Math 530 Comprehensive Exam
August 2020

Problem 1 (30 points)

Consider the number field $F = \mathbb{Q}(\alpha)$ with α a root of $f(x) = x^3 + x - 1$ (which is irreducible over $\mathbb{Q}$). For this question, you may use without proof the following facts:

- The ring of integers $\mathcal{O}_F$ of F is $\mathbb{Z}[\alpha]$.
- The discriminant d_F of F is -31.
- The polynomial $f(x)$ has precisely one root in $\mathbb{R}$.

(a). Which primes ramify in F? Give a one sentence justification for your answer.

(b). Compute the prime decomposition of 2 and of 3 in $\mathcal{O}_F$. Give generators for your prime ideals and determine their residue degrees.

(c). Is $F/\mathbb{Q}$ Galois? Briefly justify your answer.

(d). Compute the class number of F. Justify your answer.

(e). Compute the structure of $\mathcal{O}_F^\times$ (including the torsion subgroup). Justify your answer.

Problem 2 (20 points)

(a). Show that if p is a prime for which $p \equiv 2 \mod 3$, then every integer that is prime to p has a cubic root in $\mathbb{Z}_p$.

(b). Does every integer that is prime to 3 have a cubic root in $\mathbb{Q}_3$? Prove it or give an counterexample.

Problem 3 (30 points)

Let d be an odd, square-free integer. Let $F = \mathbb{Q}(\sqrt{d})$. Show that $\mathbb{Q}(\zeta)$ is the smallest cyclotomic field containing F, where ζ is a primitive $|d_F|$-th root of unity, as follows. (You may wish to use the fact that the discriminant $d_F = d$ for $d \equiv 1 \mod 4$, and $d_F = 4d$ for $d \equiv 3 \mod 4$.)

(a). Show that for any odd prime p, $\mathbb{Q}(\sqrt{p^*})$ is the unique quadratic subfield of $\mathbb{Q}(\zeta_p)$. Here $p^* = (-1)^{(p-1)/2}p$. Hint: Consider the ramification of primes. You may want to use the fact that the extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ ramifies exactly at p with ramification index $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$.

(b). Use (a) to show that F is contained in $\mathbb{Q}(\zeta)$.

(c). Show that any cyclotomic field that contains F must contain $\mathbb{Q}(\zeta)$.

Problem 4 (20 points)

Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ of degree 5 with splitting field F. Suppose that p_1, p_2, p_3 are primes that are unramified in $F/\mathbb{Q}$, and let f_i be the reduction of $f(x)$ modulo p_i for $i = 1, 2, 3$. Suppose further that

- f_1 is irreducible;
- f_2 factors (into irreducibles) as $g_1 g_2$, with $\deg(g_1) = 2$ and $\deg(g_2) = 3$;
- f_3 factors (into irreducibles) as $g_3 g_4$, with $\deg(g_3) = 1$ and $\deg(g_4) = 4$;

Show that $\mathrm{Gal}(F/\mathbb{Q})$ is either A_5 or S_5.