Comprehensive Exam

Math 518, January 2020

(1) Consider $\mathbb{R}^3$, with coordinates x, y and z, and the following objects:

$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}$

$Y = y\frac{\partial}{\partial x}$

$\alpha = x\,dy - y\,dz$

$F\colon\ \ \mathbb{R}^2 \to \mathbb{R}^3$
$\quad (u,v) \mapsto (e^u \sin v, e^u \cos v, v - u)$

Compute

(a) $Y(\alpha(X))$
(b) $\mathcal{L}_X d\alpha$
(c) the time-t flow of X
(d) $F^*(d\alpha)$

(2) Prove that

$$\{(x,y,z) \in \mathbb{R}^3 | x^2 - y^2 + 2xz - 2yz = 1,\ 2x - y + z = 0\}$$

is an embedded submanifold of $\mathbb{R}^3$.

(3) Prove that if M and N are both orientable manifolds then so is their product, $M \times N$.

(4) Consider the 2–sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and the 2–form

$$\omega = x\,dy \wedge dz - 2y\,dz \wedge dx + 3z\,dx \wedge dy.$$

Prove that the restriction of ω to S^2 represents a nontrivial class in $\mathrm{H}^2_{\text{de Rham}}(S^2)$, the second de Rham cohomology group of S^2.