Math 525 Comprehensive Exam (August 2020)

1. Let X be the space obtained from S^1 by attaching two 2-cells where the first cell is attached by a map of degree 2 and the second by a map of degree 5. Compute the groups $H_n(X, \mathbb{Z})$, for all $n \geq 0$.

2. Consider the following subsets of $\mathbb{R}^2$:

$$\ell_1 = \{(x, 0) \in \mathbb{R}^2\}$$
$$\ell_2 = \{(0, y) \in \mathbb{R}^2\}$$
$$\ell_3 = \{(x, x) \in \mathbb{R}^2\}$$
$$C = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, x^2 + y^2 = 1\} \text{ (the right semi-circle)}$$

   and let $X \subseteq \mathbb{R}^2$ be the subset

$$X = C \cup \ell_1 \cup \ell_2 \cup \ell_3.$$

   the union of the x-axis, the y-axis, the line $y = x$ and a semi-circle.

   (a) Describe the fundamental group of X in terms of generators and relations.

   (b) Classify all 2-fold covering maps of X (not necessarily connected).

3. Let (X, A) be a pair with $A \neq \varnothing$. Let $CA = A \times [0, 1]/A \times \{1\}$. Let $Y = X \cup_A CA$, where we glue CA to X by identifying A with $A \approx A \times \{0\} \subseteq CA$.

   (a) Explain why the inclusion $A \to CA \smallsetminus \{v\}$ admits a retraction, where v is image of $A \times \{1\}$ in CA.

   (b) Show that $H_*(X, A) \approx H_*(Y, \{v\})$, using only the Eilenberg-Steenrod axioms (Dimension, Sum, Homotopy, Exact Sequence, Excision).

4. Let S^2 denote the unit 2-sphere in $\mathbb{R}^3$. For $P \in S^2$, let P^* denote its antipode. Let $f \colon S^2 \to S^2$ be a continuous map with the property that $f(P) \neq f(P^*)$ for all $P \in S^2$. Show that f must be surjective.