UIUC Math 580 Comprehensive Exam, January 2020

Each problem below is worth 10 points. The exam will be scored out of 100. A good performance in both Part 1 and Part 2 is required to pass this exam.

Answers without a proof or with an incorrect proof may not receive no points.

Part 1

1. Give a combinatorial (counting) argument that proves

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1.$$

2. Let $c(m,n)$ be the number of surjective functions $f : [n] \to [m]$ where $[j] := \{1, 2, 3, 4, \ldots, j\}$. By the inclusion-exclusion principle, or otherwise, determine an explicit formula for $c(m,n)$.

3. Let F_k be the adjusted Fibonacci number, i.e.,

$$(F_1, F_2, F_3, F_4, F_5, \ldots) = (1, 1, 2, 3, 5, \ldots),$$

and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. Evaluate (with proof)

$$\sum_{k=0}^{n} F_{k+1} \binom{n}{k}.$$

4. By a combinatorial argument involving integer partitions/Ferrers diagrams, prove the following identity of ordinary generating series

$$\prod_{k=1}^{\infty} 1 + x^k = \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{t=1}^{j}(1 - x^{2t})}.$$

5. Let K_n be the *complete graph* on n vertices. A collection $\mathcal{C}$ of (undirected) paths *covers* K_n if each vertex is in exactly one of the paths. Prove that the number of ways to cover K_n, by paths with at least one vertex, is

$$\left[\frac{x^n}{n!}\right] \exp\left(\frac{x(2-x)}{2(1-x)}\right).$$

Part 2

1. Prove that a graph $G = (V, E)$ with at least 4 vertices is 3-connected if and only if for arbitrary three vertices $x, y, z \in V$ and any edge e disjoint from z there exists an x, y-path that passes through e and avoids z.

2. In a plane 3-regular connected graph G, every vertex belongs to two faces of length 4 and to one face of length 36. How many edges does G have?

3. Let $k \geq 4$ and $n - k \geq 3$. How few edges may have an n-vertex graph G with chromatic number k and minimum degree at least 2?

4. State the Szemerédi Regularity Lemma. This means that you need to define all notions used, such as ϵ-regular partitions, etc.

5. For each statement below, determine if it is true or false. If it is true, give a proof. If it is false, provide a counterexample.
a) Every 4-regular 4-connected simple graph has a perfect matching;
b) Every 3-regular 2-connected graph has a perfect matching;
c) Every 3-regular 3-connected graph has three disjoint perfect matchings;
d) Every 3-regular bipartite graph has three disjoint perfect matchings.