Math 531 Comprehensive Exam
January 2021

Problem 1

Let $f(n)$ be the arithmetic function defined by $\dfrac{n}{\phi(n)} = \sum_{d|n} f(d)$ for all n.

(i) Give an explicit formula for the values of f at prime powers p^m and use this formula to express the function f in terms of familiar arithmetic functions (such as the Moebius function).

(ii) Obtain an estimate for $\sum_{n\le x} \frac{n}{\phi(n)}$ with error term $O(\log x \log\log x)$. You may use, without proof, the estimate $n/\phi(n) = O(\log\log n)$ $(n \ge 3)$. (No need to evaluate the constant arising in this estimate.)

Problem 2

Prove an asymptotic formula for $P(x) := \sum_{p\le x} p$. You may use the prime number theorem.

Problem 3

Let f be an arithmetic function with Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, and let $M(x) = \sum_{n\le x} f(n)$. Suppose that, for some non-zero constant A, $M(x) \sim A\sqrt{x}$ as $x \to \infty$. Show that $F(s)$ converges for $\operatorname{Re} s > 1/2$, and prove rigorously (using an ϵ argument) that $F(s) \sim \frac{A}{2s-1}$ as $s \to 1/2+$.

Problem 4

(a) Prove that there are infinitely many primes that end in the digits 2021.

(b) Prove that there are infinitely many integers, each the product of two primes, that end in the digits 2021.

(c) Let k be a positive integer and χ a non-principal Dirichlet character mod k. Prove that the series $S = \sum_{n=1}^{\infty} \chi(n)n^{-1/3}$ converges, and obtain an estimate (as good as possible) for the tail sums $R(x) = \sum_{n>x} \chi(n)n^{-1/3}$.