Math 511 Comp Exam, May 2018

Justify your answers to all problems. All varieties are defined over an algebraically closed field k of characteristic zero.

Problem 1. Let $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the rational map defined by $\Phi(u : v : w) = (u : v)$.

(1) Show that Φ cannot be extended to a *regular* map $\mathbb{P}^2 \to \mathbb{P}^1$.
(2) Let C denote the curve
$$C = \{(u : v : w) \mid wv^2 = u^3\} \subset \mathbb{P}^2.$$
The restriction of Φ to C defines a rational map $\Psi = \Phi|_C : C \dashrightarrow \mathbb{P}^1$. Does Ψ extend to a regular map $C \to \mathbb{P}^1$?

Problem 2. Prove that for any $n \geq 2$, and any $x \in \mathbb{P}^n$, the open subvariety $\mathbb{P}^n \smallsetminus \{x\}$ of $\mathbb{P}^n$ is not isomorphic to any affine variety, nor is it isomorphic to any projective variety.

Problem 3. Prove that every nonsingular conic curve $X \subset \mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

Problem 4. For any finite group Γ acting by automorphisms $m_\gamma : \mathbb{A}^n \xrightarrow{\cong} \mathbb{A}^n$ (for $\gamma \in \Gamma$) of $\mathbb{A}^n$, we define the quotient variety $\mathbb{A}^n/\Gamma$ to be the affine variety whose coordinate ring is the ring $k[\mathbb{A}^n]^\Gamma$ of invariant functions, which is defined by
$$k[\mathbb{A}^n]^\Gamma = \{f \in k[\mathbb{A}^n] \mid f\big(m_\gamma(x)\big) = f(x) \text{ for all } x \in \mathbb{A}^n \text{ and } \gamma \in \Gamma\}.$$

Let the group $\{\pm 1\}$ (with group operation defined by multiplication) act on $\mathbb{A}^2$ by $m_\gamma(x, y) = (\gamma x, \gamma y)$ for $\gamma = \pm 1$.

(1) Show that $\mathbb{A}^2/\{\pm 1\}$ is isomorphic to the affine surface $Z(UW - V^2) \subset \mathbb{A}^3$.
(2) Is the quotient variety $\mathbb{A}^2/\{\pm 1\}$ nonsingular? [You may answer this even if you have not solved part (1).]

Problem 5. Consider the curve C in $\mathbb{A}^2$ given by $y^2 = x^3 - x^2$

(1) Show that the "blow up" of $\mathbb{A}^2$ at $(0,0)$ is the union $\mathrm{Spec}(k[x, y/x]) \bigcup \mathrm{Spec}(k[y, x/y]) = \mathcal{B}l_{(0,0)}(\mathbb{A}^2)$. Show that there is a birational map from this Blowup to the plane.
(2) Prove that the curve is singular and that its inverse image in the blowup is smooth.
(3) Prove that the inverse image of the curve in the blowup of the plane is its normalization.