Math 500 Comp Exam / August 2020

*Do all 5 problems, the max total is 100 points. This exam is "open book" and you may use any results from Math 500 but please make accurate references to the theorems that you will use in your solutions. Explain your answers.*

1. (a; 14 points) A finite group $G$ is called *cool* if $G$ has precisely four Sylow subgroups (over all primes $p$). The order $|G|$ of a cool group is called a *cool* number. For example, $S_3$ is a cool group and 6 is a cool number. Describe the set of all cool numbers. Hint: Use prime factorization in your description.

   (b; 5 points) For each cool number $n$ that you found in part (a), determine whether every cool group of order $n$ is nilpotent.

   (c; 5 points) For each cool number $n$ that you found in part (a), determine whether every cool group of order $n$ is solvable.

2. (18 points) Suppose a finite group $G$ acts on a set $A$ so that for every nontrivial $g \in G$ there exists a unique fixed point (i.e., there is exactly one $a \in A$, depending on $g$, such that $g(a) = a$). Prove that this fixed point is the same for all $g \in G$.

3. (a; 9 points) Compute, if possible, $\gcd(2 + 8i, 17 - 17i)$ in the ring $\mathbb{Z}[i]$ of Gaussian integers.

   (b; 9 points; 3 for each) Determine whether the following polynomials are reducible or irreducible in given rings

   (b1) $x^4 + x^2 + 1$ in $\mathbb{Z}_2[x]$, where $\mathbb{Z}_2$ is the field with 2 elements;

   (b2) $x^4 + 5x^3 + 10x^2 + 15x + 5$ in $R[x]$, where $R = \mathbb{Z}[i]$;

   (b3) $2x^4 + 4x^3 + 8x^2 + 12x + 20$ in $\mathbb{Z}[x]$.

4. (a; 10 points) Let $A$ be an $n \times n$ complex matrix and let $f$ and $g$ be the characteristic and minimal polynomials of $A$, resp. Suppose that $f(x) = g(x)(x - i)$ and $g(x)^2 = f(x)(x^2 + 1)$. Determine all possible Jordan canonical forms of $A$.

   (b; 10 points) Let $\mathbb{F}$ be a field of characteristic $p > 0$ and $p \neq 3$. If $\alpha$ is a root of the polynomial $f(x) = x^p - x + 3$, in an extension of the field $\mathbb{F}$, show that $f(x)$ has $p$ distinct roots in the field $\mathbb{F}(\alpha)$.

5. (a; 6 points) Compute a factorization for $x^{26} - 1$ into irreducible polynomials over $\mathbb{Z}$.

   (b; 14 points) Find the number of all subfields of the splitting field $K$ of $x^{26} - 1$ over $\mathbb{Q}$ and prove that all of them are Galois over $\mathbb{Q}$.