Math 540 Comprehensive Examination, August 2020

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Let f be a non-negative integrable function obeying

$$\int_{\mathbb{R}} |x|^n f(x)dx \leq C\,,\ \forall n \in \mathbb{N}\,,$$

where $C \in \mathbb{R}$ is a constant independent of n. Prove that $f(x) = 0$ a.e. $x \in (-\infty, -1) \cup (1, \infty)$.

2. Let $\theta \in [0,1] \setminus \mathbb{Q}$. On the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ consider the rotation R_θ given by $R_\theta(e^{2\pi it}) := e^{2\pi i(t+\theta)}$. Show that if $f \in L^2(\mathbb{T})$ is such that

$$f\big(R_\theta(e^{2\pi it})\big) = f(e^{2\pi it}) \quad \text{for almost every } t \in [0,1),$$

then f is a.e. constant.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, that is there exists $0 < M < \infty$ such that

$$|f(x) - f(y)| \ \leqslant\ M|x-y|, \qquad \forall x, y \in \mathbb{R}.$$

Show that the set $f(E)$ is Lebesgue measurable for every Lebesgue measurable set E.

4. Consider the set S of polynomials on $[0,1]$, which vanish at 0. Prove that S is dense in $\{f \in C[0,1] : f(0) = 0\}$.

5. Let S denote the unite square $[0,1] \times [0,1]$ in $\mathbb{R}^2$. Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions of bounded variation on a bounded closed interval $[a,b]$, $a < b$. Define the curve $\gamma : [a,b] \to \mathbb{R}^2$ by

$$\gamma\,(x) = (f\,(x)\,, g\,(x)), x \in [a,b]\,.$$

Prove that the trace of the curve γ,that is, $\gamma\,([a,b])$ cannot contain the unit square S.

6. Construct an open set $U \subseteq [0,1]$ such that (i) U is dense in $[0,1]$, (ii) Lebesgue measure $m\,(U) < 1$, (iii) for every $a, b \in \mathbb{R}$, $a < b$, Lebesgue measure $m\,(U \cap (a,b)) > 0$.