^2 + 1)}\ dx_3\ dz. \tag{2.21}$$

Reversing the order of integration on the second term in (2.21), we encounter an integral of the form examined in (2.13), so this term vanishes. Thus, rearranging the terms in (2.21) gives

$$J_3 = \frac{\pi^3}{8}.$$

Applying (2.18), we find

$$S(3) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} = \frac{1}{2(2!)} \frac{\pi^3}{8} = \frac{\pi^3}{32}.$$

□

Remark. *Our result for J_3 agrees with* (2.16)*, since* $|E_2| = 1$.

We now find $S(4)$ and $\zeta(4)$.

Theorem 2.3.2. *The values of $S(4)$ and $\zeta(4)$ are*

$$\begin{aligned} S(4) &= \sum_{n=0}^\infty \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \\ \zeta(4) &= \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}. \end{aligned}$$

Proof. Let $X_1, \ldots, X_4$ be independent, nonnegative Cauchy random variables. Defining Z_4 as their quotient, Theorem 2.2.1 implies

$$f_{Z_4}(z) = \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{z^2 x_2^2 x_3^2 x_4^2 + 1} \prod_{i=2}^4 \frac{x_i}{x_i^2 + 1}\ dx_2\ dx_3\ dx_4.$$

Applying the partial fractions identity from (2.12), we have

$$f_{Z_4}(z) = \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{x_3 x_4 \ln(z x_3 x_4)}{(z^2 x_3^2 x_4^2 - 1)(x_3^2+1)(x_4^2+1)} \, dx_3 \, dx_4,$$

which splits into three terms:

$$H_1 = \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{x_3 x_4 \ln(z)}{(z^2 x_3^2 x_4^2 - 1)(x_3^2+1)(x_4^2+1)} \, dx_3 \, dx_4, \tag{2.22}$$

$$H_2 = \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{x_3 x_4 \ln(x_3)}{(z^2 x_3^2 x_4^2 - 1)(x_3^2+1)(x_4^2+1)} \, dx_3 \, dx_4, \tag{2.23}$$

$$H_3 = \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{x_3 x_4 \ln(x_4)}{(z^2 x_3^2 x_4^2 - 1)(x_3^2+1)(x_4^2+1)} \, dx_3 \, dx_4. \tag{2.24}$$

We first evaluate H_1. Using (2.12), we see

$$H_1 = \frac{16}{\pi^4} \int_0^\infty \frac{x_4 \ln^2(z)}{(z^2 x_4^2+1)(x_4^2+1)} + \frac{16}{\pi^4} \int_0^\infty \frac{x_4 \ln(z)\ln(x_4)}{(z^2 x_4^2+1)(x_4^2+1)} \, dx_4.$$

Using (2.12) on the first term on the right hand side, we have

$$H_1 = \frac{16}{\pi^4} \frac{\ln^3(z)}{z^2-1} + \frac{16}{\pi^4} \int_0^\infty \frac{x_4 \ln(z)\ln(x_4)}{(z^2 x_4^2+1)(x_4^2+1)} \, dx_4.$$

Next, for H_2, we reverse the order of integration and use (2.12) to get

$$H_2 = \frac{16}{\pi^4} \int_0^\infty \frac{x_3 \ln(x_3)}{(z^2 x_3^2+1)(x_3^2+1)} \, dx_3.$$

Similarly, for H_3, we get

$$H_3 = \frac{16}{\pi^4} \int_0^\infty \frac{x_4 \ln(x_4)}{(z^2 x_4^2+1)(x_4^2+1)} \, dx_4.$$

Using (1.16), we have

$$\begin{aligned}
1 = \frac{16}{\pi^4} J_4 + \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{x_4 \ln(z)\ln(x_4)}{(z^2 x_4^2+1)(x_4^2+1)} \, dx_4 \, dz \\
+ \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{x_3 \ln(x_3)}{(z^2 x_3^2+1)(x_3^2+1)} \, dx_3 \, dz \\
+ \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{x_4 \ln(x_4)}{(z^2 x_4^2+1)(x_4^2+1)} \, dx_4 \, dz.
\end{aligned} \tag{2.25}$$

The third term and fourth terms on the right hand side of (2.25) vanish, which is seen upon reversing the order of integration in each integral and observing each inner integral is of the form in (2.13). To evaluate the second term, we make the substitution $z = t/x_4$, which transforms it into

$$\frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{\ln(t)\ln(x_4)}{(t^2+1)(x_4^2+1)} \, dt \, dx_4 - \frac{16}{\pi^4} \int_0^\infty \int_0^\infty \frac{\ln^2(x_4)}{(t^2+1)(x_4^2+1)} \, dt \, dx_4. \tag{2.26}$$

The first term of (2.26) is of the same form seen in (2.13), but we recognize the second term is $-8J_3/\pi^3 = -1$. Thus, we see

$$1 = \frac{16}{\pi^4} J_4 - 1,$$

so rearranging terms gives

$$J_4 = \frac{\pi^4}{8}.$$

By (2.18), we have

$$S(4) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{1}{2(3!)} \frac{\pi^4}{8} = \frac{\pi^4}{96}.$$

Finally, from (1.15), we have

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{16}{15} \left(\frac{\pi^4}{96} \right) = \frac{\pi^4}{90}.$$

□

2.4 Further Generalization to $S(k, a)$

We recall from the alternating series from (1.10), which is

$$S(k, a) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{nk}}{(an+1)^k}, \quad a > 1, \quad \frac{1}{a} \notin \mathbb{N}, \quad k \in \mathbb{N}.$$

We consider k independent, nonnegative random variables $X_1, \ldots, X_k$, each with density function

$$f_{X_i}(x_i) = \begin{cases} \dfrac{a}{\pi} \sin\left(\dfrac{\pi}{a}\right) \dfrac{1}{x_i^a + 1} & i = 1 \\ \\ \dfrac{2}{\pi} \sin\left(\dfrac{\pi}{a}\right) \dfrac{x_i^{1-\frac{2}{a}}}{x_i^2 + 1} & 2 \leq i \leq k \end{cases}, \tag{2.27}$$

where a has the same conditions that we imposed on $S(k, a)$.

Theorem 2.4.1. *All of $f_{X_1}(x_1), \ldots, f_{X_k}(x_k)$ are valid density functions.*

Proof. The crux of the proof is the application of Euler's Reflection Formula:

$$\Gamma(t)\Gamma(1-t) = \pi \csc(\pi t), \quad t > 0, \quad t \notin \mathbb{N}, \tag{2.28}$$

where

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} \, dx.$$

For $i = 1$, we see

$$\begin{aligned}
\int_0^\infty f_{X_1}(x_1)\ dx_1 &= \frac{a}{\pi}\sin\left(\frac{\pi}{a}\right)\int_0^\infty \frac{1}{x_1^a+1}\ dx_1 \\
&= \frac{a}{\pi}\sin\left(\frac{\pi}{a}\right)\int_0^\infty\int_0^\infty e^{-y(1+x_1^a)}\ dy\ dx_1 && (2.29)\\
&= \frac{a}{\pi}\sin\left(\frac{\pi}{a}\right)\int_0^\infty t^{\frac{1}{a}-1}e^{-z}\ dt\ \int_0^\infty \frac{y^{\frac{1}{a}}e^{-y}}{a}\ dy && (2.30)\\
&= \frac{a}{\pi}\sin\left(\frac{\pi}{a}\right)\frac{\Gamma\left(\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)}{a}\\
&= 1,
\end{aligned}$$

by which we made the substitution $x_1 = (t/y)^{1/a}$ on (2.29) to obtain (2.30).

For $i > 1$, we have

$$\begin{aligned}
\int_0^\infty f_{X_i}(x_i)\ dx_i &= \frac{2}{\pi}\sin\left(\frac{\pi}{a}\right)\int_0^\infty \frac{x_i^{1-\frac{2}{a}}}{x_i^2+1}\ dx_i \\
&= \frac{2}{\pi}\sin\left(\frac{\pi}{a}\right)\int_0^\infty\int_0^\infty x_i^{1-\frac{2}{a}}e^{-y(1+x_i^2)}\ dy\ dx_i && (2.31)\\
&= \frac{2}{\pi}\sin\left(\frac{\pi}{a}\right)\frac{\Gamma(\frac{1}{a})\Gamma(1-\frac{1}{a})}{2} && (2.32)\\
&= 1,
\end{aligned}$$

in which (2.32) follows from the substitution $x_i = \sqrt{t/y}$ on (2.31). $\square$

Remark. *When $a = 2$, we see $X_1, \ldots, X_k$ are Cauchy.*

Define the random variable for $k \geq 2$,

$$Z_{k,a} = X_1/(X_2 \ldots X_k)^{\frac{2}{a}}.$$

We seek the density function $f_{Z_{k,a}}(z)$.

Theorem 2.4.2. *The density function $f_{Z_{k,a}}(z)$ is*

$$f_{Z_{k,a}}(z) = \frac{a\left(2\sin\left(\frac{\pi}{a}\right)\right)^k}{2\pi^k}\int_0^\infty \ldots \int_0^\infty \frac{x_2\ldots x_k}{(z^a x_2^2\ldots x_k^2+1)(x_2^2+1)\ldots(x_k^2+1)}\ dx_2\ldots dx_k. \quad (2.33)$$

Proof. We start with the cumulative distribution function

$$F_{Z_{k,a}}(z) = \Pr(Z_{k,a} \leq z). \quad (2.34)$$

Expanding the right hand side of (2.34), we find

$$\begin{aligned}
\Pr(Z_{k,a} \leq z) &= \Pr\left(X_1 \leq z(X_2\ldots X_k)^{\frac{2}{a}}\right)\\
&= F_{X_1}\left(z(X_2\ldots X_k)^{\frac{2}{a}}\right)\\
&= \frac{a\left(2\sin\left(\frac{\pi}{a}\right)\right)^k}{2\pi^k}\int_0^\infty\cdots\int_0^\infty\int_0^{z(x_2\ldots x_k)^{\frac{2}{a}}} \frac{(x_2\ldots x_k)^{1-\frac{2}{a}}}{(x_1^a+1)(x_2^2+1)\ldots(x_k^2+1)}\ dx_1\ dx_2\ldots\ dx_k.
\end{aligned}$$

Thus, by (1.19), we have

$$f_{Z_{k,a}}(z) = \frac{a\left(2\sin\left(\frac{\pi}{a}\right)\right)^k}{2\pi^k}\int_0^\infty \dots \int_0^\infty \frac{x_2\dots x_k}{(z^a x_2^2\dots x_k^2+1)(x_2^2+1)\dots(x_k^2+1)}\ dx_2\dots dx_k.$$

Now, we consider

$$\int_0^\infty f_{Z_{k,a}}(z)\ dz, \tag{2.35}$$

whose integrand is nonnegative. Hence, Tonelli's Theorem allows us to reverse the order of integration in (2.35). We integrate with respect to z first and then with respect to the other $k-1$ variables. Making the substitution $z = t(x_2\dots x_k)^{-2/a}$, and then applying Theorem 2.4.1 several times, we see (2.35) is equal to 1, satisfying (1.16). ☐

Remark. *When $a = 2$, we recover $f_{Z_k}(z)$ from* (2.9).

In simplifying the density function when $k > 2$, we encounter integrals of the form

$$\int_0^\infty \frac{t^{m-1}}{t^n-1}\ dt = -\frac{\pi}{n}\cot\left(\frac{m\pi}{n}\right), \quad m < n, \tag{2.36}$$

as listed in Gradshteyn and Rhyzik entry [9, Section 3.241: 4]. At the end, we arrive at the logarithmic integral

$$J_{k,a} = \int_0^\infty \frac{\ln^{k-1}(z)}{z^a-(-1)^k}\ dz, \tag{2.37}$$

which generalizes J_k from (2.15). We now highlight the relation between $J_{k,a}$ and $S(k,a)$.

Theorem 2.4.3. *The following relation holds.*

$$S(k,a) = \frac{J_{k,a}}{(k-1)!}. \tag{2.38}$$

Proof. We write

$$\begin{aligned}
J_{k,a} &= \int_0^1 \frac{\ln^{k-1}(z)}{z^a-(-1)^k}\ dz + \int_1^\infty \frac{\ln^{k-1}(z)}{z^a-(-1)^k}\ dz & (2.39)\\
&= \int_0^1 \frac{\ln^{k-1}(z)}{z^a-(-1)^k}\ dz + \int_0^1 \frac{\ln^{k-1}(u)}{u^{-a}-(-1)^k}\frac{1}{u^2}\ du & (2.40)\\
&= (k-1)!\sum_{n=0}^\infty \frac{(-1)^{nk}}{(an+1)^k} + (k-1)!\sum_{n=1}^\infty \frac{(-1)^{nk}}{(1-an)^k} & (2.41)\\
&= (k-1)!\sum_{n=0}^\infty \frac{(-1)^{nk}}{(an+1)^k} + (k-1)!\sum_{n=-\infty}^{-1} \frac{(-1)^{nk}}{(an+1)^k}\\
&= (k-1)!S(k,a),
\end{aligned}$$

in which (2.40) is the result of a substitution $z = 1/u$ performed on the second term in (2.39), and (2.41) is the result of converting each integrand in (2.40) into a geometric series, exchanging summation and integration, and finally integrating by parts. ☐

2.5 Evaluating $S(2, a)$ and $S(3, a)$

With this machinery, we seek the closed forms of $S(2, a)$ and $S(3, a)$, which have manageable calculations. The former sum is a standard complex variables result. Specific examples of $S(2, a)$ for different a are provided in [5].

Theorem 2.5.1. *$S(2, a)$ is given by the formula*

$$S(2, a) = \sum_{n=-\infty}^{\infty} \frac{1}{(an+1)^2} = \frac{\pi^2}{a^2} \csc^2\left(\frac{\pi}{a}\right). \tag{2.42}$$

Proof. Let X_1 and X_2 be independent, nonnegative random variables defined in (2.27), and let $Z_{2,a} = X_1/(X_2)^{2/a}$. Then,

$$f_{Z_{2,a}}(z) = \frac{2a\sin^2(\frac{\pi}{a})}{\pi^2} \int_0^\infty \frac{x_2}{(z^a x_2^2 + 1)(x_2^2+1)}\, dx_2,$$

from Theorem 2.4.2.

Observing $z^a = \left(z^{a/2}\right)^2$, we can use (2.12) to see that

$$f_{Z_{2,a}}(z) = \frac{2a\sin^2\left(\frac{\pi}{a}\right)}{\pi^2} \frac{\ln(z^{\frac{a}{2}})}{z^a - 1} = \frac{a^2\sin^2\left(\frac{\pi}{a}\right)}{\pi^2} \frac{\ln(z)}{z^a-1}.$$

Thus, using (1.16) and then rearranging terms, we see

$$J_{2,a} = \frac{\pi^2}{a^2}\csc^2\left(\frac{\pi}{a}\right).$$

Finally, we use Theorem 2.4.3 to see

$$S(2, a) = \sum_{n=-\infty}^{\infty} \frac{1}{(an+1)^2} = \frac{1}{(2-1)!} J_{2,a} = \frac{\pi^2}{a^2}\csc^2\left(\frac{\pi}{a}\right).$$

□

Remark. *The case $a = 2$ recovers Pace's Basel Problem solution.*

Now we find $S(3, a)$.

Theorem 2.5.2. *$S(3, a)$ is given by the formula*

$$S(3, a) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(an+1)^3} = \frac{\pi^3}{2a^3}\csc^3\left(\frac{\pi}{a}\right) + \frac{\pi^3}{2a^3}\cot^2\left(\frac{\pi}{a}\right)\csc\left(\frac{\pi}{a}\right). \tag{2.43}$$

Proof. Let X_1, X_2, and X_3 be independent, nonnegative random variables defined in (2.27), and let $Z_{3,a} = X_1/(X_2X_3)^{2/a}$. Then, Theorem 2.4.2 implies

$$f_{Z_{3,a}}(z) = \frac{4a\sin^3(\frac{\pi}{a})}{\pi^3} \int_0^\infty \int_0^\infty \frac{x_2x_3}{(z^a x_2^2 x_3^2 + 1)(x_2^2+1)(x_3^2+1)}\, dx_2\, dx_3.$$

We use (2.12) twice to see

$$
\begin{aligned}
f_{Z_{3,a}}(z) &= \frac{2a\sin^3(\frac{\pi}{a})}{\pi^3}\int_0^\infty \frac{x_3\ln(z^a x_3^2)}{(x_3^2+1)(z^a x_3^2-1)}\ dx_3 \\
&= \frac{2a^2\sin^3(\frac{\pi}{a})}{\pi^3}\int_0^\infty \frac{x_3\ln(z)}{(x_3^2+1)(z^a x_3^2-1)}\ dx_3 + \frac{4a\sin^3(\frac{\pi}{a})}{\pi^3}\int_0^\infty \frac{x_3\ln(x_3)}{(x_3^2+1)(z^a x_3^2-1)}\ dx_3 \\
&= \frac{a^3\sin^3(\frac{\pi}{a})}{\pi^3}\frac{\ln^2(z)}{z^a+1} + \frac{4a\sin^3(\frac{\pi}{a})}{\pi^3}\int_0^\infty \frac{x_3\ln(x_3)}{(x_3^2+1)(z^a x_3^2-1)}\ dx_3
\end{aligned}
$$

Hence, using (1.16) we find that

$$
1 = \frac{a^3\sin^3(\frac{\pi}{a})}{\pi^3}J_{3,a} + \frac{4a\sin^3(\frac{\pi}{a})}{\pi^3}\int_0^\infty\int_0^\infty \frac{x_3\ln(x_3)}{(x_3^2+1)(z^a x_3^2-1)}\ dx_3\ dz. \tag{2.44}
$$

We focus on evaluating the second term of (2.44). Reversing the order of integration, and making the substitution $z = vx_3^{-2/a}$, we see it is of the form in (2.36). Thus, the second term is

$$
-\frac{4\sin^3(\frac{\pi}{a})\cot\left(\frac{\pi}{a}\right)}{\pi^2}\int_0^\infty \frac{x_3^{1-\frac{2}{a}}\ln(x_3)}{x_3^2+1}\ dx_3. \tag{2.45}
$$

Making the substitution, $x_3 = u^{\frac{a}{2a-2}}$, we have (2.45) is equal to

$$
\begin{aligned}
-\frac{a^2\sin^3(\frac{\pi}{a})\cot\left(\frac{\pi}{a}\right)}{(a-1)^2\pi^2}\int_0^\infty \frac{\ln(u)}{u^{\frac{a}{a-1}}+1}\ du &= -\frac{a^2\sin^3(\frac{\pi}{a})\cot\left(\frac{\pi}{a}\right)}{(a-1)^2\pi^2}\int_0^\infty \left(\frac{\ln(u)}{u^{\frac{a}{a-1}}+1} - \frac{2\ln(u)}{u^{\frac{2a}{a-1}}+1}\right)\ du \\
&= -\frac{a^2\sin^3(\frac{\pi}{a})\cot\left(\frac{\pi}{a}\right)}{(a-1)^2\pi^2}\left(J_{2,\frac{a}{a-1}} - 2J_{2,\frac{a}{a-1}}\right) \\
&= -\cos^2\left(\frac{\pi}{a}\right).
\end{aligned}
$$

Now, we rearrange terms in (2.44) and use the fact $\cot(\theta) = \cos(\theta)/\sin(\theta)$ to see

$$
J_{3,a} = \frac{\pi^3}{a^3}\csc^3\left(\frac{\pi}{a}\right) + \frac{\pi^3}{a^3}\cot^2\left(\frac{\pi}{a}\right)\csc\left(\frac{\pi}{a}\right).
$$

Thus, by Theorem 2.4.3, we have

$$
S(3,a) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(an+1)^3} = \frac{J_{3,a}}{(3-1)!} = \frac{\pi^3}{2a^3}\csc^3\left(\frac{\pi}{a}\right) + \frac{\pi^3}{2a^3}\cot^2\left(\frac{\pi}{a}\right)\csc\left(\frac{\pi}{a}\right).
$$

□

3 Uniform Random Variables

We recall the Basel Problem solution by Beukers, Calabi and Kolk [2] (see also [7, 11, 17]), whose generalization leads in to our analysis of the probability involving cyclically pairwise consecutive sums of k independent uniform random variables on $(0, 1)$.

3.1 Beukers's, Calabi's, and Kolk's Basel Problem Solution

We start with the double integral D_2 encountered in the introduction, which we modify as

$$D_2 = \int_0^1 \int_0^1 \frac{1}{1 - x_1^2 x_2^2}\, dx_1\, dx_2.$$

Rewriting the integrand as a geometric series,

$$\frac{1}{1 - x_1^2 x_2^2} = \sum_{n=0}^{\infty} (x_1 x_2)^{2n}, \tag{3.1}$$

and exchanging summation and integration, we see $D_2 = S(2)$.

Now, we modify Calabi's change of variables from (1.11),

$$x_1 = \frac{\sin\left(\frac{\pi}{2} u_1\right)}{\cos\left(\frac{\pi}{2} u_2\right)}, \quad x_2 = \frac{\sin\left(\frac{\pi}{2} u_2\right)}{\cos\left(\frac{\pi}{2} u_1\right)},$$

with Jacobian Determinant

$$\left|\frac{\partial(x_1, x_2)}{\partial(u_1, u_2)}\right| = \left(\frac{\pi}{2}\right)^2 \left(1 - x_1^2 x_2^2\right),$$

that transforms the integrand in D_2 into $(\pi/2)^2$. The resulting region of integration is the open isosceles right triangle with the shorter legs each having length 1. The area of this triangle is 1/2, so we recover $S(2) = \pi^2/8$ and $\zeta(2) = \pi^2/6$.

3.2 From Integral to Polytope

We now generalize the solution by Beukers, Calabi, and Kolk. We consider the multiple integral

$$\int_0^1 \cdots \int_0^1 \frac{1}{1 - (-1)^k \prod_{i=1}^k x_i^2}\, dx_1 \dots dx_k. \tag{3.2}$$

As before, rewriting the integrand of (3.2) as a geometric series

$$\frac{1}{1 - (-1)^k \prod_{i=1}^k x_i^2} = \sum_{n=0}^{\infty} (-1)^{nk} (x_1 \dots x_k)^{2n}, \tag{3.3}$$

exchanging summation and integration, and finally integrating term by term k successive times yields (3.2) is equal to $S(k)$.

Then, we use the change of variables (modifying those in [2, 7, 17])

$$x_i = \frac{\sin\left(\frac{\pi}{2} u_i\right)}{\cos\left(\frac{\pi}{2} u_{i+1}\right)}, \quad 1 \le i \le k, \tag{3.4}$$

where the u_i are indexed cyclically $\bmod\, k$, i.e. $u_{k+1} := u_1$. The Jacobian Determinant of (3.4) is

$$\left|\frac{\partial(x_1, \dots, x_k)}{\partial(u_1, \dots, u_k)}\right| = \frac{\pi^k}{2^k} \left(1 - (-1)^k \prod_{i=1}^k x_i^2\right), \tag{3.5}$$

and (3.4) diffeomorphically maps the open cube $(0,1)^k$ into the open polytope

$$\Delta^k = \{(u_1, \dots, u_k) \in \mathbb{R}^k : u_i + u_{i+1} < 1,\ u_i > 0, 1 \le i \le k\}.$$

The proof of (3.5) is done both in [2] and [7]. As a result,

$$S(k) = \frac{\pi^k}{2^k}\mathrm{Vol}(\Delta^k). \tag{3.6}$$

Let $U_1, \dots, U_k$ be k independent uniform random variables on $(0,1)$. Then, we see

$$\mathrm{Vol}(\Delta^k) = \Pr\left(U_1 + U_2 < 1, \dots, U_{k-1} + U_k < 1, U_k + U_1 < 1\right). \tag{3.7}$$

In the next section, we prove the formula

$$\mathrm{Vol}(\Delta^k) = \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k \sum_{n=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \sum_{\substack{(r_1,\dots,r_n)\in[k]^n: \\ |r_p - r_q| \notin \{0,1,k-1\} \\ p,q\in[n]}} \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \alpha_j},$$

in which $[k] := \{1, \dots k\}$, $\delta(a,b)$ is the Kronecker Delta Function

$$\delta(a,b) = \begin{cases} 1 & a = b \\ 0 & \text{else} \end{cases},$$

and α_j is defined

$$\alpha_j = 2 - \delta(k,2) - \sum_{m=1}^{j-1} \delta(|r_m - r_j|, 2) + \delta(|r_m - r_j|, k-2).$$

3.3 Polytope and Combinatorics

For the sake of brevity, for $n \in \mathbb{N}$, we define

$$[n] := \{1, \dots, n\}.$$

We consider the probability

$$\Pr\left(U_1 + U_2 < 1, \dots, U_{k-1} + U_k < 1, U_k + U_1 < 1\right), \tag{3.8}$$

where U_i is uniform on $(0,1)$ for each $i \in [k]$. Note that $f_{U_i}(u_i) = 1$ for each $i \in [k]$ by (1.25). If $U_{r_1}, \dots, U_{r_n}$ are n distinct random variables, we facilitate indexing by using cyclical convention: $U_{r_n+1} := U_{r_1}$. We compute (3.8) by combining the probabilities of two cases: when all $U_i < 1/2$ for each $i \in [k]$, and when $U_j \geq 1/2$ for some $j \in [k]$. The following theorem discusses the first case.

Theorem 3.3.1. *If $U_i < 1/2$ for all $i \in [k]$, then* (3.8) *is equal to* $(1/2)^k$.

Proof. For each $i \in [k]$, we have

$$U_i + U_{i+1} < \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, (3.8) is the integral

$$\int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} du_1 \dots du_k = \left(\frac{1}{2}\right)^k.$$

□

Remark. *This probability geometrically represents the volume of the cube* $(0,1/2)^k$.

The second case, namely when $U_j \geq 1/2$, for some $j \in [k]$, is nontrivial. We discuss which random variables can be simultaneously at least $1/2$.

Theorem 3.3.2. *Suppose* $U_i + U_{i+1} < 1$ *for all* $i \in [k]$. *Let there be distinct indices* $r_1, \ldots, r_n \in [k]$ *such that* $U_{r_1}, \ldots, U_{r_n} \geq 1/2$. *Then, we have that* $r_1, \ldots, r_n$ *are cyclically pairwise nonconsecutive. That is, for all distinct* $p, q \in [n]$, *we have* $|r_p - r_q| \notin \{1, k-1\}$.

Proof. Suppose on the contrary we had two distinct numbers $p, q \in [n]$ with $|r_p - r_q| \in \{1, k-1\}$ such that $U_{r_p}, U_{r_q} \geq 1/2$.

If $|r_p - r_q| = 1$, we must have r_p and r_q consecutive to one another. Hence $U_{r_p} + U_{r_q} < 1$, which is a contradiction.

If $|r_p - r_q| = k-1$, we must have $r_p = 1$ and $r_q = k$, or vice versa. But in either case, we see $U_1 + U_k < 1$, which is a contradiction. □

Now, we find the maximal n that would satisfy the previous theorem.

Theorem 3.3.3. *Reconsider the hypotheses from Theorem 3.3.2. Then we have* $n \leq \lfloor k/2 \rfloor$.

Proof. Suppose k is even and $n > \lfloor k/2 \rfloor = k/2$. Let σ be an increasing permutation map making $\sigma(r_1) < \cdots < \sigma(r_n)$. Note that $\sigma(r_1), \ldots, \sigma(r_n)$ are cyclically pairwise nonconsecutive. Hence, for all $j \in [n-1]$, there is an integer $x_j \in [k]$ such that $\sigma(r_j) < x_j < \sigma(r_{j+1})$. There is also an $x_n \in [k]$ such that either $x_n > \sigma(r_n)$ or $x_n < \sigma(r_1)$. In all we have enumerated at least $2n$ distinct integers in the set $[k]$. Since $n > k/2$, we see $2n > k$. Hence, we have enumerated more than k integers in $[k]$, which is impossible.

Now, suppose k is odd and $n > \lfloor k/2 \rfloor = (k-1)/2$. Construct an increasing permutation map σ as before. This would result in us enumerating more than $k-1$ distinct integers in the set $[k]$. Thus, we must have $2n = k$, so $2|k$. This is a contradiction to k being odd. □

Theorem 3.3.4. *Let* $y_1, \ldots, y_n \in \mathbb{R}$ *such that* $1/2 \leq y_n \leq \ldots \leq y_1 < 1$, *and let* $\beta_1, \ldots, \beta_n$ *be nonnegative integers. Define the function*

$$\psi(y_1, \ldots, y_n) = \prod_{i=1}^{n} (1 - y_i)^{\beta_i}. \tag{3.9}$$

Then the following integral formula holds

$$\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{y_1} \cdots \int_{\frac{1}{2}}^{y_{n-1}} \psi(y_1, \ldots, y_n)\ dy_n \ldots dy_1 = \left(\frac{1}{2}\right)^{n + \sum_{j=1}^{n} \beta_j} \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \beta_j}. \tag{3.10}$$

Proof. We first make the change of variables $y_j = 1 - t_j$, $j \in [n]$, which transforms the left hand side of (3.10) into

$$\int_{0}^{\frac{1}{2}} \int_{t_1}^{\frac{1}{2}} \cdots \int_{t_{n-1}}^{\frac{1}{2}} \prod_{i=1}^{n} t_i^{\beta_i}\ dt_n \ldots\ dt_1 = \int_{0}^{\frac{1}{2}} \int_{0}^{t_n} \cdots \int_{0}^{t_2} \prod_{i=1}^{n} t_i^{\beta_i}\ dt_1 \ldots\ dt_n \tag{3.11}$$

$$= \left(\frac{1}{2}\right)^{n + \sum_{j=1}^{n} \beta_j} \frac{1}{1+\beta_1} \frac{1}{2+\beta_1+\beta_2} \cdots \frac{1}{n + \sum_{j=1}^{n} \beta_j} \tag{3.12}$$

$$= \left(\frac{1}{2}\right)^{n + \sum_{j=1}^{n} \beta_j} \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \beta_j}.$$

The first equality of (3.11) follows from reversing the order of integration, and (3.12) follows upon induction on n.

$\square$

We discuss the integral bounds for the multiple integral corresponding to the probability in (3.8).

Theorem 3.3.5. *Reconsider the hypotheses from Theorem 3.3.2, but this time, suppose further that* $1/2 \leq U_{r_n} \leq \cdots \leq U_{r_1} < 1$. *For each* $j \in [n]$, *let* α_j *be the number of* $x \in \{r_j \pm 1\}$ *such that* $U_x < 1 - U_{r_j} \leq 1 - U_{r_l}$ *holds for all* $l \in [n] \setminus \{j\}$. *Then*

$$\alpha_j = 2 - \delta(k,2) - \sum_{m=1}^{j-1} \delta(|r_m - r_j|, 2) + \delta(|r_m - r_j|, k-2). \tag{3.13}$$

Proof. Suppose $k = 2$. Then by Theorem 3.3.3, exactly one of U_1, U_2 must be at least $1/2$, so $\alpha_1 = 1$ in each case. This agrees with the right hand side of (3.13), since the summation term vanishes.

Let $k > 2$. For $j = 1$, since U_{r_1} is the greatest random variable, we have $1 - U_{r_1} \leq 1 - U_{r_l}$ for each $l \in [n] \setminus \{1\}$. So $U_{r_1 - 1}, U_{r_1 + 1} < 1 - U_{r_1}$, which means $\alpha_1 = 2$. For $j > 1$, we observe if $\{r_j \pm 1\} \cap \{r_m \pm 1\}$ is nonempty for some $m \in [j-1]$, then $|r_m - r_j| \in \{2, k-2\}$. If $x \in \{r_j \pm 1\} \cap \{r_m \pm 1\}$, then $U_x < 1 - U_{r_m} \leq 1 - U_{r_j}$ since $U_{r_j} \leq U_{r_m}$. We count the number of $x \in \{r_j \pm 1\} \cap \{r_m \pm 1\}$ for some $m \in [j-1]$ by

$$\sum_{m=1}^{j-1} \delta(|r_m - r_j|, 2) + \delta(|r_m - r_j|, k-2).$$

Hence, we must have

$$\alpha_j = 2 - \sum_{m=1}^{j-1} \delta(|r_m - r_j|, 2) + \delta(|r_m - r_j|, k-2).$$

This agrees with the right hand side of (3.13) since $\delta(k,2) = 0$ for $k > 2$. $\square$

Theorem 3.3.6. *Reconsider the hypotheses from Theorem 3.3.5. Then the probability expressed by* (3.8) *is*

$$\left(\frac{1}{2}\right)^k \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \alpha_j}, \tag{3.14}$$

with α_j *defined as in* (3.13).

Proof. We have from our hypothesis the bounds

$$\begin{cases} \frac{1}{2} \leq U_{r_j} < 1 & j = 1 \\ \frac{1}{2} \leq U_{r_j} \leq U_{r_{j-1}} & j \in [n] \setminus \{1\}. \end{cases} \tag{3.15}$$

For each $j \in [n]$, there are α_j bounds of the form $0 < U_x < 1 - U_{r_j}$, with α_j constructed in (3.13). As a result, there are $k - n - \sum_{j=1}^{n} \alpha_j$ remaining bounds of the form $0 < U_x < 1/2$.

We now set up the integral corresponding to the probability (3.8). We integrate first with respect to all u_x with $0 < U_x < 1/2$. Then, we integrate with respect to all u_x in which $0 < U_x < 1 - U_{r_j}$ for some $j \in [n]$. This results in (3.8) simplifying down to

$$\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{u_{r_1}} \cdots \int_{\frac{1}{2}}^{u_{r_{n-1}}} \left(\frac{1}{2}\right)^{k-n-\sum_{j=1}^{n} \alpha_j} \phi(u_{r_1}, \ldots, u_{r_n}) \, du_{r_n} \ldots \, du_{r_1}, \tag{3.16}$$

where

$$\phi(u_{r_1}, \dots, u_{r_n}) = \prod_{i=1}^{n} (1 - u_{r_i})^{\alpha_i}.$$

Recognizing ϕ is a function of the form in (3.9), we use Theorem 3.3.4 to see (3.16) is

$$\left(\frac{1}{2}\right)^k \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \alpha_j}.$$

□

We now can prove the capstone results of our combinatorial analysis.

Theorem 3.3.7. *We have*

$$\mathrm{Vol}(\Delta^k) = \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k \sum_{n=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{\substack{(r_1,\dots,r_n)\in[k]^n: \\ |r_p - r_q| \notin \{0,1,k-1\} \\ p,q\in[n]}} \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \alpha_j}, \tag{3.17}$$

$$S(k) = \left(\frac{\pi}{4}\right)^k + \left(\frac{\pi}{4}\right)^k \sum_{n=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{\substack{(r_1,\dots,r_n)\in[k]^n: \\ |r_p - r_q| \notin \{0,1,k-1\} \\ p,q\in[n]}} \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \alpha_j}, \tag{3.18}$$

$$\zeta(2k) = \frac{\pi^{2k}}{2^{2k}-1} + \frac{\pi^{2k}}{2^{2k}-1} \sum_{n=1}^{k} \sum_{\substack{(r_1,\dots,r_n)\in[2k]^n: \\ |r_p - r_q| \notin \{0,1,2k-1\} \\ p,q\in[n]}} \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \alpha_j}, \tag{3.19}$$

with α_j defined in (3.13). *In particular the inner sums of* (3.17), (3.18), *and* (3.19) *are taken over all tuples in $[k]^n$ having cyclically pairwise nonconsecutive entries.*

Proof. The first term in (3.17) follows from Theorem 3.3.1. Next, we consider all possible tuples of length n with entries in $[k]$ satisfying the hypotheses for Theorem 3.3.2. We use Theorem 3.3.6 to compute corresponding probability (3.8) for each of those tuples. We sum of all these probabilities together to get the second term. We see (3.18) immediately follows from (3.6), and (3.19) follows directly from (1.15) and the fact $\lfloor 2k/2 \rfloor = k$. □

Remark. *When $k = 1$, Theorem 3.3.3 implies there are no random variables that can be at least* $1/2$, *which then results in the second terms of* (3.17) *and* (3.18) *vanishing. However, the first terms stay put due to Theorem 3.3.1.*

4 Cauchy Random Variables Revisited

We recall the Basel Problem solution by Zagier and Kontsevich [12, p. 9] and then show how its generalization leads us a probability involving the cyclically pairwise consecutive products k independent, nonnegative Cauchy random variables.

4.1 Zagier's and Kontsevich's Basel Problem Solution

We start with the double integral D_3, which we modify as

$$\frac{4}{\pi^2}\int_0^1\int_0^1 \frac{1}{4\sqrt{x_1x_2}(1-x_1x_2)}\,dx_1\,dx_2. \tag{4.1}$$

From a substitution $x=\sqrt{u}, y=\sqrt{v}$ on D_2, we see that (4.1) is $4S(2)/\pi^2$. Using the change of variables from (1.13),

$$x_1=\frac{\xi_1^2(\xi_2^2+1)}{\xi_1^2+1},\quad x_2=\frac{\xi_2^2(\xi_1^2+1)}{\xi_2^2+1},$$

with Jacobian Determinant

$$\left|\frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)}\right|=\frac{4\sqrt{x_1x_2}(1-x_1x_2)}{(\xi_1^2+1)(\xi_2^2+1)},$$

(4.1) becomes

$$\begin{aligned}\frac{4}{\pi^2}\int_0^\infty\int_0^{\frac{1}{\xi_1}}\frac{1}{(\xi_1^2+1)(\xi_2^2+1)}\,d\xi_2\,d\xi_1 &= \frac{2}{\pi^2}\int_0^\infty\int_0^\infty\frac{1}{(\xi_1^2+1)(\xi_2^2+1)}\,d\xi_2\,d\xi_1 \qquad (4.2)\\ &=\frac{1}{2},\end{aligned}$$

in which the equality (4.2) follows from a symmetry argument. Thus, we see

$$\frac{4}{\pi^2}S(2)=\frac{1}{2},$$

which implies $S(2)=\pi^2/8$.

4.2 From Integral to Hyperbolic Polytope

The analogue of (4.1) is

$$\frac{2^k}{\pi^k}\int_0^1\int_0^1 \frac{1}{2^k\sqrt{x_1\ldots x_k}(1-(-1)^k x_1\ldots x_k)}\,dx_1\ldots\,dx_k, \tag{4.3}$$

which is $(2/\pi)^k S(k)$ using a change of variables $x_i=\sqrt{u_i}, 1\le i\le k$ on (3.2). The change of variables generalizing (1.13) is

$$x_i=\frac{\xi_i^2(\xi_{i+1}^2+1)}{\xi_i^2+1},\quad 1\le i\le k, \tag{4.4}$$

with cyclical indexing mod $k:\xi_{k+1}:=\xi_1$. The change of variables in (4.4) has Jacobian Determinant

$$\left|\frac{\partial(x_1,\ldots,x_k)}{\partial(\xi_1,\ldots,\xi_k)}\right|=\frac{2^k\sqrt{x_1\ldots x_k}(1-(-1)^k x_1\ldots x_k)}{(\xi_1^2+1)\ldots(\xi_k^2+1)}, \tag{4.5}$$

and diffeomorphically maps $(0,1)^k$ to the hyperbolic polytope defined in (1.14), namely

$$\mathbb{H}^k=\{(\xi_1,\ldots,\xi_k)\in\mathbb{R}^k:\xi_i\xi_{i+1}<1,\ \xi_i>0, 1\le i\le k\}.$$

The proof of the Jacobian Determinant is remarkably similar in structure to the proof of (3.5).

If $\Xi_1, \ldots, \Xi_k$ are independent, nonnegative Cauchy random variables, then

$$\frac{2^k}{\pi^k} S(k) = \Pr(\Xi_1\Xi_2 < 1, \ldots, \Xi_{k-1}\Xi_k < 1, \Xi_k\Xi_1 < 1). \tag{4.6}$$

We already know the left hand side of (4.6) is $\mathrm{Vol}(\Delta^k)$, which we found by combinatorially analyzing the probability from (3.8). We show the connections between the analyses of (3.8) and

$$\Pr(\Xi_1\Xi_2 < 1, \ldots, \Xi_{k-1}\Xi_k < 1, \Xi_k\Xi_1 < 1), \tag{4.7}$$

which both lead to the formulas in (3.17), (3.18), and (3.19).

4.3 Hyperbolic Polytope and Combinatorics

We state the analogues of Theorems 3.3.1-3.3.6, and prove only those directly dealing with (4.7). The omitted proofs are similar to those in the previous section.

Theorem 4.3.1. *If $\Xi_i < 1$ for all $i \in [k]$, then* (4.7) *is equal to* $(1/2)^k$.

Proof. The integral corresponding to (4.7) is

$$\begin{aligned}\frac{2^k}{\pi^k}\int_0^1 \cdots \int_0^1 f_{\Xi_1}(\xi_1)\ldots f_{\Xi_k}(\xi_k)\ d\xi_k \ldots d\xi_1 &= \frac{2^k}{\pi^k}\left(\int_0^1 \frac{1}{\xi_1^2+1} d\xi_1\right)^k \\ &= \frac{2^k}{\pi^k}\frac{\pi^k}{4^k} = \left(\frac{1}{2}\right)^k.\end{aligned}$$

□

Theorem 4.3.2. *Suppose $\Xi_i\Xi_{i+1} < 1$ for all $i \in [k]$. Let there be distinct indices $r_1, \ldots, r_n \in [k]$ such that $\Xi_{r_1}, \ldots \Xi_{r_n}, \geq 1$ Then for all distinct $p, q \in [n]$, we have $|r_p - r_q| \notin \{1, k-1\}$.*

Proof. Similar to the proof of Theorem 3.3.2. □

Theorem 4.3.3. *Reconsider the hypotheses from Theorem 4.3.2. Then we have $n \leq \lfloor k/2 \rfloor$.*

Proof. Similar to the proof of Theorem 3.3.3. □

Theorem 4.3.4. *Given $y_1, \ldots, y_n \in \mathbb{R}$ with $1 \leq y_n \leq \cdots \leq y_1$, and nonnegative integers $\beta_1, \ldots, \beta_n$, define*

$$\phi(y_1, \ldots, y_n) = \prod_{i=1}^{n} \frac{\left(\cot^{-1}(y_i)\right)^{\beta_i}}{y_i^2+1}. \tag{4.8}$$

Then the following integral formula holds

$$\int_1^\infty \int_1^{y_1} \cdots \int_1^{y_{n-1}} \phi(y_1, \ldots, y_n)\ dy_1 \ldots\ dy_n = \left(\frac{\pi}{4}\right)^{n+\sum_{j=1}^n \beta_j} \prod_{i=1}^{n} \frac{1}{i + \sum_{j=1}^{i} \beta_j}. \tag{4.9}$$

Proof. We make the change of variables $y_i = 1/z_i, i \in [n]$, to see the left hand side of (4.9) becomes

$$\int_0^1 \int_{z_1}^1 \cdots \int_{z_{n-1}}^1 \prod_{i=1}^n \frac{\left(\tan^{-1}(z_i)\right)^{\beta_i}}{z_i^2+1}\, dz_n \dots\, dz_1 = \int_0^1 \int_0^{z_n} \cdots \int_0^{z_2} \prod_{i=1}^n \frac{\left(\tan^{-1}(z_i)\right)^{\beta_i}}{z_i^2+1}\, dz_1 \dots\, dz_n$$
$$= \left(\frac{\pi}{4}\right)^{n+\sum_{j=1}^n \beta_j} \prod_{i=1}^n \frac{1}{i+\sum_{j=1}^i \beta_j},$$

which is seen upon induction on n. □

Theorem 4.3.5. *Reconsider the hypotheses from Theorem 4.3.2, but this time, let us have $1 \le \Xi_{r_n} \le \cdots \le \Xi_{r_1}$. For each $j \in [n]$, let α_j be the number of $x \in \{r_j \pm 1\}$ such that $\Xi_x < 1/\Xi_{r_j} \le 1/\Xi_{r_l}$ holds for all $l \in [n] \setminus \{j\}$. Then we see α_j is the same as it is in* (3.13), *namely*

$$\alpha_j = 2 - \delta(k,2) - \sum_{m=1}^{j-1} \delta(|r_m - r_j|, 2) + \delta(|r_m - r_j|, k-2).$$

Proof. Similar to the proof of Theorem 3.3.5. □

Theorem 4.3.6. *Reconsider the hypotheses from Theorem 4.3.5. Then we have that* (4.7) *is equal to the right hand side of* (3.14), *namely,*

$$\left(\frac{1}{2}\right)^k \prod_{i=1}^n \frac{1}{i+\sum_{j=1}^i \alpha_j},$$

with α_j defined as in (3.13).

Proof. We have from our hypothesis the bounds

$$\begin{cases} 1 \le \Xi_{r_j} < \infty & j = 1 \\ 1 \le \Xi_{r_j} \le \dfrac{1}{\Xi_{r_{j-1}}} & j \in [n] \setminus \{1\}. \end{cases} \tag{4.10}$$

For each $j \in [n]$, there are α_j bounds of the form $0 < \Xi_x < 1/\Xi_{r_j}$, with α_j constructed in (3.13), and $k - n - \sum_{j=1}^n \alpha_j$ bounds of the form $0 < \Xi_x < 1$.

We now set up the integral for (4.7), whose integrand is $f_{\Xi_1}(\xi_1) \dots f_{\Xi_k}(\xi_k)$. We integrate first with respect to all ξ_x with bounds of the form $0 < \Xi_x < 1$. Then, we integrate with respect to all ξ_x with $0 < \Xi_x < 1/\Xi_{r_j}$ for some $j \in [n]$. Lastly, we integrate with respect to $\xi_{r_j}, \dots, \xi_{r_1}$ in that order with the bounds of the form in (4.10).

Integrating with respect to the first two groups of variables, (4.7) becomes

$$\frac{2^k}{\pi^k} \int_1^\infty \int_1^{\frac{1}{\xi_{r_1}}} \cdots \int_1^{\frac{1}{\xi_{r_{n-1}}}} \left(\frac{\pi}{4}\right)^{k-n-\sum_{j=1}^n \alpha_j} \phi(\xi_{r_1}, \dots, \xi_{r_n})\, d\xi_{r_n} \dots\, d\xi_{r_1}, \tag{4.11}$$

where

$$\phi(\xi_{r_1}, \dots, \xi_{r_n}) = \prod_{i=1}^n \frac{\left(\cot^{-1}(\xi_i)\right)^{\alpha_i}}{\xi_i^2+1}.$$

Recognizing ϕ is a function of the form (4.8), we use Theorem 4.3.4 to simplify (4.11) to

$$\left(\frac{1}{2}\right)^k \prod_{i=1}^n \frac{1}{i+\sum_{j=1}^i \alpha_j}.$$

□

Therefore,

$$S(k) = \frac{\pi^k}{2^k}\mathrm{Vol}(\Delta^k) = \frac{\pi^k}{2^k}\int_{\mathbb{H}^k} f_{\Xi_1}(\xi_1)\dots f_{\Xi_k}(\xi_k)\ d\xi_1\dots\ d\xi_k,$$

and the capstone results from the previous section are immediate consequences.

As promised, we have shown $S(k)$ and $\zeta(2k)$ are the consequences of two equal probabilities: the probability k uniform random variables on $(0,1)$ have cyclically pairwise consecutive sums less than 1, and the probability k independent, nonnegative Cauchy random variables have cyclically pairwise consecutive products less than 1.

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