MATH 518 — Comp Exam — Jan 2021

Instructions:

- Five problems, 20 points each. Maximum 100 points. Duration: 3 hours.
- You are allowed to consult one book: Lee's *Introduction to smooth manifolds.* It's available through the library catalog: `https://i-share-uiu.primo.exlibrisgroup.com/permalink/01CARLI_UIU/gpjosq/alma99785532212205899`.
- If you use results from the book please quote them precisely (e.g., " by Lemma 11.1 ...").
- You **may not** ask or receive help from any human. You **may not** use search engines, Siri, Google, Alexa etc., various websites or books other than the one mentioned above.
- Justify all your answers. Please make sure your scan is readable.

1. (20 points) Consider the smooth vector field on $\mathbb{R}^3$ given by

$$V = -y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

and let:

$$M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + yz = 1\right\}.$$

(a) Show that M is a submanifold of $\mathbb{R}^3$.

(b) Show that that V is tangent to M, so the restriction $X = V|_M$ defines a vector field on M.

(c) Find the flow of X.

2. (20 points) On $\mathbb{R}^3$ consider the vector field and the differential form given by:

$$V = xy\frac{\partial}{\partial z} - \frac{\partial}{\partial y}, \qquad \omega = z\,\mathrm{d}x \wedge \mathrm{d}y + \,\mathrm{d}y \wedge \mathrm{d}z.$$

Compute the following:

(a) The contraction $\imath_V\omega$;

(b) The exterior derivative $\mathrm{d}\omega$;

(c) The Lie derivative $L_V\omega$;

(d) The pullback $\Phi^*\omega$ by the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, $\Phi(s, t) = (s + t, s, t)$.

3. (20 points) Consider the 2-form on $\mathbb{R}^3 \setminus \{0\}$

$$\alpha = \frac{x-y}{x^2+y^2+z^2}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{y-z}{x^2+y^2+z^2}\,\mathrm{d}y \wedge \mathrm{d}z + \frac{z-x}{x^2+y^2+z^2}\,\mathrm{d}z \wedge \mathrm{d}x.$$

Compute the integral

$$\int_{\mathbb{S}^2} \alpha$$

with respect to the orientation of your choice. Here, of course, $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$ centered at 0.

4. (20 points) Let $n > 1$ and denote by $\mathbb{S}^{n-1}$ be unit sphere in $\mathbb{R}^n$. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, \ldots, x_n) = x_1^3 + \cdots + x_n^3.$$

Prove that

$$M = \{x \in \mathbb{S}^{n-1} \mid f(x) = 0\}$$

is an embedded submanifold of $\mathbb{S}^{n-1}$. What is its dimension? Explain.

5. (20 points) Give an example of a nowhere vanishing vector field on $\mathbb{RP}^5$ – the 5-dimensional real projective space – or prove that any vector field on $\mathbb{RP}^5$ has to have a zero.