Math 542 **Comprehensive Examination** **August 2020**

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems. Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3, and 4. Each problem is worth 10 points. Justify all your answers. This is an open book exam, so you can consult your textbook but you cannot communicate with any other person. Good Luck!

Notation:
We denote the set of complex numbers by $\mathbb{C}$. We denote the set of natural numbers by $\mathbb{N}$. We write $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ for the unit disk in $\mathbb{C}$.

1. Let the function f be entire and such that for all real x, the value $f(x)$ is real and the value $f(ix)$ is purely imaginary. Prove that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

2. Showing the details of your calculations, evaluate the integral
$$\int_0^\infty \frac{\cos x \, dx}{(x^2+1)(x^2+4)}.$$

3. Let the functions f and g be analytic in a neighborhood of 0 and let $f(0) = g(0) = 0$. Prove that
$$\lim_{z \to 0} \frac{f(z)}{g(z)} = \lim_{z \to 0} \frac{f'(z)}{g'(z)}.$$
(Here the limit may be finite or infinite.)

4. Let C be an arbitrary path in $\mathbb{C}$ from 0 to 1, avoiding the points $\pm i$, and consisting of a finite number of line segments. Determine all possible values of the integral
$$\int_C \frac{dz}{1+z^2}$$
and prove that your answer is correct.

5. Let $\mathcal{F}$ consist of all functions f that are analytic in $\mathbb{D}$, and satisfy $f(\mathbb{D}) \subset \mathbb{D}$ and $f(1/2) = f'(1/2) = 0$. We define $M = \sup\{|f''(1/2)| \colon f \in \mathcal{F}\}$.

   (a) Determine M.

   (b) Does there exist $f \in \mathcal{F}$ such that $|f''(1/2)| = M$? If yes, give an explicit example of such a function f and prove that it has all the required properties. If not, prove that there is no such function f.

6. Prove that there is $N \in \mathbb{N}$ such that for every $n \geq N$ all zeros of the function

$$f_n(z) = 1 + \frac{1}{1!\,z} + \frac{1}{2!\,z^2} + \ldots + \frac{1}{n!\,z^n}$$

are contained in $\mathbb{D}$.