Math 500 Comprehensive Exam, August 2021

Complete all 4 problems for a maximum total of 100 points. You may use any result from Math 500, but you must make accurate references to the theorems used in your solutions and explain your answers.

1. [20 points total] Let G be a non-trivial finite group acting on a finite set X. We assume that for all $g \in G \setminus \{e\}$ there exists a unique $x \in X$ such that $g \cdot x = x$.

(a) (4 points) Let $Y = \{x \in X \mid G_x \neq \{e\}\}$ where G_x denotes the stabilizer of x. Show that Y is stable under the action of G.

(b) (10 points) Let $y_1, y_2, \ldots, y_n$ be a set of orbit representatives of Y/G (with $|Y/G| = n$), and let $m_i = |G_{y_i}|$. Show that:

$$1 - \frac{1}{|G|} = \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right)$$

(c) (6 points) Show that X has (at least) a fixed point under the action of G.

2. [30 points total]

(a) (7 points) Show that $x^6 + 69x^5 - 511x + 363$ is irreducible over the integers.

(b) (7 points) Show that $x^4 + 5x + 1$ is irreducible over the rationals.

(c) (7 points) Show that $x^4 + x^3 + x^2 + 6x + 1$ is irreducible over the rationals.

(d) (9 points) Calculate the number of distinct, irreducible polynomials over $\mathbb{Z}_5$ that have the form:

$$f(x) = x^2 + ax + b, \quad \text{or} \quad g(x) = x^3 + \alpha x^2 + \beta x + \gamma \qquad a, b, \alpha, \beta, \gamma \in \mathbb{Z}_5.$$

3. [25 points] Find possible Jordan canonical forms of an 8×8 matrix M over the field $\mathbb{F}_5$ with five elements if it is known that the characteristic polynomial of M is $(x^2+1)^4$ and the minimal polynomial of M is $(x^2+1)^2(x+2)$.

4. [25 points total] Let F be a field, $F[x]$ be the ring of polynomials over F, and $F(x)$ be the field of fractions of (the integral domain) $F[x]$. The map $F \to F(x)$ is an injective field homomorphism, so we view F as a subfield of $F(x)$; in this way, $F \subseteq F(x)$ is a field extension. $Gal(F(x)/F)$ will denote the Galois group of the field extension $F \subseteq F(x)$. In what follows, provide justification.

(a) (9 points) Prove that the function $\sigma : F(x) \to F(x)$ given by

$$\sigma\left(\frac{f(x)}{g(x)}\right) := \frac{f(x+1)}{g(x+1)}$$

is a well-defined automorphism of the field $F(x)$. Prove that $\sigma \in Gal(F(x)/F)$.

(b) (8 points) Let G be the (cyclic) subgroup of $Gal(F(x)/F)$ generated by σ. What is the order of G?

(c) (8 points) Let $F := \mathbb{F}_2$, the field of order 2, and $E \subseteq \mathbb{F}_2(x)$ be the intermediate field corresponding to the subgroup $G \leq Gal(\mathbb{F}_2(x)/\mathbb{F}_2)$ as in (b). Prove that $[E : \mathbb{F}_2] \geq 2$.