Sharp Bounds for the Directed Postman Problem

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Introduction

Postman Problem

The Chinese Postman Problem was introduced by Guan Meigu in 1960 and refers to the shortest route a postal worker could possibly take to deliver mail to every street in a city [3]. In an unweighted graph, this solution is a closed walk repeating the fewest number of edges possible while still traversing each edge at least once. We call such a closed walk a *post tour.*

This problem relies on the well-known Eulerian circuit condition. In a post tour of a connected graph, edges are repeated until each vertex has an even degree. If we add the repeated edges to the graph, the augmented graph has an Eulerian circuit and thus the original graph has a post tour.

Directed Postman Problem

For the Postman Problem on a directed graph, a directed edge (u, v) can only be repeated in the specified direction. In the undirected Postman Problem each edge is repeated at most once in a post tour, which means the number of repeated edges is at most the total number of edges in the graph. However, in the directed Postman Problem, edges may need to be repeated multiple times, resulting in significantly longer post tours.

A directed graph has a post tour if and only if that graph is *strongly connected*, i.e. there is a directed path between any two vertices. An *ear decomposition* of a connected graph is a sequence of subgraphs starting with a cycle and successively adding paths, called *ears*, that only meet the previous subgraph at the ear's endpoints. A directed graph has an ear decomposition if and only if that graph is strongly connected [6].

Definition 1. For a strongly connected directed graph G, we define $\rho(G)$ to be the number of edges in a post tour of G minus the number of edges in G.

The goal of this project was to find sharp upper bounds for $\rho(G)$ on several types of strongly connected directed graphs.

Results

Directed multigraphs

A *directed multigraph* is a directed graph that may have multiple directed edges (u, v) from vertex u to vertex v.

Theorem 1. *If G is a strongly connected directed multigraph with m edges and n vertices, then*

$$\rho(G) \leq (m-n)(n-1),$$

and this bound is sharp.

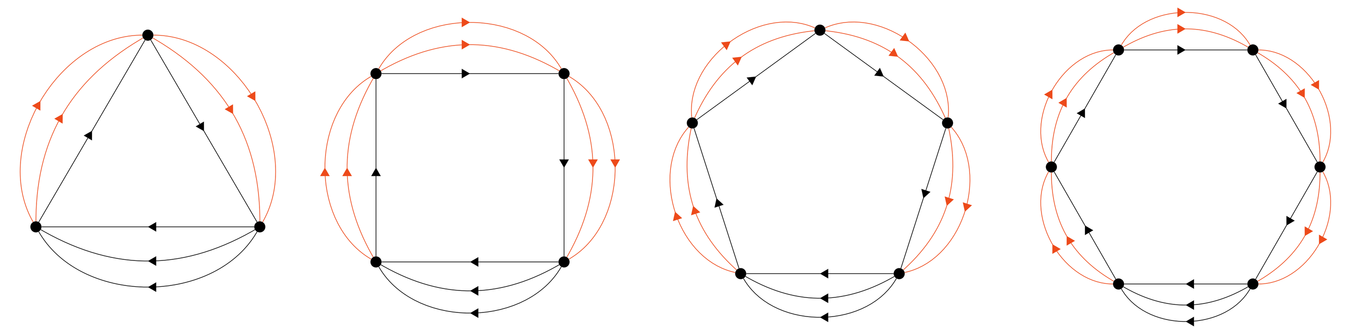

Figure 1: *Directed multigraphs with $\rho(G) = (m-n)(n-1)$*

In Theorem 1, $m - n$ is the number of ears in the graph and if j is the length of the longest cycle, $j - 1$ is the most $\rho(G)$ can increase with the addition of an ear. Thus, $\rho(G)$ is maximized when the longest cycle has length n. If the $m - n$ extra edges all repeat the same edge, as in Figure 1, $\rho(G) = (m - n)(n - 1)$.

Oriented graphs

An *oriented graph* is a directed graph with only one edge between any two vertices u and v, i.e. there cannot be both edges (u, v) and (v, u).

Theorem 2. *If G is a strongly connected oriented graph with m edges, n vertices, and a cycle of length n, then*

$$\rho(G) \leq \sum_{k=2}^{n-2} \max\left(0, \min\left(k, m-n+1-\frac{k}{2}(k-1)\right)\right)(n-k),$$

and this bound is sharp.

The formula above is illustrated in Figure 3 for several values of m and n. Figure 2 depicts an oriented complete graph on 8 vertices with maximum $\rho(G)$. Note that for an oriented complete graph, Theorem 2 reduces to

$$\rho(G) \leq \sum_{k=2}^{n-2} k(n-k).$$

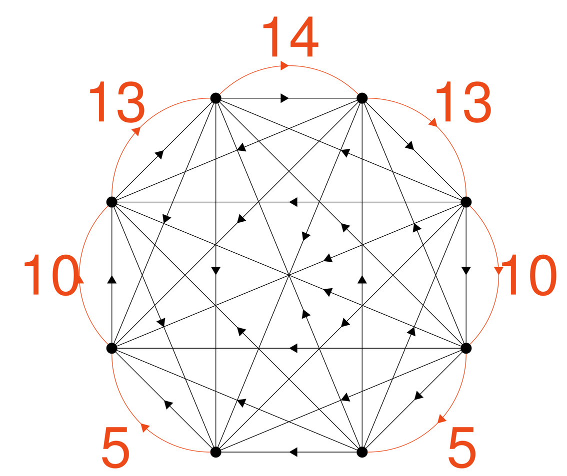

Figure 2: *G with 8 vertices and $\rho(G) = 70$*

m	$\rho(G)$
$n+1$	$(n-2)$
$n+2$	$2(n-2)$
$n+3$	$2(n-2)+(n-3)$
$n+4$	$2(n-2)+2(n-3)$
$n+5$	$2(n-2)+3(n-3)$
$n+6$	$2(n-2)+3(n-3)+(n-4)$
$n+7$	$2(n-2)+3(n-3)+2(n-4)$
$n+8$	$2(n-2)+3(n-3)+3(n-4)$
$n+9$	$2(n-2)+3(n-3)+4(n-4)$

Figure 3: *$\rho(G)$ for oriented G*

Since G has a cycle of length n, we consider adding edges to a base n-cycle. Because G is oriented, adding a single edge increases $\rho(G)$ by at most $n-2$. As illustrated in the first graph of Figure 4, we can add only two edges that simultaneously increase $\rho(G)$ by $n-2$. Similarly, there are another three edges that can increase $\rho(G)$ by $n-3$, four edges that can increase $\rho(G)$ by $n-4$, etc. (See Figure 4). Moreover, if all of these edges are added to the base n-cycle as in Figure 2, $\rho(G)$ attains the maximum as written in Theorem 2.

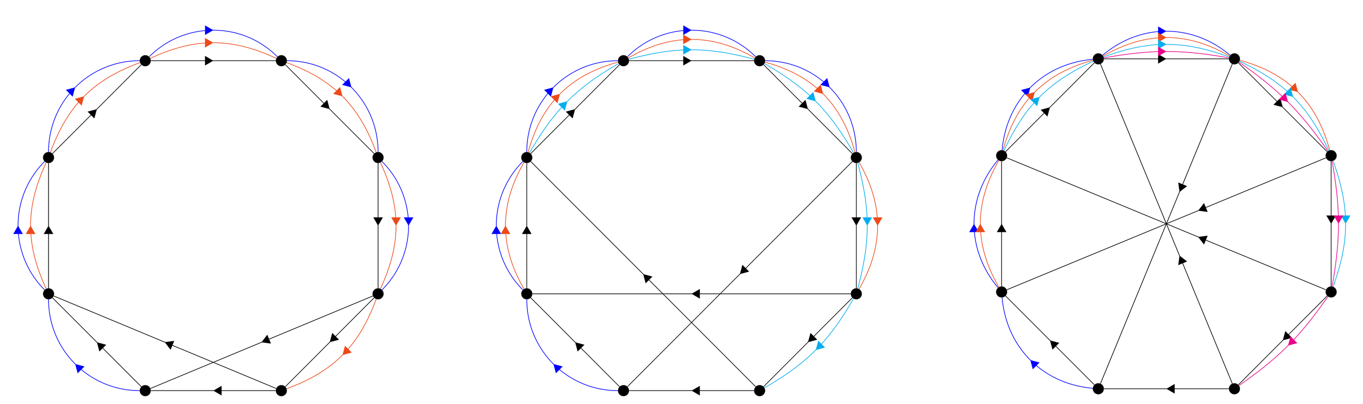

Figure 4: *Edges that increase $\rho(G)$ by $n-2$, $n-3$, and $n-4$*

Simple directed graphs

A *simple directed graph* is a directed graph where there is only one edge (u, v) for any two vertices u and v. Note that there can be both edge (u, v) and edge (v, u).

Corollary 3. *If G is a strongly connected simple directed graph with m edges, n vertices, and a cycle of length n, then*

$$\rho(G) \leq \sum_{k=2}^{n-1} \max\left(0, \min\left(k, m-n+1-\frac{k}{2}(k-1)\right)\right)(n-k),$$

and this bound is sharp.

When there is an edge (u, v), an edge (v, u) can increase $\rho(G)$ by at most 1. Therefore, when adding a set number of edges to a cycle of length n, all edges that can be placed so that they increase $\rho(G)$ by more than 1 should be added and only $n-1$ remaining edges can be added so that they increase $\rho(G))$ by 1.

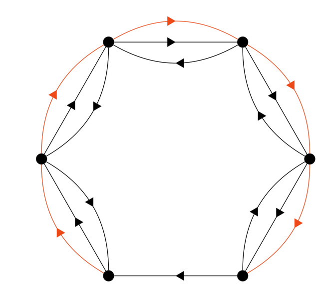

Figure 5: *The $n-1$ edges that each increase $\rho(G)$ by 1*

Future Directions

Conjecture 4. *Theorem 2 and Corollary 3 give upper bounds for all strongly connected oriented and simple directed graphs, respectively (not only for graphs with a cycle of length n).*

All of our constructions rely on having a large base cycle, and thus we suspect that graphs with cycles including all the vertices will have the largest $\rho(G)$. Theorem 1 is sharp for all graphs, and any maximum construction must contain an n-cycle.

Question 1: If we restrict the degree of the vertices of G, what new bounds do we discover for $\rho(G)$?

Question 2: In what cases does an oriented graph have the same post tour as the underlying undirected graph?

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