# EFFECTIVE ESTIMATES FOR THE SMALLEST PARTS FUNCTION

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ABSTRACT. We give a substantial improvement for the error term in the asymptotic formula for the smallest parts function spt(n) of Andrews. Our methods depend on an explicit bound for sums of Kloosterman sums of half integral weight on the full modular group.

### 1. INTRODUCTION

The smallest parts function  $\operatorname{spt}(n)$ , introduced by Andrews [And08], is defined for any integer  $n \geq 1$  as the number of smallest parts among the integer partitions of n. For example, the partitions of n = 4 are (with the smallest parts underlined)

 $\underline{4}, \\
 3 + \underline{1}, \\
 \underline{2} + \underline{2}, \\
 2 + \underline{1} + \underline{1}, \\
 \underline{1} + \underline{1} + \underline{1} + \underline{1},$ 

and so spt(4) = 10. Apart from its combinatorial significance, this function is also of interest because the generating function is closely related to a weak harmonic Maass form (see (1.3)), and it has been the topic of much recent study. Define

$$\lambda(n) := \frac{\pi}{6}\sqrt{24n-1}.\tag{1.1}$$

Refining an asymptotic result of Bringmann [Bri08], Locus Dawsey and Masri used the algebraic formula for the smallest parts function ([AA16, Thm. 2]) and traces of singular moduli to prove the following asymptotic formula for the smallest parts function spt(n).

**Theorem 1.1** ([LDM19], Thm. 1.1). Let  $\lambda(n)$  be as in (1.1). Then for all  $n \geq 1$ , we have

$$\operatorname{spt}(n) = \frac{\sqrt{3}}{\pi\sqrt{24n-1}}e^{\lambda(n)} + E_s(n),$$

where

$$|E_s(n)| < (3.59 \times 10^{22}) 2^{q(n)} (24n-1)^2 e^{\frac{\lambda(n)}{2}}$$

and

$$q(n) := \frac{\log(24n - 1)}{|\log(\log(24n - 1)) - 1.1714|}$$

In Theorem 1.2 we give a substantial improvement to Theorem 1.1. Our methods rely on the exact formula (2.5) and an explicit bound for sums of Kloosterman sums (Theorem 1.6).

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**Theorem 1.2.** Let  $\lambda(n)$  be as in (1.1). Then for all  $n \ge 1$ , we have

$$\operatorname{spt}(n) = \frac{\sqrt{3}}{\pi\sqrt{24n-1}}e^{\lambda(n)} + E_s(n),$$

where

$$|E_s(n)| < 4.1e^{\frac{\lambda(n)}{2}}.$$

In Figure 1 we present some data that shows how close  $E_s(n)$  is to the bound given in Theorem 1.2. The values of spt(n) were obtained by using the recurrence given in [AA15, Thm. 1].

n	$\operatorname{spt}(n)$	$ E_s(n) $	$4.1e^{\frac{\lambda(n)}{2}}$
1 000	$6.0 \times 10^{32}$	$2.1 \times 10^{15}$	$1.7 \times 10^{18}$
10000	$2.8\times10^{108}$	$8.0 \times 10^{52}$	$2.1\times10^{56}$
100000	$6.8 \times 10^{348}$	$7.0 \times 10^{172}$	$5.7 \times 10^{176}$
1000000	$1.1\times10^{1110}$	$1.6\times10^{553}$	$4.1 \times 10^{557}$
5000000	$5.0 \times 10^{2486}$	$2.2 \times 10^{1241}$	$1.3 \times 10^{1246}$

### FIGURE 1.

Using our method we can obtain more terms in the asymptotic expansion of spt(n). For example, we can prove the following theorem.

**Theorem 1.3.** Let  $\lambda(n)$  be as in (1.1). Then for all  $n \ge 1$ , we have

$$\operatorname{spt}(n) = \frac{\sqrt{3}}{\pi\sqrt{24n-1}}e^{\lambda(n)} + \frac{(-1)^n\sqrt{6}}{\pi\sqrt{24n-1}}e^{\frac{\lambda(n)}{2}} + E_{s_2}(n)$$

where

$$|E_{s_2}(n)| < 8e^{\frac{\lambda(n)}{3}}.$$

In 2014, Chan and Mao [CM14] raised the following question (later stated as a conjecture in [Che17]) regarding spt(n):

$$\frac{\sqrt{6}}{\pi}\sqrt{n}\,p(n) < \operatorname{spt}(n) < \sqrt{n}\,p(n).$$

Locus Dawsey and Masri [LDM19, Thm. 1.3] proved the following stronger result: for each  $\varepsilon > 0$ , there is an  $N(\varepsilon) > 0$  such that for all  $n \ge N(\varepsilon)$  we have

$$\frac{\sqrt{6}}{\pi}\sqrt{n}\,p(n) < \operatorname{spt}(n) < \left(\frac{\sqrt{6}}{\pi} + \varepsilon\right)\sqrt{n}\,p(n).$$

In the following corollary we improve this result.

**Corollary 1.4.** Let  $\lambda(n)$  be as in (1.1). Then for all  $n \ge 1$ , we have

$$\operatorname{spt}(n) = \frac{\sqrt{24n-1}}{2\pi} p(n) + \frac{6\sqrt{3}}{\pi^2(24n-1)} e^{\lambda(n)} + E(n)$$

where

$$|E(n)| < 4.11e^{\frac{\lambda(n)}{2}}.$$

Much of the interest in the smallest parts function arises from its connection to a harmonic Maass form. Let  $\eta$  be the Dedekind eta function

$$\eta(z) := e\left(\frac{z}{24}\right) \prod_{n=1}^{\infty} \left(1 - e(nz)\right), \qquad \text{Im}(z) > 0$$
 (1.2)

with  $e(x) := e^{2\pi i x}$ . Define a weak harmonic Maass form of weight 3/2 on  $SL_2(\mathbb{Z})$  with multiplier  $\overline{\chi}$  by

$$F(z) := \sum_{n=1}^{\infty} \operatorname{spt}(n) q^{n-\frac{1}{24}} - \frac{1}{12} \cdot \frac{E_2(z)}{\eta(z)} + \frac{\sqrt{3i}}{2\pi} \int_{-\overline{z}}^{i\infty} \frac{\eta(w)}{(\tau+w)^{\frac{3}{2}}} dw$$
$$= \sum_{n=0}^{\infty} S(n) q^{n-\frac{1}{24}} + \frac{\sqrt{3i}}{2\pi} \int_{-\overline{z}}^{i\infty} \frac{\eta(w)}{(z+w)^{\frac{3}{2}}} dw.$$
(1.3)

Here  $\chi$  is as in (2.2), and  $E_2$  is the usual weight two Eisenstein series given by

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

where  $\sigma_1(n) := \sum_{d|n} d$ . We prove the following effective asymptotic formula for S(n).

**Theorem 1.5.** For all  $n \ge 1$  we have

$$S(n) = 2\sqrt{3}e^{\lambda(n)} + E_S(n)$$

where  $\lambda(n)$  is as in (1.1) and

$$E_S(n) \le 44.11e^{\frac{\lambda(n)}{2}}.$$

This improves [LDM19, Thm. 1.4], where a bound of size

$$(4.30 \times 10^{23})2^{q(n)}(24n-1)^2 e^{\frac{\lambda(n)}{2}}$$

was obtained (with q(n) as in Theorem 1.1).

Our methods rely on an explicit bound for the sums  $\sum_{c \leq x} \frac{A_c(n)}{c}$ , where the Kloosterman sum  $A_c(n)$  is given by

$$A_{c}(n) := \sum_{\substack{d \bmod c \\ (d,c)=1}} e^{\pi i s(d,c)} e^{-2\pi i \frac{dn}{c}},$$
(1.4)

and s(d, c) is the Dedekind sum defined by

$$s(d,c) := \sum_{r=1}^{c-1} \frac{r}{c} \left( \frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2} \right).$$

$$(1.5)$$

These sums exhibit good cancelation. We give a brief summary of known bounds for sums of such Kloosterman sums. For individual Kloosterman sums Lehmer [Leh38, Thm. 8] proved

$$|A_c(n)| < 2^{\omega_o(c)} c^{\frac{1}{2}} \le \tau(c) c^{\frac{1}{2}}, \tag{1.6}$$

where  $\omega_o(c)$  is the number of distinct odd primes dividing c and  $\tau(c)$  is the number of divisors of c. Using (1.6) one obtains

$$\sum_{c \le x} \frac{A_c(n)}{c} \ll_{\epsilon} x^{\frac{1}{2} + \epsilon}.$$

The work of Goldfeld-Sarnak [GS83] yields

$$\sum_{c \le x} \frac{A_c(n)}{c} \ll_{n,\epsilon} x^{\frac{1}{6}+\epsilon}.$$
(1.7)

Ahlgren-Andersen [AA18] (with improvements by Dunn in the *n*-aspect [Dun18]) replaces the bound in (1.7) by  $\ll_{\epsilon} \left(x^{\frac{1}{6}} + n^{\frac{1}{4}}\right) (nx)^{\epsilon}$ . It is conjectured (generalization of Linnik-Selberg) that the bound can be replaced by  $\ll_{\epsilon} (nx)^{\epsilon}$ . Our methods depend on an explicit version of the work of Goldfeld-Sarnak and Pribitkin for sums of Kloosterman sums of half integral weight on the full modular group.

For any  $\delta > 0$  we have  $2^{\omega_o(c)} \ll_{\delta} c^{\delta}$ . For  $\delta > 0$  let  $\ell(\delta)$  be a constant such that for all  $c \in \mathbb{N}$  we have

$$2^{\omega_o(c)} \le \ell(\delta) c^{\delta}. \tag{1.8}$$

Then we have the following bound.

**Theorem 1.6.** Let  $0 < \delta \leq 1/4$ . For any  $x \geq 1$  and any integer  $n \geq 1$  we have

$$\left|\sum_{c \le x} \frac{A_c(n)}{c}\right| \le \left(652.33\zeta^2(1+\delta)\tau((24n-23)^2)|\log\delta|(n-1/24)^{\frac{1}{4}} + 3\ell(\delta)\log x\right)x^{\frac{1}{6}+\delta},$$

where  $\ell(\delta)$  is as in (1.8).

For example, we may take  $\ell(1/4) = 8.447$  and  $\ell(1/5) = 28.117$ . This follows from  $2^{\omega_o(c)} \leq \tau(c)$  and [Nic88, page 221]. For s > 1 we have

$$\zeta^{2}(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{s}}.$$
(1.9)

The special case of Theorem 1.6 with  $\delta = 1/4$  is given as the following corollary.

Corollary 1.7 ( $\delta = 1/4$ ).

$$\left|\sum_{c \le x} \frac{A_c(n)}{c}\right| \le \left(19094.8 \ \tau ((24n - 23)^2)(n - 1/24)^{\frac{1}{4}} + 25.35 \log x\right) x^{\frac{5}{12}}$$

In the next section we give some background material. In Section 3 we calculate the inner product of two Poincaré series in order to obtain an expression for the Kloosterman zeta function. We write the inner product as a main term plus an error term  $Q_{1,n}$ . In Section 4 we obtain a bound for the error term. In Section 5 we give bounds for the norm of the Poincaré series. In order to do this we require some results on the K Bessel functions. In Section 6 we prove a bound on the Kloosterman zeta function. Theorem 6.1 is a quantitative version of Pribitkin's main theorem [Pri00]. In Section 7 we prove Theorem 1.6 using the bound on the Kloosterman zeta function and the Phragmén-Lindelöf principle. Finally in Section 8 we use Theorem 1.6 and the exact formula (2.5) for the smallest parts function to prove Theorem 1.2 and Corollary 1.4.

## 2. Preliminaries

Let  $\Gamma = \Gamma_0(N)$  for some  $N \ge 1$ . We say that  $\nu : \Gamma \to \mathbb{C}^{\times}$  is a multiplier system of weight  $k \in \mathbb{R}$  if

(i)  $|\nu| = 1$ ,

(ii) 
$$\nu(-I) = e^{-\pi i k}$$
, and  
(iii)  $\nu(\gamma_1 \gamma_2) j(\gamma_1 \gamma_2, \tau)^k = \nu(\gamma_1) \nu(\gamma_2) j(\gamma_2, \tau)^k j(\gamma_1, \gamma_2 \tau)^k$  for all  $\gamma_1, \gamma_2 \in \Gamma$ .

If  $\nu$  is a multiplier system of weight k, then it is also a multiplier system of weight k' for any  $k' \equiv k \pmod{2}$ , and the conjugate  $\overline{\nu}$  is a multiplier system of weight -k. Define  $\alpha_{\nu} \in [0, 1)$  by the condition  $\nu\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) = e(-\alpha_{\nu})$ . For  $n \in \mathbb{Z}$ , define  $n_{\nu} := n - \alpha_{\nu}$ . The Kloosterman sum for a general multiplier  $\nu$  is given by

$$S(m, n, c, \nu) := \sum_{\substack{0 \le a, d < c \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma}} \overline{\nu}(\gamma) e\left(\frac{m_{\nu}a + n_{\nu}d}{c}\right).$$
(2.1)

We are interested in the multiplier system  $\chi$  of weight 1/2 on  $SL_2(\mathbb{Z})$  given by

$$\eta(\gamma z) = \chi(\gamma)\sqrt{cz+d}\,\eta(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$
(2.2)

where  $\eta$  is as in (1.2). Rademacher (see (74.11), (74.12), and (71.21) of [Rad73]) showed that for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with c > 0 we have

$$\chi(\gamma) = \sqrt{-i} e^{-\pi i s(d,c)} e\left(\frac{a+d}{24c}\right), \qquad (2.3)$$

where s(d, c) is as in (1.5). From (2.3) we have  $\chi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = e(1/24)$ , so

$$\alpha_{\chi} = \frac{23}{24}$$
 and  $\alpha_{\bar{\chi}} = \frac{1}{24}$ 

For the eta-multiplier, (2.1) and (2.3) give

$$S(m, n, c, \chi) = \sqrt{i} \sum_{\substack{d \mod c \\ (d,c)=1}} e^{\pi i s(d,c)} e\left(\frac{(m-1)\overline{d} + (n-1)d}{c}\right),$$

so the sums  $A_c(n)$  are given by

$$A_c(n) = \sqrt{-i} S(1, 1 - n, c, \chi).$$
(2.4)

Recently, Ahlgren and Andersen gave the following Rademacher-type exact formula for the smallest parts function as a conditionally convergent infinite sum of I-Bessel functions and Kloosterman sums ([AA16, Thm. 1]):

$$\operatorname{spt}(n) = \frac{\pi}{6} (24n - 1)^{\frac{1}{4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} \left( I_{\frac{1}{2}} - I_{\frac{3}{2}} \right) \left( \frac{\pi\sqrt{24n - 1}}{6c} \right).$$
(2.5)

We also have ([AA16])

$$S(n) = 2\pi (24n - 1)^{\frac{1}{4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{\frac{1}{2}} \left( \frac{\pi \sqrt{24n - 1}}{6c} \right),$$
(2.6)

and

$$\operatorname{spt}(n) = \frac{1}{12}S(n) - \frac{24n-1}{12}p(n).$$
 (2.7)

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# 3. The inner product $I_{m,n}(s,w)$

We obtain an expression for the inner product of two Poincaré series by unfolding. Let  $z = x + iy \in \mathbb{H}$  and  $s = \sigma + it \in \mathbb{C}$ . For m > 0, define the Poincaré series  $\mathcal{U}_m(z, s, \frac{1}{2}, \chi)$  by

$$\mathcal{U}_m(z,s,\frac{1}{2},\chi) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \overline{\chi(\gamma)} j(\gamma,z)^{-\frac{1}{2}} \operatorname{Im}(\gamma z)^s e(m_\chi \gamma z), \qquad \sigma > 1,$$
(3.1)

where  $e(x) = e^{2\pi i x}$ . Selberg [Sel65] proved that  $\mathcal{U}_m(z, s, \frac{1}{2}, \chi)$  has an analytic continuation to a meromorphic function. Let  $n \leq 0$  and define

$$Z_{m,n}(s) := \sum_{c>0} \frac{S(m, n, c, \chi)}{c^{2s}}.$$
(3.2)

Note that  $Z_{m,n}(s)$  converges absolutely for  $\operatorname{Re}(s) > 1$  and is analytic in this half-plane. Selberg proved that  $Z_{m,n}(s)$  has an analytic continuation to a function which is meromorphic in the whole plane.

Define the Petersson inner product by

$$\langle f,g\rangle := \int_{\Gamma \setminus \mathbb{H}} f(\tau) \overline{g(\tau)} \, \frac{dx \, dy}{y^2}.$$
 (3.3)

Let  $\operatorname{Re}(w) > 1$ . Define

$$I_{m,n}(s,w) := \left\langle \mathcal{U}_m(z,s,\frac{1}{2},\chi), \overline{\mathcal{U}_{1-n}(z,w,-\frac{1}{2},\overline{\chi})} \right\rangle$$
(3.4)

and

$$Q_{m,n}(s,w) := Z_{m,n}(s) \int_0^\infty y^{w-s-1} e^{2\pi n_\chi y} \\ \times \left( \int_{-\infty}^\infty \frac{(u+i)^{-\frac{1}{2}}}{(u^2+1)^{s-\frac{1}{4}}} e(-n_\chi y u) \left( e\left(\frac{-m_\chi}{c^2 y(u+i)}\right) - 1 \right) du \right) dy.$$
(3.5)

We have the following expansion for the inner product for  $m_{\chi} > 0$  and  $n_{\chi} < 0$ . The case  $m_{\chi} > 0$  and  $n_{\chi} > 0$  is given in [Pri00].

**Lemma 3.1.** Let  $m_{\chi} > 0$  and  $n_{\chi} < 0$ . For  $s = \sigma + it$  with  $\sigma > 1$  and  $\operatorname{Re}(w) > \sigma$  we have

$$I_{m,n}(s,w) = \frac{\Gamma(w+s-1)\Gamma(w-s)}{\Gamma\left(w+\frac{1}{4}\right)\Gamma\left(s-\frac{1}{4}\right)} (-1)^{-s-w} Z_{m,n}(s) \pi^{s-w+1} (-i)^{-\frac{1}{2}} (n_{\chi})^{s-w} 4^{-w+1} + Q_{m,n}(s,w) + Q_$$

*Proof.* From the proof of [AA18, Lemma 4.2], we have

$$I_{m,n}(s,w) = Z_{m,n}(s) \int_{-\infty}^{\infty} \left(\frac{u+i}{|u+i|}\right)^{-\frac{1}{2}} (u^2+1)^{-s} \int_{0}^{\infty} y^{w-s-1} e\left(\frac{-m_{\chi}}{c^2 y(u+i)} - n_{\chi} yu\right) e^{2\pi n_{\chi} y} \, dy du.$$
  
Therefore

Therefore,

$$\begin{split} I_{m,n}(s,w) &= Z_{m,n}(s) \int_0^\infty y^{w-s-1} e^{2\pi n_\chi y} \int_{-\infty}^\infty \frac{(u+i)^{-\frac{1}{2}}}{(u^2+1)^{s-\frac{1}{4}}} e\left(\frac{-m_\chi}{c^2 y(u+i)}\right) e(-n_\chi y u) \, du dy \\ &= Z_{m,n}(s) \int_0^\infty y^{w-s-1} e^{2\pi n_\chi y} \int_{-\infty}^\infty \frac{(u+i)^{-\frac{1}{2}}}{(u^2+1)^{s-\frac{1}{4}}} e(-n_\chi y u) \, du dy + Q_{m,n}(s,w). \end{split}$$

Next we use the formula ([GR14, 3.384 # 9])

$$\int_{-\infty}^{\infty} (b+ix)^{-2\mu} (c-ix)^{-2\nu} e^{-ipx} dx$$
$$= 2\pi (b+c)^{-\mu-\nu} \frac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)} \exp\left(\frac{b-c}{2}p\right) W_{\mu-\nu,\frac{1}{2}-\nu-\mu}(-bp-cp),$$

with parameters  $p = 2\pi n_{\chi} y$ , b = c = 1,  $\mu = \frac{s}{2} - \frac{1}{8}$ , and  $\nu = \frac{s}{2} + \frac{1}{8}$ . Here W is the Whittaker function. We obtain

$$\int_{-\infty}^{\infty} \frac{(u+i)^{-\frac{1}{2}}}{(u^2+1)^{s-\frac{1}{4}}} e(-n_{\chi}yu) \, du = \frac{(-1)^{2s}\pi(-i)^{-\frac{1}{2}}(-\pi n_{\chi}y)^{s-1}}{\Gamma\left(s-\frac{1}{4}\right)} W_{-\frac{1}{4},s-\frac{1}{2}}(-4\pi n_{\chi}y).$$

It follows that

$$\begin{split} I_{m,n}(s,w) &- Q_{m,n}(s,w) \\ &= Z_{m,n}(s)(-1)^{-2s} \frac{\pi(-i)^{-\frac{1}{2}}}{\Gamma\left(s-\frac{1}{4}\right)} (-\pi n_{\chi})^{s-1} \int_{0}^{\infty} y^{w-2} e^{2\pi n_{\chi} y} W_{-\frac{1}{4},s-\frac{1}{2}}(-4\pi n_{\chi} y) \, dy \\ &= Z_{m,n}(s)(-1)^{-2s} \frac{\pi(-i)^{-\frac{1}{2}}}{\Gamma\left(s-\frac{1}{4}\right)} (-\pi n_{\chi})^{s-1} (-4\pi n_{\chi})^{-w+1} \int_{0}^{\infty} y^{w-2} e^{-\frac{y}{2}} W_{-\frac{1}{4},s-\frac{1}{2}}(y) \, dy. \end{split}$$

Now we use  $[Obe74, \S1.13, 13.52]$  to obtain

$$\int_0^\infty y^{w-2} e^{-\frac{y}{2}} W_{-\frac{1}{4},s-\frac{1}{2}}(y) \, dy = \frac{\Gamma(w+s-1)\Gamma(w-s)}{\Gamma\left(w+\frac{1}{4}\right)}.$$

Lemma 3.1 follows.

## 4. Bounds for Q

In this section we obtain a bound for the error term  $Q_{1,n}(s, s+2)$  in Lemma 3.1. We begin with a preliminary lemma.

Lemma 4.1. Let  $\operatorname{Re}(z) \leq 0$ . Then,

$$|e^z - 1| \le 1.682 |z|^{\frac{1}{4}}.$$

*Proof.* Let  $f(z) = \frac{e^z - 1}{z^{\frac{1}{4}}}$ , f(0) = 0. On  $\operatorname{Re}(z) = 0$  we have

$$|f(z)| = \frac{2|\sin(y/2)|}{|y^{\frac{1}{4}}|} \le \max\left(\frac{2}{|y|^{\frac{1}{4}}}, \frac{|y|}{|y|^{\frac{1}{4}}}\right).$$

Thus,  $|f(z)| \leq 2^{\frac{3}{4}}$  on  $\operatorname{Re}(z) = 0$ . Since the same bound holds trivially on  $\operatorname{Re}(z) = n$  with  $n \leq -2$ , the result follows by the Phragmén-Lindelöf principle.

**Lemma 4.2.** Let  $s = \sigma + it$  with  $\sigma = 1/2 + \delta/2$  and  $0 < \delta \le 1/4$ , and let  $n \le 0$  be an integer. Then

$$|Q_{1,n}(s,s+2)| \le 0.414\zeta^2(1+\delta)|n_{\chi}|^{-\frac{\gamma}{4}}.$$

*Proof.* From (3.5) we have

$$|Q_{1,n}(s,s+2)| \le |Z_{1,n}(s)| \int_0^\infty y e^{2\pi n_\chi y} \int_{-\infty}^\infty \frac{1}{(u^2+1)^\sigma} \left| e\left(\frac{-1}{24c^2 y(u+i)}\right) - 1 \right| du dy.$$

By Lemma 4.1 we obtain

$$\begin{aligned} |Q_{1,n}(s,s+2)| &\leq 1.682 |Z_{1,n}(s)| \int_0^\infty y e^{2\pi n_\chi y} \left(\frac{\pi}{12c^2 y}\right)^{\frac{1}{4}} \int_{-\infty}^\infty \frac{1}{(u^2+1)^{\sigma+\frac{1}{8}}} \, du dy \\ &\leq 2.136 \, \frac{\Gamma\left(\sigma-\frac{3}{8}\right)}{\Gamma\left(\sigma+\frac{1}{8}\right)} \sum_{c>0} \frac{|S(1,n,c,\chi)|}{c^{2\sigma+\frac{1}{2}}} \int_0^\infty y^{\frac{3}{4}} e^{2\pi n_\chi y} \, dy, \end{aligned}$$

where in the last line we use [GR14, 3.251 # 2]. Note that

$$\int_0^\infty y^{\frac{3}{4}} e^{2\pi n_\chi y} \, dy = (-2\pi n_\chi)^{-\frac{7}{4}} \, \Gamma\left(\frac{7}{4}\right).$$

Using (1.9) and the fact that  $\frac{\Gamma(\delta/2+1/8)}{\Gamma(\delta/2+5/8)}$  is decreasing in this range of  $\delta$ , we get

$$|Q_{1,n}(s,s+2)| \le 2.136 \sum_{c>0} \frac{\tau(c)}{c^{1+\delta}} (-2\pi n_{\chi})^{-\frac{7}{4}} \frac{\Gamma\left(\frac{7}{4}\right) \Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{5}{8}\right)} \le 0.414 \zeta^2 (1+\delta) |n_{\chi}|^{-\frac{7}{4}}.$$

# 5. Bounds on $U_m$

In this section we give a bound for the norm of the Poincaré series. The proofs use similar techniques as in [Yos11, Lemma 3.2]. We will need the following bounds for the K-Bessel function.

Lemma 5.1. For y > 0 we have

$$K_0(y) < 0.975y^{-\frac{1}{2}} \tag{5.1}$$

and

$$K_0(y) < 1.7y^{-\frac{7}{2}}. (5.2)$$

*Proof.* From [Luk72, (6.28)] we have

$$K_0(y) < \frac{\sqrt{\frac{\pi}{2}}(16y+7)}{e^y y^{\frac{1}{2}}(16y+9)}$$

for y > 0. Since  $\frac{\sqrt{\pi}(16y+7)}{\sqrt{2}e^{y}(16y+9)} < \frac{7\sqrt{\pi}}{9\sqrt{2}}$ , the first inequality follows. To obtain the second inequality, we use that  $e^{y} > 0.74y^{3}$ . Then

$$K_0(y) < \frac{\sqrt{\pi}(16y+7)}{0.74\sqrt{2}y^{\frac{7}{2}}(16y+9)} < \frac{\sqrt{\pi}}{0.74\sqrt{2}y^{\frac{7}{2}}} < 1.7y^{-\frac{7}{2}}.$$

We will also need the following integral representation of  $K_0$  ([DLMF, (10.32.10)]):

$$K_0(z) = \frac{1}{2} \int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right) \frac{dt}{t}.$$
 (5.3)

**Proposition 5.2.** For  $s = \sigma + it$  with  $\sigma = 1/2 + \delta/2$  and  $0 < \delta \le 1/4$ , we have  $||U_1(z, s+1, 1/2, \chi)|| \le 4.73.$ 

*Proof.* Unfolding as in the proof of [Pri00, Lemma 1] we have

$$\begin{aligned} \|U_1(z,s+1,1/2,\chi)\|^2 &= \left(\frac{\pi}{6}\right)^{-1-2\sigma} \Gamma(2\sigma+1) + \sum_{c>0} \frac{S(1,1,c,\chi)}{c^{2s+2}} \\ &\times \int_0^\infty \int_{-\infty}^\infty \frac{y^{-2it-1}}{(x^2+1)^{s+1}} \left[\frac{x+i}{(x^2+1)^{\frac{1}{2}}}\right]^{-\frac{1}{2}} e\left(\frac{-1}{24yc^2(x+i)} - \frac{y(x-i)}{24}\right) \, dxdy. \end{aligned}$$

Taking absolute values we see that

If absolute values we see that  

$$\|U_1(z,s+1,1/2,\chi)\|^2 \le 4.85 + \sum_{c>0} \frac{|S(1,1,c,\chi)|}{c^{2\sigma+2}} \times \int_0^\infty \int_{-\infty}^\infty \frac{y^{-1}}{(x^2+1)^{\sigma+1}} \exp\left(\frac{-\pi y}{12} - \frac{\pi}{12yc^2(x^2+1)}\right) dxdy$$

$$\le 2\int_{-\infty}^\infty \frac{1}{(x^2+1)^{\sigma+1}} K_0\left(\frac{\pi}{6c(x^2+1)^{\frac{1}{2}}}\right) dx,$$

where in the last inequality we used (5.3). Using Lemma 5.1 we obtain

$$||U_1(z,s+1,1/2,\chi)||^2 \le 4.85 + 6.46 \sum_{c>0} \frac{|S(1,1,c,\chi)|}{c^{2\sigma+2}} c^{\frac{1}{2}}.$$

The result follows by (1.6) and (1.9).

**Proposition 5.3.** Let  $s = \sigma + it$  with  $\sigma = 1/2 + \delta/2$  and  $0 < \delta \leq 1/4$ . For any integer  $n \leq 0$  we have

$$||U_{1-n}(z,s+2,-1/2,\overline{\chi})|| \le 0.156\zeta(1+\delta)\tau((1-24n)^2)|n_{\chi}|^{-\frac{7}{4}}.$$

*Proof.* Recalling that  $(1-n)_{\overline{\chi}} = -n_{\chi} = |n_{\chi}|$ , we have

$$\begin{aligned} \|U_{1-n}(z,s+2,-1/2,\overline{\chi})\|^2 &= (4\pi|n_{\chi}|)^{-3-2\sigma}\Gamma(2\sigma+3) + \sum_{c>0} \frac{S(1-n,1-n,c,\overline{\chi})}{c^{2s+4}} \\ &\times \int_0^\infty \int_{-\infty}^\infty \frac{y^{-2it-1}}{(x^2+1)^{s+2}} \left[\frac{x+i}{(x^2+1)^{\frac{1}{2}}}\right]^{\frac{1}{2}} e\left(\frac{-|n_{\chi}|}{yc^2(x+i)} - |n_{\chi}| \cdot y(x-i)\right) \, dxdy. \end{aligned}$$

Since  $|n_{\chi}| \geq 23/24$ , we see that the absolute value is bounded by

$$\frac{3}{128\pi^4} |n_{\chi}|^{-4} + \sum_{c>0} \frac{|S(1-n,1-n,c,\overline{\chi})|}{c^{2\sigma+4}} \int_0^\infty \int_{-\infty}^\infty \frac{y^{-1}}{(x^2+1)^{\sigma+2}} \\ \times \exp(2\pi n_{\chi} y) \exp\left(\frac{2\pi n_{\chi}}{yc^2(x^2+1)}\right) dxdy.$$
(5.4)

By (5.3), the double integral in (5.4) becomes

$$2\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{\sigma+2}} K_0\left(\frac{4\pi|n_{\chi}|}{c(x^2+1)^{\frac{1}{2}}}\right) dx.$$
(5.5)

Using Lemma 5.1 and estimating with  $\sigma = 1/2$ , we see that (5.5) is bounded by 0.0026  $\left(\frac{c}{|n_{\chi}|}\right)^{\frac{1}{2}}$ . Thus,

$$\|U_{1-n}(z,s+2,-1/2,\overline{\chi})\|^2 \le \frac{3}{128\pi^4} |n_{\chi}|^{-4} + 0.0026 |n_{\chi}|^{-\frac{7}{2}} \sum_{c>0} \frac{|S(1-n,1-n,c,\chi)|}{c^{\frac{3}{2}+\delta}}.$$
 (5.6)

From an argument as in [IK04, page 413] using [AA18, (2.27), (2.29)] we get

$$\sum_{c=1}^{\infty} \frac{|S(1-n,1-n,c,\chi)|}{c^{\frac{3}{2}+\delta}} \le \frac{16}{\sqrt{3}} \zeta^2 (1+\delta)\tau ((1-24n)^2)^2.$$

Note that  $|n_{\chi}| \geq 23/24$ . From this and (5.6) we obtain

$$||U_{1-n}(z,s+2,-1/2,\overline{\chi})||^{2} \leq \frac{3}{128\pi^{4}}|n_{\chi}|^{-4} + 0.0241|n_{\chi}|^{-\frac{7}{2}}\zeta^{2}(1+\delta)\tau((1-24n)^{2})^{2}$$
$$\leq 0.0242|n_{\chi}|^{-\frac{7}{2}}\zeta^{2}(1+\delta)\tau((1-24n)^{2})^{2}.$$

Proposition 5.3 follows.

# 6. Bounds for the Kloosterman zeta function

Now we can give a bound for the Kloosterman zeta function. Pribitkin's main theorem [Pri00] is an ineffective version of this result valid in more generality. In this section we will prove the following theorem.

**Theorem 6.1.** Let  $n \leq 0$  and  $s = \sigma + it$  with  $\sigma = 1/2 + \delta/2$  and  $0 < \delta \leq 1/4$ . Then

$$|Z_{1,n}(s)| \le 189.91\zeta^2(1+\delta)\tau((1-24n)^2)(1+|t|)^{\frac{1}{2}}|n_{\chi}|^{\frac{1}{4}}.$$

The proof of Theorem 6.1 requires some preliminary results. Let  $\mathcal{L}_{\frac{1}{2}}(N,\chi)$  denote the  $L^2$ -space of automorphic functions with respect to the Petersson inner product given by (3.3). Define  $\Delta_{\frac{1}{2}} := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{iy}{2}\frac{\partial}{\partial x}$ . Then  $\Delta_{\frac{1}{2}}$  has a unique self-adjoint extension to  $\mathcal{L}_{\frac{1}{2}}(N,\chi)$ . Denote by  $\lambda_0(1/2) \leq \lambda_1(1/2) \leq \cdots$  the discrete spectrum of  $\Delta_{\frac{1}{2}}$ . From [Sar84, Prop. 1.2] we have  $\lambda_0(1/2) = 3/16$ .

By Lemma 3.1 we find that

$$Z_{1,n}(s) = (I_{1,n}(s,w) - Q_{1,n}(s,w)) \frac{\Gamma\left(w + \frac{1}{4}\right)\Gamma\left(s - \frac{1}{4}\right)}{\Gamma(w + s - 1)\Gamma(w - s)} (-1)^{s+w} \pi^{w-s-1} (-i)^{\frac{1}{2}} 4^{w-1} |n_{\chi}|^{w-s}.$$

Now let w = s + 2. Then

$$|Z_{1,n}(s)| \le (|I_{1,n}(s,s+2)| + |Q_{1,n}(s,s+2)|) \frac{|\Gamma\left(s+\frac{9}{4}\right)\Gamma\left(s-\frac{1}{4}\right)|}{|\Gamma(2s+1)|} 4^{\sigma+1}\pi |n_{\chi}|^{2}.$$
(6.1)

We have the following bound for  $|I_{1,n}(s, s+2)|$ .

**Proposition 6.2.** Let  $s = \sigma + it = 1/2 + \delta/2 + it$  with  $0 < \delta \le 1/4$ . Then

$$|I_{1,n}(s,s+2)| \le 0.674\tau ((1-24n)^2)\zeta^2 (1+\delta)|n_{\chi}|^{-\frac{\gamma}{4}}.$$

*Proof.* From [GS83, (2.4)] and [Sar90, (A.2.9)] we see that

$$\begin{aligned} |I_{1,n}(s,s+2)| &= |\langle \mathcal{U}_1(z,s,1/2,\chi), \overline{\mathcal{U}_{1-n}(z,s+2,-1/2,\overline{\chi})} \rangle| \\ &\leq (\pi/6)|s-1/4||R_{s(1-s)}| \|\mathcal{U}_1(z,s+1,1/2,\chi)\| \|\mathcal{U}_{1-n}(z,s+2,-1/2,\overline{\chi})\| \\ &\leq \frac{(\pi/6)|s-1/4|}{\operatorname{distance}(s(1-s),\operatorname{spectrum}(\Delta_{\frac{1}{2}}))} \|\mathcal{U}_1(z,s+1,1/2,\chi)\| \|\mathcal{U}_{1-n}(z,s+2,-1/2,\overline{\chi})\|, \end{aligned}$$

where  $R_{s(1-s)} = (\Delta_{\frac{1}{2}} + s(1-s))^{-1}$  is the resolvent of  $\Delta_{\frac{1}{2}}$ . For |t| > 1 we see that  $|s - 1/4| \le 1.07|t|$  and as in [GS83],

$$\operatorname{dist}(s(1-s),\operatorname{spectrum}(\Delta_{\frac{1}{2}})) \ge |t(2\sigma-1)|.$$

Also we have  $\zeta(1+\delta)/\delta \leq \zeta^2(1+\delta)$ , so Propositions 5.2 and 5.3 give us

$$|I_{1,n}(s,s+2)| \le 0.414\tau((1-24n)^2)\zeta^2(1+\delta)|n_{\chi}|^{-\frac{7}{4}}.$$

For  $|t| \leq 1$  we see that  $\text{Re}(s(1-s)) \leq 5/4$ . Since  $\lambda_1(1/2) > 3.86$  ([AA18, Corollary 5.3]) and  $\lambda_0(1/2) = 3/16$  we have

distance
$$(s(1-s), \operatorname{spectrum}(\Delta_{\frac{1}{2}})) = |s(1-s) - \frac{3}{16}| = |s - 1/4| |s - 3/4| \ge \frac{1}{8}|s - 1/4|.$$

Since  $\zeta(1+\delta) \ge \zeta(\frac{5}{4}) \ge 4.59$ , for these values of t we have

$$|I_{1,n}(s,s+2)| \le 0.674\tau ((1-24n)^2)\zeta^2 (1+\delta)|n_{\chi}|^{-\frac{7}{4}}.$$

To obtain a bound for the  $\Gamma$  functions appearing in (6.1), we use the following lemma.

**Lemma 6.3** ([Rad73], §34, Thm. A). Let  $0 \le c \le 1$ . Then for  $\text{Re}(s) \ge (1-c)/2$  we have

$$\left|\frac{\Gamma(s+c)}{\Gamma(s)}\right| \le |s|^c$$

**Proposition 6.4.** Let  $s = \sigma + it = 1/2 + \delta/2 + it$  with  $0 < \delta \le 1/4$ . Then

$$\frac{\Gamma\left(s+\frac{9}{4}\right)\Gamma\left(s-\frac{1}{4}\right)|}{|\Gamma(2s+1)|} \le 5.84(1+|t|)^{\frac{1}{2}}.$$

*Proof.* Apply the duplication formula to obtain

$$|\Gamma(2s+1)| = |\Gamma(s+1/2)\Gamma(s+1)|\pi^{-\frac{1}{2}}4^{\sigma}.$$

Use the functional equation to obtain

$$\Gamma\left(s-\frac{1}{4}\right) = \left(s-\frac{1}{4}\right)^{-1}\Gamma\left(s+\frac{3}{4}\right),$$
  
$$\Gamma\left(s+\frac{9}{4}\right) = \left(s+\frac{5}{4}\right)\Gamma\left(s+\frac{5}{4}\right).$$

Then by Lemma 6.3,

$$\frac{\left|\Gamma(s+\frac{9}{4})\Gamma\left(s-\frac{1}{4}\right)\right|}{\left|\Gamma(2s+1)\right|} = \pi^{\frac{1}{2}}4^{-\sigma}\frac{\left|s+\frac{5}{4}\right|\left|\Gamma\left(s+\frac{5}{4}\right)\Gamma\left(s+\frac{3}{4}\right)\right|}{\left|s-\frac{1}{4}\right|\left|\Gamma(s+1)\right|\Gamma\left(s+\frac{1}{2}\right)\right|}$$
$$\leq \pi^{\frac{1}{2}}4^{-\frac{1}{2}}\frac{\left|s+\frac{5}{4}\right|\left|s+1\right|^{\frac{1}{4}}\left|s+\frac{1}{2}\right|^{\frac{1}{4}}}{\left|s-\frac{1}{4}\right|}.$$

If 
$$|t| \leq 1$$
 we see that  $\frac{|\Gamma(s+\frac{9}{4})\Gamma(s-\frac{1}{4})|}{|\Gamma(2s+1)|} \leq \frac{|\Gamma(\frac{11}{4})\Gamma(\frac{1}{4})|}{|\Gamma(2)|} \leq 5.84$ . If  $|t| \geq 1$  we have

$$\frac{\left|\Gamma\left(s+\frac{9}{4}\right)\Gamma\left(s-\frac{1}{4}\right)\right|}{\left|\Gamma(2s+1)\right|} \le \pi^{\frac{1}{2}}4^{-\frac{1}{2}}\frac{\left|s+\frac{5}{4}\right|^{\frac{3}{2}}}{\left|s-\frac{1}{4}\right|} \le \pi^{\frac{1}{2}}4^{-\frac{1}{2}}\left|1+\frac{6}{4s-1}\right|^{\frac{3}{2}}\left|s-\frac{1}{4}\right|^{\frac{1}{2}} \le 2.51|t|^{\frac{1}{2}}.$$
proposition follows.

The proposition follows.

Proof of Theorem 6.1. The theorem follows from Lemma 4.2, (6.1) and Propositions 6.2 and 6.4. 

# 7. Proof of Theorem 1.6

We use Perron's formula as in [Dav80, §17]. Let  $f(s) = Z_{1,n}\left(\frac{1+s}{2}\right)$ . We see that

$$\sum_{c \le x} \frac{S(1, n, c, \chi)}{c} = \frac{1}{2\pi i} \int_{v - i\infty}^{v + i\infty} f(s) \frac{x^s}{s} ds,$$
(7.1)

where v > 1/2. Now for T > 0 and  $x \in \mathbb{Z} + 1/2$  we have

$$\left| \sum_{c \le x} \frac{S(1, n, c, \chi)}{c} - \frac{1}{2\pi i} \int_{v-iT}^{v+iT} f(s) \frac{x^s}{s} ds \right| = \frac{1}{2\pi} \left| \int_{v-i\infty}^{v+i\infty} f(s) \frac{x^s}{s} ds - \int_{v-iT}^{v+iT} f(s) \frac{x^s}{s} ds \right|$$
$$< \sum_{c=1}^{\infty} \frac{|S(1, n, c, \chi)|}{c} \left(\frac{x}{c}\right)^v \min\left(1, T^{-1} \left|\log \frac{x}{c}\right|^{-1}\right). \quad (7.2)$$

Let  $v = 1/2 + \delta$  where  $\delta$  is as in Theorem 1.6. Then

$$\sum_{c=1}^{\infty} \frac{|S(1,n,c,\chi)|}{c} \left(\frac{x}{c}\right)^v \min(1,T^{-1}\left|\log\frac{x}{c}\right|^{-1}) \le \frac{x^{\frac{1}{2}+\delta}}{T} \sum_{c=1}^{\infty} \frac{|S(1,n,c,\chi)|}{c^{\frac{3}{2}+\delta}} \left|\log\frac{x}{c}\right|^{-1}.$$

Now we split the sum into the ranges  $c \leq \frac{3}{4}x$ ,  $\frac{3}{4}x < c < x$ ,  $x < c < \frac{5}{4}x$ , and  $c \geq \frac{5}{4}x$ . Note that  $\frac{1}{\log x - \log c} \leq \frac{x}{x-c}$  when c < x. So for  $x \geq 10000$  we have

$$\sum_{c=\lfloor\frac{3x}{4}\rfloor+1}^{x-\frac{1}{2}} \frac{|S(1,n,c,\chi)|}{c^{\frac{3}{2}+\delta}} \left|\log\frac{x}{c}\right|^{-1} \le \sum_{c=\lfloor\frac{3x}{4}\rfloor+1}^{x-\frac{1}{2}} \frac{|S(1,n,c,\chi)|}{c^{\frac{3}{2}+\delta}} \frac{x}{x-c}$$
(7.3)  
$$\le \frac{4\ell(\delta)}{3} \int_{\frac{3x}{4}}^{x-\frac{1}{2}} \frac{dt}{x-t} + \frac{8\ell(\delta)}{3} \le 1.523\ell(\delta)\log x.$$

Similarly, for  $x \ge 10000$  we see that

$$\sum_{c=x+\frac{1}{2}}^{\left\lceil\frac{5x}{4}\right\rceil-1} \frac{|S(1,n,c,\chi)|}{c^{\frac{3}{2}+\delta}} \left|\log\frac{x}{c}\right|^{-1} \le \ell(\delta) \sum_{c=x+\frac{1}{2}}^{\left\lceil\frac{5x}{4}\right\rceil-1} \frac{1}{c-x} \le 1.142\ell(\delta)\log x.$$
(7.4)

We have

$$\sum_{c \le \frac{3x}{4}} \frac{|S(1, n, c, \chi)|}{c^{\frac{3}{2} + \delta}} \left| \log \frac{x}{c} \right|^{-1} \le \sum_{c \le \frac{3x}{4}} \frac{|S(1, n, c, \chi)|}{c^{\frac{3}{2} + \delta}} \left| \log \frac{4}{3} \right|^{-1} \le 3.5 \sum_{c \le \frac{3x}{4}} \frac{\tau(c)}{c^{1 + \delta}}$$

and

$$\sum_{c \ge \frac{5x}{4}} \frac{|S(1, n, c, \chi)|}{c^{\frac{3}{2} + \delta}} \left| \log \frac{x}{c} \right|^{-1} \le \sum_{c \ge \frac{5x}{4}} \frac{|S(1, n, c, \chi)|}{c^{\frac{3}{2} + \delta}} \left| \log \frac{4}{5} \right|^{-1} \le 4.5 \sum_{c \ge \frac{5x}{4}} \frac{\tau(c)}{c^{1 + \delta}}.$$

By (1.9) we obtain

$$\sum_{c \le \frac{3x}{4}} + \sum_{c \ge \frac{5x}{4}} \frac{|S(1, n, c, \chi)|}{c^{\frac{3}{2} + \delta}} \left| \log \frac{x}{c} \right|^{-1} \le 4.5 \sum_{c=1}^{\infty} \frac{\tau(c)}{c^{1+\delta}} = 4.5 \zeta^2 (1+\delta).$$
(7.5)

Therefore, by (7.2), (7.3), (7.4), and (7.5) we obtain

$$\left|\sum_{c \le x} \frac{S(1, n, c, \chi)}{c}\right| \le \left|\int_{\frac{1}{2} + \delta - iT}^{\frac{1}{2} + \delta + iT} f(s) \frac{x^s}{s} ds\right| + \left(4.5\zeta^2 (1+\delta) + 3\ell(\delta)\log x\right) \frac{x^{\frac{1}{2} + \delta}}{T}.$$
 (7.6)

From [GS83, (3.2)] we see that  $Z_{m,n}\left(\frac{1+s}{2}\right)$  is holomorphic for  $\operatorname{Re}(s) > 0$  (the only possible pole at s = 1/2 does not arise since  $n \leq 0$ ). Thus

$$\int_{\partial E} Z_{m,n}\left(\frac{1+s}{2}\right)\frac{x^s}{s}ds = 0,$$
(7.7)

where E is the rectangle  $[\delta, 1/2 + \delta] \times [-T, T]$ . We obtain a bound for  $Z_{1,n}\left(\frac{1+\delta+it}{2}\right)$  using Theorem 6.1 and a bound for  $Z_{1,n}\left(\frac{\frac{3}{2}+\delta+it}{2}\right)$  using the Weil bound. We require the Phragmén-Lindelöf principle for a strip.

**Proposition 7.1** ([IK04], Thm. 5.53). Let f be a function holomorphic on an open neighborhood of a strip  $a \leq \sigma \leq b$ , for some real numbers a < b, such that  $|f(s)| \ll \exp(|s|^A)$  for some  $A \geq 0$  and  $a \leq \sigma \leq b$ . Assume that

$$|f(a+it)| \le M_a(1+|t|)^{\alpha},$$
  
 $|f(b+it)| \le M_b(1+|t|)^{\beta}$ 

for  $t \in \mathbb{R}$ . Then

$$|f(\sigma+it)| \le M_a^{d(\sigma)} M_b^{1-d(\sigma)} (1+|t|)^{\alpha d(\sigma)+\beta(1-d(\sigma))}$$

for all s in the strip, where d is the linear function such that d(a) = 1, d(b) = 0.

**Proposition 7.2.** Let  $n \leq 0$  and  $f(s) = Z_{1,n}(\frac{1+s}{2})$ . For  $\delta \leq \sigma \leq 1/2 + \delta$  with  $0 < \delta \leq 1/4$  we have

$$|f(\sigma+it)| \le 189.91\zeta^2(1+\delta)\tau((1-24n)^2)|n_{\chi}|^{\frac{1}{4}}(1+|t/2|)^{-\sigma+\frac{1}{2}+\delta}.$$

*Proof.* By Theorem 6.1, we have

$$|f(\delta + it)| \le 189.91\zeta^2 (1+\delta)\tau ((1-24n)^2) |n_{\chi}|^{\frac{1}{4}} (1+|t/2|)^{\frac{1}{2}}.$$
(7.8)

Also, by (1.9)

$$\left| f\left(\frac{1}{2} + \delta + it\right) \right| = \left| \sum_{c=1}^{\infty} \frac{S(1, n, c, \chi)}{c^{\frac{3}{2} + \delta + it}} \right| \le \sum_{c=1}^{\infty} \frac{c^{\frac{1}{2}}\tau(c)}{c^{\frac{3}{2} + \delta}} = \zeta^2 (1 + \delta).$$

Note that the line d such that  $d(\delta) = 1$  and  $d(1/2 + \delta) = 0$  is  $d(\sigma) = -2\sigma + 1 + 2\delta$ . By Phragmén-Lindelöf, for  $\delta \leq \sigma \leq 1/2 + \delta$  we have

$$|f(\sigma+it)| \le \left(189.91\zeta^2 \left(1+\delta\right)\tau \left(\left(1-24n\right)^2\right)|n_{\chi}|^{\frac{1}{4}}\right)^{-2\sigma+1+2\delta} \left(\zeta^2 \left(1+\delta\right)\right)^{2\sigma-2\delta} \left(1+|t/2|\right)^{-\sigma+\frac{1}{2}+\delta} = \zeta^2 \left(1+\delta\right) \left(189.91\tau \left(\left(1-24n\right)^2\right)|n_{\chi}|^{\frac{1}{4}}\right)^{-2\sigma+1+2\delta} \left(1+|t/2|\right)^{-\sigma+\frac{1}{2}+\delta}.$$

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. For x < 10000 this follows from  $\left|\sum_{c \leq x} \frac{A_c(n)}{c}\right| \leq x$ . Let  $x \geq 10000$  and set  $T = x^{\frac{1}{3}}$ . By Proposition 7.2 we have

$$\begin{aligned} \left| \int_{\frac{1}{2}+\delta+iT}^{\delta+iT} f(s) \frac{x^s}{s} ds \right| &= \left| \int_{\delta}^{\frac{1}{2}+\delta} f(\sigma+iT) \frac{x^{\sigma+iT}}{\sigma+iT} d\sigma \right| \\ &\leq 189.91 \zeta^2 (1+\delta) \tau ((1-24n)^2) |n_{\chi}|^{\frac{1}{4}} (1+T/2)^{\frac{1}{2}+\delta} \int_{\delta}^{\frac{1}{2}+\delta} (1+T/2)^{-\sigma} \frac{x^{\sigma}}{|\sigma+iT|} d\sigma \\ &\leq \frac{189.91 \zeta^2 (1+\delta) |n_{\chi}|^{\frac{1}{4}} \tau ((1-24n)^2) (1+T/2)^{\frac{1}{2}+\delta}}{T} \cdot \frac{\left(\frac{x}{1+T/2}\right)^{\frac{1}{2}+\delta} - \left(\frac{x}{1+T/2}\right)^{\delta}}{\log\left(\frac{x}{1+T/2}\right)}. \end{aligned}$$

Disregarding the negative term and using the estimate  $\log\left(\frac{x}{1+T/2}\right) \ge 6.74$  gives

$$\left| \int_{\frac{1}{2}+\delta+iT}^{\delta+iT} f(s) \frac{x^s}{s} ds \right| \le 28.18\zeta^2 (1+\delta)\tau ((1-24n)^2) |n_{\chi}|^{\frac{1}{4}} x^{\frac{1}{6}+\delta}.$$
(7.9)

The same bound holds for the bottom of the rectangle. To estimate the integral over the left side of the rectangle we use (7.8) to see that  $\int_{\delta+iT}^{\delta-iT} f(s) \frac{x^s}{s} ds$  is bounded by

$$189.91\zeta^{2}(1+\delta)\tau((1-24n)^{2})|n_{\chi}|^{\frac{1}{4}}x^{\delta}\left(2\int_{2}^{T}\frac{(1+t/2)^{\frac{1}{2}}}{|\delta+it|}dt+2\int_{0}^{2}\frac{(1+t/2)^{\frac{1}{2}}}{|\delta+it|}dt\right).$$
 (7.10)

Using the inequality  $\int_2^T \left(\frac{1}{2t} + \frac{1}{t^2}\right)^{\frac{1}{2}} dt \leq \int_2^T \frac{1}{(2t)^{\frac{1}{2}}} dt$  we see that the first term in the expansion of (7.10) is bounded by

$$537.15\zeta^2(1+\delta)\tau((1-24n)^2)|n_{\chi}|^{\frac{1}{4}}x^{\delta}T^{\frac{1}{2}}.$$
(7.11)

Note that

$$\int_{0}^{2} \left(\frac{1+|t|/2}{\delta^{2}+t^{2}}\right)^{\frac{1}{2}} dt \leq \sqrt{2} \int_{0}^{2} \left(\frac{1}{\delta^{2}+t^{2}}\right)^{\frac{1}{2}} dt = \frac{1}{\sqrt{2}} \log\left(\delta^{2}+4\sqrt{\delta^{2}+4}+8\right) - \sqrt{2} \log\left(\delta\right).$$
  
Since  $0 < \delta \leq 1/4$  we have  $\frac{1}{\sqrt{2}} \log\left(\delta^{2}+4\sqrt{\delta^{2}+4}+8\right) \leq 1.77$ , so

$$\int_0^2 \left(\frac{1+|t|/2}{\delta^2+t^2}\right)^{\frac{1}{2}} dt \le 2.7|\log \delta|.$$

Thus, the second term in the expansion of (7.10) is bounded by

$$1025.52\zeta^2(1+\delta)\tau((1-24n)^2)|n_{\chi}|^{\frac{1}{4}}x^{\delta}|\log\delta|.$$
(7.12)

Thus (recall that  $x \ge 10000$  and  $0 < \delta \le 1/4$ ),

$$\left| \int_{\delta+iT}^{\delta-iT} f(s) \frac{x^s}{s} ds \right| \le \zeta^2 (1+\delta) \tau ((1-24n)^2) |n_{\chi}|^{\frac{1}{4}} \left( 537.15x^{\frac{1}{6}+\delta} + 1025.52x^{\delta} |\log \delta| \right) \le 608.42 \zeta^2 (1+\delta) \tau ((1-24n)^2) |\log \delta| |n_{\chi}|^{\frac{1}{4}} x^{\frac{1}{6}+\delta}.$$
(7.13)

Hence, using (7.7), (7.9), and (7.13) we get

$$\left| \int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT} f(s) \frac{x^s}{s} ds \right| \le 649.08\zeta^2 (1+\delta)\tau ((1-24n)^2) |\log \delta| |n_{\chi}|^{\frac{1}{4}} x^{\frac{1}{6}+\delta}.$$
(7.14)

The result follows from (2.4) and (7.6).

# 8. Proof of Theorem 1.2

We begin with a lemma.

**Lemma 8.1.** Let  $y \ge 2$  and  $q \le 1$ . Then

$$\int_{y}^{\lambda(n)} e^{\frac{\lambda(n)}{t}} t^{-\frac{3}{2}+q} dt \le \frac{4}{3-2q} \lambda(n)^{-1} y^{1/2+q} e^{\frac{\lambda(n)}{y}}.$$

*Proof.* Using the Taylor expansion for  $e^{\frac{\lambda(n)}{t}}$  we have

$$\int_{y}^{\lambda(n)} e^{\frac{\lambda(n)}{t}} t^{-\frac{3}{2}+q} dt \le \lambda(n)^{-1} y^{1/2+q} \sum_{m=1}^{\infty} \frac{\left(\frac{\lambda(n)}{y}\right)^{m+1}}{(m+1)!} \frac{4}{3-2q} \le \frac{4}{3-2q} \lambda(n)^{-1} y^{1/2+q} e^{\frac{\lambda(n)}{y}}.$$

Next is the proof of Theorem 1.5.

Proof of Theorem 1.5. By partial summation and (2.6), we have

$$S(n) = \sqrt{24\pi\lambda}(n)^{\frac{1}{2}} \left( I_{\frac{1}{2}}(\lambda(n)) - I_{\frac{1}{2}}\left(\frac{\lambda(n)}{2}\right) - \int_{2}^{\infty} \left( I_{\frac{1}{2}}\left(\frac{\lambda(n)}{t}\right) \right)' \sum_{c \le t} \frac{A_{c}(n)}{c} dt \right).$$
(8.1)

Note that the convergence of the integral in (8.1) follows from Theorem 1.6. From (5.4) of [AA16] we have

$$\left| \left( I_{\frac{1}{2}} \left( \frac{\lambda(n)}{x} \right) \right)' \right| = \frac{\lambda(n)}{2x^2} \left( I_{-\frac{1}{2}} \left( \frac{\lambda(n)}{x} \right) + I_{\frac{3}{2}} \left( \frac{\lambda(n)}{x} \right) \right),$$

 $\mathbf{SO}$ 

$$\left| \int_{2}^{\infty} \left( I_{\frac{1}{2}} \left( \frac{\lambda(n)}{t} \right) \right)' \sum_{c \leq t} \frac{A_{c}(n)}{c} dt \right|$$
$$\leq \int_{2}^{\infty} \frac{\lambda(n)}{2t^{2}} \left( I_{-\frac{1}{2}} \left( \frac{\lambda(n)}{t} \right) + I_{\frac{3}{2}} \left( \frac{\lambda(n)}{t} \right) \right) \left| \sum_{c \leq t} \frac{A_{c}(n)}{c} \right| dt. \quad (8.2)$$

Let  $f(x) = \left(I_{-\frac{1}{2}}(x) + I_{\frac{3}{2}}(x)\right) x^{\frac{1}{2}} e^x$ . We have f'(x) > 0, so  $f(x) \le f(1) < 4.146$  for  $x \le 1$  and

$$I_{-\frac{1}{2}}(x) + I_{\frac{3}{2}}(x) < 4.146x^{-\frac{1}{2}}e^{-x} \quad \text{for } x \le 1.$$
(8.3)

Let  $g(x) = \left(I_{-\frac{1}{2}}(x) + I_{\frac{3}{2}}(x)\right) x^{\frac{1}{2}} e^{-x}$ . We have g'(x) > 0, so  $g(x) \le \lim_{x \to \infty} g(x) < 0.798$  for  $x \ge 1$ . Thus,

$$I_{-\frac{1}{2}}(x) + I_{\frac{3}{2}}(x) < 0.798x^{-\frac{1}{2}}e^x \quad \text{for } x \ge 1.$$
(8.4)

Using (8.2), (8.3), and (8.4) we see that

$$\left| \int_{2}^{\infty} \left( I_{\frac{1}{2}} \left( \frac{\lambda(n)}{t} \right) \right)' \sum_{c \leq t} \frac{A_{c}(n)}{c} dt \right|$$

$$\leq 0.399 \sqrt{\lambda(n)} \int_{2}^{\lambda(n)} e^{\frac{\lambda(n)}{t}} t^{-\frac{3}{2}} \left| \sum_{c \leq t} \frac{A_{c}(n)}{c} \right| dt + 2.073 \sqrt{\lambda(n)} \int_{\lambda(n)}^{\infty} t^{-\frac{3}{2}} \left| \sum_{c \leq t} \frac{A_{c}(n)}{c} \right| dt. \quad (8.5)$$

Using the trivial bound  $|A_c(n)| \leq c$  and Lemma 8.1 we see that the first term in (8.5) is bounded by

$$4.515 \frac{e^{\frac{\lambda(n)}{2}}}{\sqrt{\lambda(n)}}.$$
(8.6)

From Corollary 1.7 we see that the second term in (8.5) is bounded by

$$24.88 \left( 19094.8\tau ((24n-23)^2)(n-1/24)^{\frac{1}{4}} + 25.35(12+\log\lambda(n)) \right) \lambda(n)^{\frac{5}{12}}.$$
 (8.7)

We can suppose  $n \ge 10000$ . Then by (8.7) the second term in (8.5) is bounded by

$$\frac{0.001e^{\frac{\lambda(n)}{2}}}{\sqrt{\lambda(n)}}.$$
(8.8)

From (8.5), (8.6), and (8.8) we obtain

$$\left| \int_{2}^{\infty} I_{\frac{1}{2}} \left( \frac{\lambda(n)}{t} \right)' \sum_{c \le t} \frac{A_{c}(n)}{c} dt \right| \le \frac{4.516e^{\frac{\lambda(n)}{2}}}{\sqrt{\lambda(n)}}.$$

Noting that

$$|E_S(n)| \le \sqrt{24\pi\lambda} \langle n \rangle^{\frac{1}{2}} \left( \left| I_{\frac{1}{2}} \left( \frac{\lambda(n)}{2} \right) \right| + \left| \int_2^\infty I_{\frac{1}{2}} \left( \frac{\lambda(n)}{t} \right)' \sum_{c \le t} \frac{A_c(n)}{c} dt \right| \right),$$

the Theorem follows for  $\lambda(n) > 256$ . For  $\lambda(n) \le 256$  it can be verified by direct computation.

We have the following result for p(n).

**Lemma 8.2.** For  $n \ge 1$  we have

$$p(n) = \frac{2\sqrt{3}}{24n - 1} \left(1 - \frac{1}{\lambda(n)}\right) e^{\lambda(n)} + E_p(n)$$

where

$$E_p(n) \le \frac{5e^{\frac{\lambda(n)}{2}}}{24n-1}.$$

*Proof.* For  $\lambda(n) > 573$  this follows from [Leh38, (4.14)]. For  $\lambda(n) \leq 573$  it can be verified by direct computation.

*Proof of Theorem 1.2.* The theorem follows by (2.7), Theorem 1.5, and Lemma 8.2.

*Proof of Theorem 1.3.* The proof is similar to the proof of Theorem 1.2.

Proof of Corollary 1.4. From Lemma 8.2 we see that

$$\frac{\sqrt{24n-1}}{2\pi}p(n) = \frac{\sqrt{3}}{\pi\sqrt{24n-1}}e^{\lambda(n)} - \frac{\sqrt{3}}{\pi\sqrt{24n-1}} \cdot \frac{e^{\lambda(n)}}{\lambda(n)} + \frac{\sqrt{24n-1}}{2\pi}E_p(n).$$

By Lemma 8.2 and Theorem 1.2 the result follows.

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