Math 542 **Comprehensive Examination** **January 2021**

Instructions. Do all problems. Each problem is worth 10 points. Justify all your answers. Good Luck!

Notation:
We denote the set of complex numbers by $\mathbb{C}$.

1. Recall $\cosh z = \frac{1}{2}(e^z + e^{-z})$ and $\sinh z = \frac{1}{2}(e^z - e^{-z})$. Find all complex solutions $z = x + iy$ of the inequality
$$|\cosh z|^2 - |\sinh z|^2 \geq 1.$$

2. Evaluate
$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)\,dx}{x(x^2 - 2x + 2)}.$$

3. Find a conformal mapping of the domain $D = \{z \in \mathbb{C} : |z| > 1,\ |z - i| < \sqrt{2}\}$ onto the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$.

   Hint. You can find the mapping as a composition of linear fractional transformations (Möbius transformations) and powers of z. Carefully sketch all your intermediate domains, if any.

4. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be an **injective** analytic function. Suppose $\lim_{z\to\infty} f(z) = 0$. Prove $f(z) = a/z$, where $a \in \mathbb{C}$ is a constant.

   Hint. Determine the type of the singularity of f at 0.

5. Write Log for the principal branch of the logarithm.

   (a) Prove
$$\operatorname{Log} z = -2\left[\left(\frac{1-z}{1+z}\right) + \frac{1}{3}\left(\frac{1-z}{1+z}\right)^3 + \frac{1}{5}\left(\frac{1-z}{1+z}\right)^5 + \dots\right]$$
   when $\operatorname{Re} z > 0$.

   (b) Justify that the series converges normally.

6. Let $a \in \mathbb{C}$ be a constant with $|a| \geq e$. Suppose n is a positive integer.

   (a) Prove the equation $az^n = \cosh z$ has n solutions (counting multiplicity) in the unit disk $\{z : |z| < 1\}$.

   (b) Show that if z_0 is a solution, then its multiplicity (order) cannot equal 3 or higher.