**Math 542** **Comprehensive Examination** **August 2021**

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems. Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1. 2. 3. and 4. Each problem is worth 10 points. Justify all your answers. Good Luck!

**Notation:**
We denote the set of complex numbers by $\mathbb{C}$. We write $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ for the unit disk in $\mathbb{C}$.

1. We define the domain $D$ by $D = \mathbb{D} \setminus (-1, -1/2]$. Find a conformal mapping $f$ of $D$ onto $\mathbb{D}$ such that $f(0) = 0$. Explain the process that leads to your answer. You may express your answer as a composite function without simplifying the resulting expression further.

2. Showing the details of your calculations, evaluate the integral

$$\int_0^\infty \frac{\cos x \, dx}{(x^2+4)(x^2+9)}.$$

3. Does there exist an entire function $f$ such that $f(0) = 0$, $f(i) = i$, and $|f(z)| \le |z|^{2/3}$ for all $z \in \mathbb{C}$ ? Prove that your answer is correct.

4. Determine the Laurent series of the function $f(z) = \frac{1}{1-z^2}$ in terms of the powers of $z-2$ in the annulus $\{z \in \mathbb{C}: 1 < |z-2| < 3\}$.

5. Let $\mathcal{F}$ consist of all functions $f$ that are analytic in $\mathbb{D}$ with $f(0) = 2$, and satisfy $\operatorname{Re} f(z) > 0$ for all $z \in \mathbb{D}$. We define $M = \sup\{|f'(0)|: f \in \mathcal{F}\}$.

   (a) Determine $M$.

   (b) Does there exist $f \in \mathcal{F}$ such that $|f'(0)| = M$ ? If yes, give an explicit example of such a function $f$ and prove that it has all the required properties. If not, prove that there is no such function $f$.

6. Let $f$ be analytic and not constant in the disk $\{z \in \mathbb{C}: |z| < 2\}$. Suppose that for all $z$ with $|z| = 1$, we have $|f(z)| = 1$. Prove that $f$ has at least one zero in $\mathbb{D}$.