MATH 500 — January 2020

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. Let G be a finite group of order 100.

   (a) Show that G is solvable. (Feel free to use that groups of order p^2 are abelian for p a prime number)

   (b) Show, by giving a counterexample, that G need not be nilpotent.

2. Decide which of the following sets are ideals of the ring $\mathbb{Z}[x]$. Provide justification.

   (a) The set of all polynomials whose coefficient of x^2 is a multiple of 3.

   (b) $\mathbb{Z}[x^2]$, the set of all polynomials in which only even powers of x appear.

   (c) The set of polynomials whose coefficients sum to zero.

3. Find the possible Jordan canonical forms of 7×7 matrices M with entries in $\mathbb{C}$ satisfying the following criteria:

   - the characteristic polynomial of M is $(z-3)^4(z-5)^3$,
   - the minimal polynomial of M is $(z-3)^2(z-5)^2$, and
   - the $\mathbb{C}-$vector space dimension of the nullspace of $3 \cdot \mathrm{Id} - M$ is 2.

4. Determine if the following polynomials are irreducible over $\mathbb{Z}$.

   (a) $x^3 - 5x - 1$.

   (b) $x^4 + 10x^2 + 5$.

5. (a) Describe the subgroups of S_4 that can occur as Galois group of an irreducible quartic polynomial.

   (b) Determine the Galois group of the irreducible polynomial $x^4 + 2x^2 + 4$. (Feel free to use that a quartic polynomial $f(x) = x^4 + qx^2 + rx + s$ has resolvent cubic $g(x) = x^3 - 2qx^2 + (q^2 - 4s)x + r^2$.)