Math 540 Comprehensive Examination, January 2021

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Suppose g is a measurable function on $[0,1]$, and fg is integrable whenever f is integrable. Prove that g is essentially bounded.

2. Let $f:[0,1]\to\mathbb{R}$ be continuous. Evaluate $\lim_{n\to\infty}\int_0^1 x^n f(x)\,dx$.

3. Suppose (f_n) is a sequence of real-valued functions on $[0,1]$, s.t. $\lim_{n,k}\int|f_n - f_k|^2\,dm = 0$. Suppose, furthermore, that K is a continuous function on $[0,1]\times[0,1]$. For each n and $x\in[0,1]$, define $g_n(x)=\int_0^1 K(x,y)f_n(y)\,dy$. Prove that the sequence (g_n) converges uniformly on $[0,1]$.

4. Let E be a Lebesgue measurable set in $\mathbb{R}$, of positive Lebsegue measure.

(a) Prove that there exists $\delta>0$ so that for every $x\in\mathbb{R}$ with $|x|<\delta$, we have $m\big(E\cap(E+x)\big)>0$. Here $E+x=\{y+x : y\in E\}$.

(b) Show that $E-E$ contains an open (non-empty) interval. Here $E-E$ is defined by $\{x-y : x,y\in E\}$.

5. For a measurable function f on $[0,1]$, define its *non-increasing rearrangement* f^* (also on $[0,1]$) by setting

$$f^*(t)=\inf\left\{c>0 : m(\{s : |f(s)|>c\})<t\right\}.$$

(a) Prove that, for any $\lambda>0$, $m(\{t : |f(t)|>\lambda\}) = m(\{t : f^*(t)>\lambda\})$.

(b) Prove that f is integrable iff f^* is, and moreover, $\int|f|\,dm=\int f^*\,dm$.

(c) Suppose f and g belong to $L^2(m)$. Prove that $\int f^*g^*\,dm \ge \int|fg|\,dm$.

6. A set $E\subset\mathbb{R}$ is said to have Property (*) if for every $\varepsilon>0$ there exist **finitely many** open intervals $I_1,\dots,I_n$ so that $E\subset\cup_{k=1}^n I_k$, and $\sum_{k=1}^n \ell(I_k)<\varepsilon$ (here $\ell(I)$ refers to the length of the interval I).

(a) Prove that, if $E\subset[0,1]$ has Property (*), then $[0,1]\backslash E$ contains an open interval.

(b) Prove that any compact set of Lebesgue measure 0 has Property (*).