Logic Comprehensive Exam (Math 570), January 27, 2021

Do all four problems. Explain your answers except when asked to "indicate" something. The four problems have equal weight. Throughout: m, n range over $\mathbb{N} := \{0, 1, 2, 3, \dots\}$; L is a language; given a set Σ of L-sentences, $\mathrm{Th}(\Sigma)$ is the set of L-sentences σ such that $\Sigma \vdash \sigma$; for an L-structure $\mathcal{A}$, $\mathrm{Th}(\mathcal{A})$ is the set of L-sentences true in $\mathcal{A}$; *computable* has the same meaning as *recursive*, and *computably generated* the same as *recursively enumerable* (for those used to other terminology).

1. Let L have just the binary relation symbol $<$. Let σ be the sentence $\forall x \exists y (x < y)$.

(i) Indicate a finite set Σ of L-sentences whose models are exactly the (nonempty) totally ordered sets $(A; <)$. Here "ordered" is taken in the strict sense where $a < b$ implies $a \neq b$.

(ii) Show that σ is not Σ-equivalent to any existential L-sentence.

(iii) Show that σ is not Σ-equivalent to any universal L-sentence.

2. Let L have just the unary relation symbol P.

(i) Indicate a set Σ of L-sentences whose models are exactly the L-structures $\mathcal{A} = (A; P)$ such that $P \subseteq A$ is infinite.

(ii) Determine the countable models of Σ up to isomorphism.

(iii) Show that Σ is not complete.

(iv) Indicate a family $(\Sigma_i)_{i \in I}$ where each $\Sigma_i \supseteq \Sigma$ is a complete set of L-sentences and every model of Σ is a model of Σ_i for exactly one $i \in I$.

(v) Show that $\mathrm{Th}(\Sigma)$ is decidable. (You can argue informally using "decidable" intuitively.)

3. Let $\mathcal{N} = (\mathbb{N}; <, 0, S, +, \cdot)$ be the standard model of arithmetic. Let PA be the usual set of axioms of (first-order) Peano Arithmetic; recall that PA includes an induction scheme.

(i) $\mathcal{A} \equiv \mathcal{N}$ for all $\mathcal{A} \models \mathrm{PA}$. True or false?

(ii) Is there a model $\mathcal{A}$ of PA such that $\mathrm{Th}(\mathcal{A})$ is decidable?

(iii) Show that there is a countable model $\mathcal{A} = (A; <, \dots)$ of PA with an element $a \in A$ such that $n < a$ and $a \in nA$ for all n; here $\mathbb{N}$ is identified with its image in A via the embedding $n \mapsto (S^n 0)^{\mathcal{A}} : \mathcal{N} \to \mathcal{A}$ and $nA := \{n \cdot a : \ a \in A\}$.

(iv) Let $\mathcal{A}$ be as in (iii). Show that the subset $\mathbb{N}$ of A is not definable in $\mathcal{A}$.

4. Let $f, g : \mathbb{N} \to \mathbb{N}$ be computable such that f is injective, $f(\mathbb{N})$ is computable, and $f(n) \leq g(n)$ for all n.

(i) Show that $g(\mathbb{N})$ is computable. (You can argue informally using "computable" intuitively.)

Let $A, B \subseteq \mathbb{N}$. (Continued on other side.)

(ii) Show that if A, B are computably generated, then there are disjoint computably generated sets $A^* \subseteq A$ and $B^* \subseteq B$ such that $A^* \cup B^* = A \cup B$.

(iii) Suppose $A \cap B = \emptyset$ and the complements of A and B are computably generated. Use (ii) to show there is a computable set $S \subseteq \mathbb{N}$ such that $A \subseteq S$ and $S \cap B = \emptyset$.