Probability Comprehensive Exam, August 2020

1. (20 points) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed positive random variables with $P(X_1 > x) = e^{-x}$ for all $x \geq 0$. Show that

$$\limsup_{n\to\infty} \frac{X_n - \ln n}{\ln \ln n} = 1$$

almost surely.

2. (20 points) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $EX_1 = 0$ and $EX_1^2 = 1$. Show that

$$\frac{\sqrt{n}\sum_{m=1}^{n} X_m}{\sum_{m=1}^{n} X_m^2}$$

converges weakly to a standard normal random variable.

3. (20 points) Let $Y_1, Y_2, \ldots$ be nonnegative independent and identically distributed random variables with $E(Y_1) = 1$ and $P(Y_1 = 1) < 1$. Put $\mathcal{F}_n = \sigma\{Y_1, \ldots, Y_n\}$. (i) Show that $X_n = \prod_{m \leq n} Y_m$ is a martingale with respect to $\mathcal{F}_n$. (ii) Show that X_n converges to zero almost surely as $n \to \infty$.

4. (20 points) Let $\xi_{i,n}$, $i, n \geq 0$ be independent identically distributed non-negative *integer* valued random variables with a common expectation μ and common variance $\sigma^2 \in (0, \infty)$. Define a sequence $Z_n, n \geq 0$ by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} \xi_{1,n+1} + \cdots + \xi_{Z_n, n+1}, & Z_n > 0 \\ 0, & Z_n = 0. \end{cases}$$

Put $\mathcal{F}_n = \sigma(\xi_{i,m} : i \geq 1, 1 \leq m \leq n)$ and $X_n = \frac{Z_n}{\mu^n}$. (a) Show that X_n is a martingale with respect to $\mathcal{F}_n$. (b) Show that if $\mu \leq 1$, then $Z_n = 0$ for all n sufficiently large. (c) Show that

$$EX_n^2 = 1 + \sigma^2 \sum_{k=2}^{n+1} \mu^{-k}$$

by proving the identities

$$E(X_n^2 | \mathcal{F}_{n-1}) = X_{n-1}^2 + E((X_n - X_{n-1})^2 | \mathcal{F}_{n-1})$$

and

$$E((X_n - X_{n-1})^2 | \mathcal{F}_{n-1}) = \sigma^2 \mu^{-2n} Z_{n-1}.$$

(d) Show that, if $\mu > 1$, X_n converges in L^2.

5. (20 points) Suppose that X_n is a nonnegative submartingale with respect to a filtration $\mathcal{F}_n$. Show that for any $a > 0$ and any positive integer N we have the following Doob's inequality

$$P(\max_{1 \leq n \leq N} X_n \geq a) \leq \frac{1}{a} \int_{\{\max_{1 \leq n \leq N} X_n \geq a\}} X_N dP.$$