BILLIARDS AND BRAIDS

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A *billiard* is dynamical system of particles in a planar domain. Each particle moves in a straight line until it strikes the boundary, reflecting with angle of incidence equal to angle of reflection.

By graphing the positions of the balls over time, one obtains a *braid*, an entangled collection of paths in three-dimensional space. With just two balls, the braiding is measuring by an integer, the *winding number*, which counts the number of times one ball encircles the other.

The goal of this project is to predict the long-term braiding behavior based on the shape of the billiard domain. For the two-ball case, we seek to calculate the limit

$$\lim_{t\to\infty} \frac{\omega(t)}{t},$$

the *asymptotic winding number*. Owing to the high degree of symmetry in the circular billiard, we make the following conjecture

Conjecture 0.1. *For a generic two-ball circular billiard, the asymptotic winding number is a constant depending only on chords, up to rotation, determined by the trajectories of the balls.*

The conjecture was "experimentally" verified by computer simulation implemented with the mathematical software Sage. The initial positions and velocities of the balls are chosen at random. For an iteration of the simulation, we compute the time needed for each ball to collide with the boundary and advance each ball by the lesser collision time, keeping track of the winding number with each iteration.

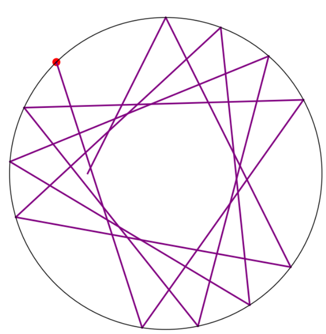

FIGURE 1. *Circular billiard simulation*

We found that the winding number grows linearly as a function of time. Furthermore, the asymptotic winding does not change if one ball is allowed to iterate many times before the simulation begins, thus indicating that the asymptotics depends only on the trajectory chords up to rotation.

We have verified computationally the following conjectured formula for the asymptotic winding number, based on Birkhoff's ergodic theorem:

Conjecture 0.2. *The asymptotic winding number of a generic two-ball circular billiard is given by*

$$\frac{1}{\ell_1 \ell_2 2\pi} \int_0^{\ell_2} \int_0^{\ell_1} \int_0^{2\pi} \frac{e^{i\alpha} - 1}{x_2(s_2, \alpha) - x_1(s_1)} \, d\alpha \, ds_1 \, ds_2 \tag{0.3}$$

where x_i are the positions of the balls, viewed as complex numbers, and l_i are the lengths of the trajectory chords.

The Bunimovich stadium is a planar domain consisting of a rectangle capped with semi-circles on opposite ends. The stadium lacks the symmetry of the circular billiard, and therefore is conjectured to have an oscillating winding number:

Conjecture 0.4. *For a generic two-ball billiard in the Bunimovich stadium, the asymptotic winding number is 0*

The conjecture was verified computationally. Moreover, we demonstrated that asymptotic winding number tends to be normally distributed about 0. The following figure is a histogram of the quantity $\frac{\omega}{t}$ of a billiard iterated 10^8 times in which the winding number and time is reset every 10^5 iterations.

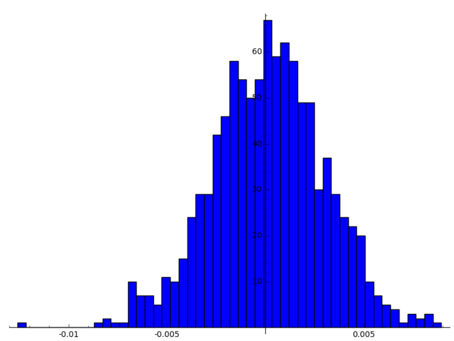

FIGURE 2. *Histogram of stadium asymptotic winding numbers reset every* 10^5 *iterations*

In our ongoing work we seek to prove that the asymptotic winding number exists and is indeed given by the conjectured formula 0.3 above. The proof will involve showing that a generic two-ball circular billiard is ergodic (which is already known for the single-ball case). We hope to accomplish this for the stadium billiard and other asymmetric domains as well. Furthermore, billiards with more than two balls can exhibit more complicated braiding behavior. In future work we seek to characterize the asymptotics of higher order braiding.