Math 540 Comprehensive Examination, August 2021

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Suppose f is a continuous function on $[0,1]$ and $f(x) > 0$ for all $x \in (0,1]$. Prove that

$$\int_0^1 \frac{f(x)}{x^{1/3}}\,dx \le 2^{2/3}\left(\int_0^1 |f(x)|^3\,dx\right)^{1/3}.$$

Can the equality be attained?

2. Let $E_1, \cdots, E_J$ be Lebesgue measurable sets in $[0,1]$. Suppose that

$$\sum_{j=1}^{J} m(E_j) > J - 1\,.$$

Prove that $m(\cap_{j=1}^{J} E_j) > 0$.

3. Let $f \in L^1(\mathbb{R}, \mathcal{B}, m)$ (here $\mathcal{B}$ is the Borel σ-algebra) and $\epsilon > 0$. Prove that

$$\lim_{n\to\infty} n^{-\epsilon} f(nx) = 0\,,$$

for a.e. $x \in \mathbb{R}^1$.

4. Let $p \in (1,\infty)$ and $f : [0,1] \to \mathbb{R}$ be a function such that

$$\sum_{i=1}^{n} \frac{|f(b_i) - f(a_i)|^p}{(b_i - a_i)^{p-1}} \le 2021$$

whenever $(a_1, b_1), \ldots, (a_n, b_n)$ are disjoint intervals in $[0,1]$.

(i) Prove that f is absolutely continuous on $[0,1]$.

(ii) Prove that $\int_0^1 |f'|^p\,dm < \infty$.

5. Let $E \subseteq \mathbb{R}$ be a non-empty Lebesgue measurable set and $f \in L^1(E)$. Prove that for every λ satisfying $0 \le \lambda \le \int_E |f|\,dm$, there is a non-empty Lebesgue measurable set $E_\lambda \subseteq E$ such that $\int_{E_\lambda} |f|\,dm = \lambda$.

6. Let $(V, \|*\|)$ be a normed vector space and $F : V \to \mathbb{R}$. Recall that F has the directional derivative at $f \in V$ in the direction of $g \in V$ if the following limit exists

$$DF(f;g) = \lim_{t\to 0} \frac{F(f+tg) - F(f)}{t}$$

relative to the norm $\|*\|$. Let $V = L^1(\mathbb{R})$, $\|*\| = \|*\|_{L^1(\mathbb{R})}$ and $F(f) = \|f\|_{L^1(\mathbb{R})}$. Give necessary and sufficient condition for f such that the directional derivative $DF(f;g)$ exists for all $g \in L^1(\mathbb{R})$. Find the directional derivative $DF(f;g)$ when exists. Justify your answers.