Math 540 Comprehensive Examination, January 2020

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Let I be a subset of $\mathbb{R}$. Suppose that for any $\alpha \in I$, f_α is measurable. Is $\sup\{f_\alpha(x) : \alpha \in I\}$ measurable? Justify your answer.

2. Let f, f_k be measurable functions on E with $\mu(E) < \infty$. Suppose that $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. $x \in E$. Prove that there exists a collection of subsets E_n's of E, such that f_k converges to f uniformly on each E_n, and

$$\mu\big(E \setminus \cup_{n=1}^{\infty} E_n\big) = 0\,.$$

3. Let f be differentiable on $[a, b] \subset \mathbb{R}$. Recall that the total variation of f on $[a, b]$ is given by

$$V(f) = \sup\Big\{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| : a = x_0 < x_1 < \cdots < x_n = b\Big\}\,.$$

Prove that if f' is Riemann-integrable on $[a, b]$, then

$$V(f) = \int_a^b \big|f'(x)\big| dx\,.$$

4. (i) (3 pts.) Show that $g(x) = \frac{\sin x}{x}$ is not Lebesgue integrable on $([0, \infty), m)$.
(ii) (7 pts.) Employ the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt \quad (x > 0)$$

to evaluate the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\, dx \;=\; \lim_{L\to\infty} \int_0^L \frac{\sin x}{x}\, dx.$$

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K : X \times Y \to \mathbb{C}$ be measurable with respect to the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$. Assume there is a constant $0 < C < \infty$ such that

$$\forall x \in X, \quad \int_Y |K(x, y)|\, d\nu(y) \leqslant C$$

and

$$\forall y \in Y, \quad \int_X |K(x, y)|\, d\mu(x) \leqslant C.$$

Let $p \in [1, \infty)$ and for $f \in L^p(\mu)$ define

$$(Tf)(y) := \int_X f(x)K(x,y)\, d\mu(x).$$

Prove that $Tf \in L^p(\nu)$ and

$$\|Tf\|_{L^p(\nu)} \leqslant C\|f\|_{L^p(\mu)}.$$

6. Let E be a Lebesgue measurable subset of $\mathbb{R}$. Define the function $f_E : [0, +\infty) \to \mathbb{R}$ by

$$f_E(x) = \mu(E \cap (-x, x)).$$

where μ denotes Lebesgue's measure on $\mathbb{R}$. Prove:
(a) f is a uniformly continuous function from $[0, +\infty)$ to $[0, \mu(E)]$.
(b) $\lim\limits_{x \to \infty} f_E(x) = \mu(E)$.