DEFECTIVE DP-COLORINGS OF SPARSE MULTIGRAPHS

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ABSTRACT. DP-coloring (also known as correspondence coloring) is a generalization of list coloring developed recently by Dvořák and Postle. We introduce and study (i, j)-defective DP-colorings of multigraphs. We concentrate on sparse multigraphs and consider $f_{DP}(i, j, n)$ — the minimum number of edges that may have an n-vertex (i, j)-*critical multigraph*, that is, a multigraph G that has no (i, j)-defective DP-coloring but whose every proper subgraph has such a coloring. For every i and j, we find linear lower bounds on $f_{DP}(i, j, n)$ that are exact for infinitely many n.

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1. INTRODUCTION

1.1. **Defective Coloring.** A *proper k-coloring* of a graph G is a partition of $V(G)$ into k independent sets $V_1, \ldots, V_k$. A $(d_1, \ldots, d_k)$-*defective coloring* (or simply $(d_1, \ldots, d_k)$-*coloring*) of a graph G is a partition of $V(G)$ into sets $V_1, V_2, \ldots, V_k$ such that for every $i \in [k]$, every vertex in V_i has at most d_i neighbors in V_i. In particular, a proper k-coloring is a $(0, 0, \ldots, 0)$-defective coloring. A number of significant results on defective colorings of graphs were obtained in [1, 9, 11, 13, 15, 17, 20, 21, 22, 26].

While it is easy to check whether a graph is $(0, 0)$-colorable (i.e., bipartite), for every $(i, j) \neq (0, 0)$, it is an NP-complete problem to decide whether a graph G has an (i, j)-coloring. In particular, Esperet, Montassier, Ochem, and Pinlou [15] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0, 1)$-coloring is NP-complete. In view of this, there was a series of papers estimating how sparse can be graphs not admitting (i, j)-coloring for given i and j, see e.g. [3, 4, 5, 6, 7, 8, 18, 19]. One of often used measures of sparsity is the *maximum average degree*, $mad(G) = \max_{G' \subseteq G} \frac{2|E(G')|}{|V(G')|}$. In this paper we restrict ourselves to coloring with 2 colors. One of the ways to handle the problem is to study (i, j)-*critical* graphs, that is, the graphs that do not have (i, j)-coloring but every proper subgraph of which has such a coloring. Let $f(i, j, n)$ denote the minimum number of edges in an (i, j)-critical n-vertex graph. For example, since every acyclic graph is $(0, 0)$-colorable, for odd n we have $f(0, 0, n) = n$. In the above papers, a number of interesting bounds on $f(i, j, n)$

were proved. In particular, for $j \geq 2i + 2$ and also for $(i, j) \in \{(0, 1), (1, 1)\}$ lower bounds were proved that are exact for infinitely many n.

1.2. **Defective List Coloring.** A *list-assignment* of a graph G is a function $L : V(G) \to \mathcal{P}(\mathbb{N})$ that assigns to each $v \in V(G)$ a list $L(v)$ of 'colors'. L is an *ℓ-list assignment* if the list of every vertex is of size ℓ. An *L-coloring* of G is a function $\phi : V(G) \to \bigcup_{v \in V(G)} L(v)$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$ and $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$. A graph G is *k-choosable* if G has an L-coloring for every k-list assignment L. The following notion was introduced in [14, 24] and studied in [25, 28, 16, 17]: A *d-defective list L-coloring* of G is a function $\phi : V(G) \to \bigcup_{v \in V(G)} L(v)$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$ and every vertex has at most d neighbors of the same color. If G has a d-defective list L-coloring from every k-list assignment L, then it is called *d-defective k-choosable*. As in the case of ordinary coloring, a direction of study is showing that "sparse" graphs are d-defective k-choosable. As mentioned before, in this paper we consider only $k = 2$. The best known bounds on maximum average degree that guarantee that a graph is d-defective 2-choosable are due to Havet and Sereni [16] (a new proof of the lower bound is due to Hendrey and Wood [17]):

Theorem A ([16]). *For every $d \geq 0$, if $mad(G) < \frac{4d+4}{d+2}$, then G is d-defective 2-choosable. On the other hand, for every $\epsilon > 0$, there is a graph G_ϵ with $mad(G_\epsilon) < 4 + \epsilon - \frac{2d+4}{d^2+2d+2}$ that is not (d, d)-colorable.*

1.3. **Defective DP-Coloring.** In order to solve some problems on list coloring, Dvořák and Postle [12] introduced and studied the more general notion of DP-coloring. This notion was extended to multigraphs by Bernshteyn, Kostochka and Pron [2].

Definition 1. Let G be a multigraph. A *cover* of G is a pair $\mathscr{H} = (L, H)$, consisting of a graph H (called the *cover graph* of G) and a function $L : V(G) \to \mathrm{Pow}(V(H))$, satisfying the following requirements:

    (1) the family of sets $\{L(u) : u \in V(G)\}$ forms a partition of $V(H)$;
    (2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
    (3) if $E(H[L(u), L(v)]) \neq \varnothing$, then either $u = v$ or $uv \in E(G)$;
    (4) if the multiplicity of an edge $uv \in E(G)$ is k, then $H[L(u), L(v)]$ is the union of at most k matchings connecting $L(u)$ with $L(v)$. (For simplicity, we only consider, throughout our paper, perfect matching whenever there is an edge between u and v.)

A cover $\mathscr{H} = (L, H)$ of G is *k-fold* if $|L(u)| = k$ for every $u \in V(G)$.

In this paper, we consider only 2-fold covers and by **graphs** below we always mean *multigraphs with no loops*.

For a graph G with a cover $\mathscr{H} = (L, H)$, the set $V(H)$ is partitioned into two parts P and R such that for every $v \in V(G)$, $|L(v) \cap P| = |L(v) \cap R| = 1$. The vertices in P are called *poor*, those in R are called *rich*. For every vertex $v \in V(G)$, denote the poor vertex in $L(v)$ by $p(v)$, the rich one by $r(v)$.

Definition 2. Let G be a graph and $\mathscr{H} = (L, H)$ be a cover of G. An *$\mathscr{H}$-map* of G is an injection $\phi : V(G) \to V(H)$, such that $\phi(v) \in L(v)$ for every $v \in V(G)$. The subgraph of H induced by $\phi(V(G))$ is called the *ϕ-induced graph*, denoted by H_ϕ.

Definition 3 (An (i, j)-coloring)**.** Let $0 \leq i \leq j$. Let G be a graph and $\mathscr{H} = (L, H)$ be its cover. An $\mathscr{H}$-map ϕ of G is an (i, j)-*defective-DP-coloring* of H if the degree of every poor vertex in H_ϕ is at most i and the degree of every rich vertex in H_ϕ is at most j. We say that G is (i, j)-*defective-DP-colorable* if for every 2-fold cover $\mathscr{H} = (L, H)$ of G, H admits an (i, j)-defective-DP-coloring.

For brevity, in the rest of the paper we call an (i, j)-defective-DP-coloring simply (i, j)-*coloring*, and instead of "(i, j)-defective-DP-colorable" say "(i, j)-colorable".

Definition 4 ((i, j)-critical graphs)**.** Given $0 \leq i \leq j$, a multigraph G is (i, j)-*critical*, if G is not (i, j)-colorable, but every proper subgraph of G is. Let $f_{DP}(i, j, n)$ be the minimum number of edges in an n-vertex (i, j)-critical multigraph.

The goal of our paper is to find linear lower bounds for $f_{DP}(i, j, n)$ that are exact for all $i \leq j$ for infinitely many n. Since every not (i, j)-colorable graph contains an (i, j)-critical subgraph, this will yield best possible bounds on sparseness of graphs that provides the existence of (i, j)-colorings.

2. Results

The goal of this paper is to prove the following extremal result.

Theorem 2.1.   (1) *If $i = 0$ and $j \geq 1$, then $f_{DP}(0, j, n) \geq n + j$. This is sharp for every $j \geq 1$ and every $n \geq 2j + 2$.*
  (2) *If $i \geq 1$ and $j \geq 2i + 1$, then $f_{DP}(i, j, n) \geq \frac{(2i+1)n - (2i - j)}{i+1}$. This is sharp for each such pair (i, j) for infinitely many n.*
  (3) *If $i \geq 1$ and $i + 2 \leq j \leq 2i$, then $f_{DP}(i, j, n) \geq \frac{2jn + 2}{j+1}$. This is sharp for each such pair (i, j) for infinitely many n.*
  (4) *If $i \geq 1$, then $f_{DP}(i, i+1, n) \geq \frac{(2i^2 + 4i + 1)n + 1}{i^2 + 3i + 1}$. This is sharp for each $i \geq 1$ for infinitely many n.*
  (5) *If $i \geq 1$, then $f_{DP}(i, i, n) \geq \frac{(2i+2)n}{i+2}$. This is sharp for each $i \geq 1$ for infinitely many n.*

Note that depending on the relations between i and j, we have five different (exact) bounds. Since every non-(i, j)-colorable graph contains an (i, j)-critical subgraph, Theorem 2.1 yields the following.

Corollary 2.2. *Let G be a multigraph.*
  (1) *If $j \geq 1$ and for every subgraph H of G, $|E(H)| \leq |V(H)| + j - 1$, then G is $(0, j)$-colorable. This is sharp.*
  (2) *If $i \geq 1$, $j \geq 2i + 1$ and for every subgraph H of G, $|E(H)| \leq \frac{(2i+1)|V(H)| - (2i - j + 2)}{i+1}$, then G is (i, j)-colorable. This is sharp.*
  (3) *If $i \geq 1$, $i + 2 \leq j \leq 2i$ and for every subgraph H of G, $|E(H)| \leq \frac{2j|V(H)| + 1}{j+1}$, then G is (i, j)-colorable. This is sharp.*
  (4) *If $i \geq 1$ and for every subgraph H of G, $|E(H)| \leq \frac{(2i^2 + 4i + 1)}{i^2 + 3i + 1}|V(H)|$, then G is $(i, i+1)$-colorable. This is sharp.*
  (5) *If $i \geq 1$ and for every subgraph H of G, $|E(H)| \leq \frac{(2i+2)|V(G)| - 1}{i+2}$, then G is (i, i)-colorable. This is sharp.*

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Since a version of our construction in Section 10 for $(0, j)$-colorings is a simple graph, Part 1 of Corollary 2.2 implies the following result.

Corollary 2.3. *Let G be a simple graph and $j \geq 1$ be integers. If for every subgraph H of G, $|E(H)| \leq |V(H)| + j - 1$, then G is $(0, j)$-colorable. This is sharp for all $j \geq 1$ and each $n \geq 3j + 3$.*

In the next section we prove the lower bound in Part 5 of Theorem 2.1. For other lower bounds we will use a more general framework. It will be introduced in Section 4, and in the subsequent five sections we prove the more general versions of the four other lower bounds. In the last section, we present constructions showing that our bounds are sharp for each $i \leq j$ for infinitely many n.

3. PROOF OF THE LOWER BOUND IN THEOREM 2.1 FOR (i, i)-COLORINGS

In this section, we prove the lower bound in Part 5 of Theorem 2.1. The proof adjusts to DP-coloring the idea of Hendrey and Wood in [17, Theorem 7] for list coloring.

Proposition 3.1. *Let $i \geq 1$ be an integer, and G be an (i, i)-critical graph. Then $|E(G)| \geq \frac{2i+2}{i+2}|V(G)|$.*

Lemma 3.2. *Let $k \geq 0$. If a cover $\mathscr{H} = (L, H)$ of a graph G satisfies*

$$(1) \qquad \text{for every } u \in V(G),\ d(u) + 1 \leq |L(u)|(i + 1),$$

then $\mathscr{H}$ is (i, i)-colorable.

Proof. Choose an $\mathscr{H}$-map ϕ with minimum $|E(H_\phi)|$. Suppose there is $v \in V(G)$ such that $d_{H_\phi}(\phi(v)) \geq i + 1$. By (1), there is $\alpha \in L(v) - \phi(v)$ such that

$$|N(\alpha) \cap V(H_\phi)| \leq \left\lfloor \frac{d(v)}{|L(v)|} \right\rfloor \leq \left\lfloor \frac{|L(v)|(i+1) - 1}{|L(v)|} \right\rfloor = i.$$

Define a map ϕ' as follows: $\phi'(v) = \alpha$, and $\phi'(u) = \phi(u)$ for every $u \in V(G) \setminus \{v\}$. Then $|E(H_{\phi'})| \leq |E(H_\phi)| - 1$, a contradiction. $\square$

Let G be an (i, i)-critical graph. For $X \subset V(G)$ and $v \in V(G)$, let $d_X(v) = |N(v) \cap X|$.

Lemma 3.3. *For every partition $V(G) = A \sqcup B$ with $A \neq \emptyset$ and $B \neq \emptyset$, there is $v \in B$ such that $(i + 1)d_A(v) + d_B(v) \geq 2i + 2$.*

Proof. Suppose there is a partition $A \sqcup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that

$$(2) \qquad \text{for every } v \in B,\ (i + 1)d_A(v) + d_B(v) \leq 2i + 1.$$

Let $\mathscr{G} = (L, H)$ be a 2-fold cover on G such that H does not have an (i, i)-coloring. Let H^A (respectively, H^B) denote the subgraph of H corresponding to $G[A]$ (respectively, $G[B]$). Since G is (i, i)-critical, H^A has an (i, i)-coloring ϕ. For every $v \in B$, form $L'(v)$ from $L(v)$ by excluding from it every v_α such that v_α has a neighbor in H_ϕ^A. Then $|L'(v)| \geq 2 - d_A(v)$ for each $v \in B$. By (2), this is at least $2 - \frac{2i+1-d_B(v)}{i+1} = \frac{d_B(v)+1}{i+1}$. Hence by Lemma 3.2, H^B has an (i, i)-coloring ϕ'. Then the representative map ψ defined by $\psi(w) = \phi(w)$ for $w \in A$ and $\psi(w) = \phi'(w)$ for $w \in B$ is an (i, i)-coloring on H, a contradiction to the choice of G. $\square$

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Let $v_1, \ldots, v_p \in V(G)$ be a maximal sequence such that for every $k \in [p]$,

$$(i+1)d_{A_k}(v_k) + d_{B_k}(v_k) \geq 2i + 2, \text{ where } A_k := \{v_1, \ldots, v_k\}, B_k := V(G) \setminus A_k.$$

By Lemma 3.3, $p = |V(G)|$. Then

$$\sum_{k=1}^{|V(G)|} \left((i+1)d_{A_k}(v_k) + d_{B_k}(v_k) \right) \geq (2i+2)|V(G)|.$$

On the other hand, every edge of G contributes to the sum $\sum_{k=1}^{|V(G)|} \left((i+1)d_{A_k}(v_k) + d_{B_k}(v_k) \right)$ exactly $i + 2$. It follows that $(i + 2)|E(G)| \geq (2i + 2)|V(G)|$, as claimed. This proves Proposition 3.1.

4. A MORE GENERAL MODEL

When $j > i$, we will need the following more general framework. Instead of (i, j)-colorings of a cover $\mathscr{H}$ of a graph G, we will consider $\mathscr{H}$-maps ϕ with variable restrictions on the degrees of the vertices in H_ϕ. Furthermore, we will define potentials of vertex subsets of G so that the lower is a potential of a set W, the larger is the average degree of $G[W]$. We will prove existence of our variable colorings in graphs with no subsets of "low" potential, and will derive our main result, Theorem 2.1, as a partial case of our bounds.

For a graph G, a *toughness function on* G is a mapping $\mathbf{t} : V(G) \to \{0, 1, \ldots, j+1\}$. A pair $(G, \mathbf{t})$ where G is a graph and $\mathbf{t}$ is a toughness function will be called a *weighted pair*.

Definition 5 (An $(i, j, \mathbf{t})$-coloring). *Given a weighted pair $(G, \mathbf{t})$ and a cover $\mathscr{H} = (L, H)$ of G, an $(i, j, \mathbf{t})$-coloring of H is a $\mathscr{H}$-map ϕ such that the degree of every poor vertex $p(v)$ in H_ϕ is at most $i - \mathbf{t}(v)$ and the degree of every rich vertex $r(v)$ in H_ϕ is at most $j - \mathbf{t}(v)$. (If $i - \mathbf{t}(v) < 0$ (respectively, $j - \mathbf{t}(v) < 0$), this means $\phi(v)$ cannot be $p(v)$ (respectively, $\phi(v)$ cannot be $r(v)$).)*

A vertex $v \in V(G)$ is k-tough in $(G, \mathbf{t})$ if $\mathbf{t}(v) = k$.

If $\mathbf{t} \equiv 0$, then any $(i, j, \mathbf{t})$-coloring of a graph G is an (i, j)-coloring in the sense of Definition 3. So, Definition 5 is a refinement of Definition 3. Similarly the next definition refines Definition 4.

Definition 6 ((i, j)-critical pairs.). *Given $0 \leq i \leq j$ and a weighted pair $(G, \mathbf{t})$, we say that $(G, \mathbf{t})$ is (i, j)-critical, if G is not $(i, j, \mathbf{t})$-colorable, but every proper subgraph of G is.*

We will measure the sparsity of our graphs with so called potential function.

Definition 7. *If $j \neq i + 1$ or $j \leq 2$, given a weighted pair $(G, \mathbf{t})$, the $(i, j, \mathbf{t})$-potential of a vertex $v \in V(G)$ is defined by*

$$(3) \qquad \rho_{G,\mathbf{t}}(v) := a_{i,j} + \mathbf{t}(v) \cdot (a_{i,j} - 2b_{i,j}),$$

where $a_{i,j} := b_{i,j} := 1$ when $i = 0$, $a_{i,j} := 2i + 1$ and $b_{i,j} := i + 1$ when $i \geq 1$ and $j \geq 2i + 1$, $a_{i,j} := 2j$ and $b_{i,j} := j + 1$ when $i \geq 1$ and $2i \geq j \geq i + 2$.

In other words,

$$(4) \qquad \rho_{G,\mathbf{t},i,j}(v) := \begin{cases} 1 - \mathbf{t}(v), & \text{if } i = 0; \\ 2i + 1 - \mathbf{t}(v), & \text{if } i \geq 1 \text{ and } j \geq 2i + 1; \\ 2j - 2\mathbf{t}(v), & \text{if } i \geq 1 \text{ and } i + 2 \leq j \leq 2i. \end{cases}$$

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For a subset $S \subseteq V(G)$, the $(i, j, \mathbf{t})$-potential of S is defined by

$$(5) \qquad \rho_{G,\mathbf{t},i,j}(S) := \sum_{v \in S} \rho_{G,\mathbf{t}}(v) - b_{i,j} \cdot |E(G[S])|.$$

The $(i, j, \mathbf{t})$-potential of a graph G is defined by $\rho_{\mathbf{t},i,j}(G) := \min_{S \subseteq V(G)} \rho_{G,\mathbf{t},i,j}(S)$.

When i and j are clear from the context, we will drop these subscripts from the notation $\rho_{G,\mathbf{t},i,j}(S)$ and will call the $(i, j, \mathbf{t})$-potential of S simply *the potential of S*. Let $w_k(i, j) = a_{i,j} + k(a_{i,j} - 2b_{i,j})$, i.e., $w_k(i, j)$ is the $(i, j, \mathbf{t})$-potential of a k-tough vertex in $(G, \mathbf{t})$.

In the next four sections we prove the following theorem.

Theorem 4.1. *Let $(G, \mathbf{t})$ be an (i, j)-critical weighted pair, where $j \neq i + 1$ and $\mathbf{t}$ is an arbitrary toughness function. Then $\rho_{\mathbf{t},i,j}(G) \leq w_{j+1}(i, j)$. In particular,*
  (1) *If $i = 0$ and $j \geq 1$, then $\rho_{\mathbf{t},i,j}(G) \leq -j$.*
  (2) *If $i \geq 1$ and $j \geq 2i + 1$, then $\rho_{\mathbf{t},i,j}(G) \leq 2i - j$.*
  (3) *If $i \geq 1$ and $2i \geq j \geq i + 2$, then $\rho_{\mathbf{t},i,j}(G) \leq -2$.*

Observe that if we take $\mathbf{t} \equiv 0$, then Theorem 4.1 implies the lower bounds of Parts 1, 2 and 3 of Theorem 2.1. In other words, we are proving a generalization of these parts of Theorem 2.1.

5. Preliminaries

For a graph G and disjoint sets $U, W \subset V(G)$, $E_G(U, W)$ denotes the set of the edges of G with one end in U and one in W. If $U = \{u\}$ and $W = \{w\}$, then instead of $E_G(U, W)$ we write $E_G(u, w)$. If $e \in E_G(u, v)$ and $\mathscr{H} = (L, H)$ is a cover of G, then $M_H(e)$ (or simply $M(e)$ when H is clear from the context) denotes the matching between $L(u)$ and $L(v)$ in H corresponding to e.

For $e \in E_G(u, v)$, a matching $M(e)$ is *even* if its edges are $r(u)r(v)$ and $p(u)p(v)$, and is *odd* otherwise, i.e., if its edges are $r(u)p(v)$ and $p(u)r(v)$.

We will use the following lemmas at various points:

Lemma 5.1. *For nonnegative integers i, j with $i \leq j$, suppose Theorem 4.1 does not hold, and $(G, \mathbf{t})$ is an (i, j)-critical pair of minimum order with potential larger than $w_{j+1}(i, j)$. Then every nonempty $S \subsetneq V(G)$ with $\rho_{G,\mathbf{t}}(S) \leq w_j$ consists of a single j-tough vertex.*

Proof. Suppose the lemma fails. Choose a maximum $S \subsetneq V(G)$ with $\rho_{G,\mathbf{t}}(S) \leq w_j$. Let $\mathscr{H} = (L, H)$ be a cover on G such that H does not have an (i, j)-coloring. Note that for every $v \in V(G) \setminus S$, we have $|N(v) \cap S| \leq 1$, since otherwise

$$\rho(S \cup \{v\}) \leq w_j + (a_{i,j} - 2b_{i,j}) = w_{j+1},$$

which contradicts the assumption on $(G, \mathbf{t})$. Form a pair $(G', \mathbf{t}')$ from $(G, \mathbf{t})$ as follows:
(a) Let $V(G') = V(G) \setminus S \cup \{v^*\}$;
(b) let $\mathbf{t}'(u) = \mathbf{t}(u)$ for every $u \in V(G') \setminus \{v^*\}$ and $\mathbf{t}'(v^*) = j$;
(c) for each edge $e \in E_G(uz)$ with $u \in S$ and $z \in V(G) \setminus S$, add an edge between v^* and z.

If $\rho(G', \mathbf{t}') \leq w_{j+1}$, let $S' \subsetneq V(G')$ be a maximal subset with $\rho_{G',\mathbf{t}'}(S') \leq w_{j+1}$. By construction of $(G', \mathbf{t}')$, $v^* \in S'$. Let $S'' = S' \setminus \{v^*\}$. Then

$$\rho_{G,\mathbf{t}}(S'' \cup S) = \rho_{G,\mathbf{t}}(S) + \rho_{G',\mathbf{t}'}(S') - \rho_{G',\mathbf{t}'}(v^*) \leq w_j + w_{j+1} - w_j = w_{j+1},$$

a contradiction to $\rho(G, \mathbf{t}) > w_{j+1}$. Hence such S' does not exist, and $\rho(G', \mathbf{t}') > w_{j+1}(i, j)$.

Denote the subgraph of H induced by $H[L(S)]$ by $H[S]$. Since $(G, \mathbf{t})$ is (i, j)-critical, $H[S]$ has an $(i, j, \mathbf{t})$-coloring ϕ. For every $z \in N_G(S)$ and its neighbor $u \in S$, for each $e \in E_G(u, z)$, denote the neighbor of $\phi(u)$ in $M(e)$ by $z_a(e)$, and the other vertex in $L(z)$ by $z_b(e)$. Let $\mathscr{G}'' = (L', H')$ be a cover of G', such that :

1) $L'(v^*) = \{p(v^*), r(v^*)\}$;

2) for every $z \in N(v^*)$ and every edge $e \in E_G(S, z)$, $p(v^*)$ is adjacent to $z_b(e)$ and $r(v^*)$ is adjacent to $z_a(e)$;

3) for every edge $xy \in E(G')$ such that neither of x nor y is equal to v^*, $H'[\{x, y\}] = H[\{x, y\}]$.

Then by the minimality of $(G, \mathbf{t})$, H' has an $(i, j, \mathbf{t})$-coloring ψ. Since $\mathbf{t}'(v^*) = j$, $\psi(v^*) = r(v^*)$ and $r(v^*)$ has degree 0 in H'_ψ. Now we define an $\mathscr{H}$-map σ by $\sigma(z) = \phi(z)$ for every $z \in S$, and $\sigma(z) = \psi(z)$ for every $z \in V(G) \setminus S$. By the construction of G', for every $vu \in E(G)$ such that $v \in S$ and $u \in V(G) \setminus S$, $\sigma(v)$ is not adjacent to $\sigma(u)$. Hence σ is an $(i, j, \mathbf{t})$-coloring of H, a contradiction. $\qquad\square$

Lemma 5.2. *If $(G, \mathbf{t})$ is an (i, j)-critical pair and $v \in V(G)$ with $d(v) = 1$, then $\mathbf{t}(v) \geq i+1$.*

Proof. Suppose $\mathbf{t}(v) \leq i$, $d(v) = 1$ and $N(v) = \{u\}$. Given an arbitrary cover $\mathscr{H} = (L, H)$ of G, we consider the graph $H' = H - L(v)$. Since $(G, \mathbf{t})$ is (i, j)-critical, H' has an $(i, j, \mathbf{t})$-coloring ϕ. We extend ϕ to H by letting $\phi(v)$ be the vertex in $L(v)$ not adjacent to $\phi(u)$. $\qquad\square$

6. Proof of Theorem 4.1 for $(0, j)$-colorings

In this section, we prove Part 1 of Theorem 4.1:

Proposition 6.1. *Let $j \geq 1$ be an integer, and let $(G, \mathbf{t})$ be a $(0, j)$-critical pair. Then $\rho(G, \mathbf{t}) \leq -j$.*

Recall that $a_{0,j} = b_{0,j} = 1$. By (4), $w_k = 1 - k$, for every $k \in \{0, 1, \ldots, j+1\}$. Suppose the proposition does not hold, and $(G, \mathbf{t})$ is a $(0, j)$-critical pair with potential larger than $w_{j+1}(0, j) = -j$ with the minimum $|V(G)| + |E(G)|$. Let $\mathscr{H} = (L, H)$ be a cover of G such that H does not have a $(0, j, \mathbf{t})$-coloring. First, we analyse the structure of G and $\mathscr{H}$.

Lemma 6.2. *For every edge $e \in E(G)$, matching $M(e)$ is even.*

Proof. Suppose there exists $e \in E(G)$ such that $M(e)$ is odd. For definiteness, suppose $e \in E_G(u, v)$.

Case 1: There is an $e' \in E_G(u, v) - e$ such that $M(e')$ is even. Let $G' = G - e - e'$ and define $\mathbf{t}'(x) = \mathbf{t}(x)$ for every $x \in V(G) - \{u, v\}$, and $\mathbf{t}'(x) = \mathbf{t}(x) + 1$ when $x \in \{u, v\}$. We claim that

$$(6) \qquad \rho(G', \mathbf{t}') \geq 1 - j = w_j(0, j).$$

Indeed, assume $\rho_{G', \mathbf{t}'}(S) \leq -j$. By the definition of $\mathbf{t}'$, $S \cap \{u, v\} \neq \emptyset$, say $u \in S$. If also $v \in S$, then $|E(G[S])| - |E(G'[S])| = 2$, and hence by the definition of potentials, $\rho_{G, \mathbf{t}}(S) = \rho_{G', \mathbf{t}'}(S) \leq -j$, a contradiction. Thus $v \notin S$ and hence $\rho_{G, \mathbf{t}}(S) = \rho_{G', \mathbf{t}'}(S) + 1 \leq -j + 1$. Then by Lemma 5.1, $S = \{u\}$ and $\mathbf{t}(u) = j$. But in this case,

$$\rho_{G, \mathbf{t}}(\{u, v\}) = \rho_{G, \mathbf{t}}(u) + \rho_{G, \mathbf{t}}(v) - |E_G(u, v)| \leq (1 - j) + 1 - 2 = -j,$$

a contradiction. This proves (6).

Let H' be the cover graph on G' obtained from H by deleting $M(e)$ and $M(e')$. By the minimality of $(G, \mathbf{t})$, H' has a $(0, j, \mathbf{t}')$-coloring σ. For $x \in \{u, v\}$, since $\mathbf{t}'(x) \geq 1$, we have $\sigma(x) = r(x)$ and $d_{H'_\sigma}(x) \leq j - \mathbf{t}(x) - 1$. Since only one edge in $M_H(e) \cup M_H(e')$ connects $r(u)$ with $r(v)$, $d_{H_\sigma}(x) \leq j - \mathbf{t}(x)$, and hence σ is a $(0, j, \mathbf{t})$-coloring on H, a contradiction.

Case 2: For every $e' \in E_G(u, v)$, $M(e')$ is odd. Form G'' from G by deleting all edges between u and v and gluing u and v into a new vertex v^*. Let $\mathbf{t}''(x) = \mathbf{t}(x)$ for every $x \in V(G'') - v^*$, and $\mathbf{t}''(v^*) = \max\{\mathbf{t}(u), \mathbf{t}(v)\}$. Since

$$\rho_{G'', \mathbf{t}''}(\{v^*\}) = 1 - \max\{\mathbf{t}(u), \mathbf{t}(v)\} \geq (1 - \mathbf{t}(u)) + (1 - \mathbf{t}(v)) - |E_G(u, v)| = \rho_{G, \mathbf{t}}(\{u, v\}),$$

we get $\rho(G'', \mathbf{t}'') \geq \rho(G, \mathbf{t}) \geq 1 - j$. Let H'' be the cover graph on G'' obtained from H by deleting the edges between $L(u)$ and $L(v)$ and by gluing $r(u)$ with $r(v)$ into the new vertex $r(v^*)$ and gluing $p(u)$ with $p(v)$ into the new vertex $p(v^*)$. By the minimality of $(G, \mathbf{t})$, H'' has a $(0, j, \mathbf{t}'')$-coloring ψ. Then the map ϕ, where $\phi(x) = \psi(x)$ for every $x \in V(G) - u - v$ and $\phi(u) = \phi(v) = \psi(v^*)$, is a $(0, j, \mathbf{t})$-coloring of H, a contradiction. $\qquad\square$

Lemma 6.3. *For every 0-tough $v \in V(G)$, $|N_G(v)| \geq 3$.*

Proof. Suppose $|N_G(v)| \leq 2$ for some 0-tough $v \in V(G)$.

Case 1: $|N_G(v)| = 1$, say $N_G(v) = \{u\}$. Since $(G, \mathbf{t})$ is $(0, j)$-critical, $H - L(v)$ has a $(0, j, \mathbf{t})$-coloring σ. By Lemma 6.2, for every $e \in E_G(v, u)$, matching $M(e)$ is even. Extend σ to v by choosing $\sigma(v) \in L(v)$ not adjacent to $\sigma(u)$. Then σ is a $(0, j, \mathbf{t})$-coloring on H, a contradiction.

Case 2: $|N_G(v)| = 2$, say $N_G(v) = \{u, w\}$, and u is not adjacent to w in G. Let G' be obtained from $G - v$ by gluing u with w into the new vertex u^*. Let $\mathbf{t}'(x) = \mathbf{t}(x)$ for every $x \in V(G') - u^*$, and $\mathbf{t}'(u^*) = \max\{\mathbf{t}(u), \mathbf{t}(w)\}$. Since

$$\rho_{G', \mathbf{t}'}(\{u^*\}) = 1 - \max\{\mathbf{t}(u), \mathbf{t}(w)\} \geq (1 - \mathbf{t}(u)) + (1 - \mathbf{t}(w)) + (1 - \mathbf{t}(v)) - 2$$

$$\geq (1 - \mathbf{t}(u)) + (1 - \mathbf{t}(w)) + (1 - \mathbf{t}(v)) - |E_G(v, \{u, w\})| = \rho_{G, \mathbf{t}}(\{v, u, w\}),$$

we get $\rho(G', \mathbf{t}') \geq \rho(G, \mathbf{t}) \geq 1 - j$. Let H' be the cover graph of G' obtained from $H - L(v)$ by gluing $r(u)$ with $r(w)$ into the new vertex $r(u^*)$ and gluing $p(u)$ with $p(w)$ into the new vertex $p(u^*)$. By the minimality of G, H' has a $(0, j, \mathbf{t}')$-coloring ψ. Define $\phi(x) = \psi(x)$ for every $x \in V(G) - u - w - v$, $\phi(u) = \phi(w) = \psi(u^*)$, and choose $\phi(v) \in L(v)$ not adjacent to $\phi(u)$. By Lemma 6.2, ϕ is a $(0, j, \mathbf{t})$-coloring of H, a contradiction.

Case 3: $|N_G(v)| = 2$, say $N_G(v) = \{u, w\}$, and u is adjacent to w in G, say $e \in E_G(u, w)$. Let G'' be obtained from $G - v$ by adding an extra edge e' connecting u and w. Let $\mathbf{t}''(x) = \mathbf{t}(x)$ for every $x \in V(G'')$. Suppose $\rho_{G'', \mathbf{t}''}(S) \leq -j$ for some $S \subset V(G'')$. Since $\mathbf{t}''(x) = \mathbf{t}(x)$ for every $x \in V(G'')$, $u, w \in S$ and $\rho_{G, \mathbf{t}}(S) \leq \rho_{G', \mathbf{t}'}(S) + 1 \leq 1 - j$. Then

$$\rho_{G, \mathbf{t}}(S \cup \{v\}) \leq \rho_{G, \mathbf{t}}(S) + \rho_{G, \mathbf{t}}(v) - |E_G(v, S)| \leq (1 - j) + 1 - 2 = -j,$$

a contradiction. Thus $\rho(G'', \mathbf{t}'') \geq 1 - j$. Let H'' be the cover graph on G'' obtained from $H - L(v)$ by adding an odd matching connecting $L(u)$ and $L(w)$. By the minimality of $(G, \mathbf{t})$, H'' has a $(0, j, \mathbf{t}'')$-coloring ψ. Since H'' has both, odd and even, matchings connecting $L(u)$ and $L(w)$, $\psi(u) = r(u)$ and $\psi(w) = r(w)$. Then by choosing $\psi(v) = p(v)$ we get a $(0, j)$-coloring on H, a contradiction. $\qquad\square$

If $d(v) \leq j - \mathbf{t}(v)$ for each $v \in V(G)$, then we color each $v \in V(G)$ with $r(v)$ and obtain a $(0, j, \mathbf{t})$-coloring of G. Thus there is a vertex $v_0 \in V(G)$ such that

$$ (7) \qquad\qquad d(v_0) \geq j + 1 - \mathbf{t}(v_0). $$

By (5), every edge e contributes potential $-b_{0,j} = -1$ to the potential of a subset S containing the ends of e. We will view this as if each edge e has charge $ch(e) = -1$ and each vertex v has charge $ch(v) = 1 - \mathbf{t}(v)$. By the choice of G, $\sum_{x \in V(G) \cup E(G)} ch(x) \geq 1 - j$. We will use discharging to show that this is not the case. The discharging rules are as follows.

(R1) Every edge incident to v_0 gives charge -1 to v_0.

(R2) Every edge not incident to v_0 gives charge $-1/2$ to each of its ends.

Denote the new charge of a vertex $v \in V(G)$ by $\mu(v)$. Note that after discharging, every edge has charge 0. So

$$ \rho(G, \mathbf{t}) = \sum_{v \in V(G)} \mu(v). $$

By (7) and (R1),

$$ \mu(v_0) = \rho_{G,\mathbf{t}}(v_0) - d(v_0) \leq (1 - \mathbf{t}(v_0)) - (j - \mathbf{t}(v_0) + 1) = -j. $$

If $v \neq v_0$ is 0-tough, then by Lemma 6.3, it has at least two neighbors distinct from v_0. So by (R2),

$$ \mu(v) \leq \rho_{G,\mathbf{t}}(v) - 2 \times 1/2 \leq 1 - 1 = 0. $$

Finally, if v is k-tough for some $k \geq 1$, then $\mu(v) = \rho_{G,\mathbf{t}}(v) = 1 - k \leq 0$. Therefore

$$ \rho(G, \mathbf{t}) = \sum_{v \in V(G)} \mu(v) \leq \mu(v_0) = -j, $$

a contradiction. This proves the proposition.

7. PROOF OF THEOREM 4.1 FOR $i \geq 1$ AND $j \geq 2i + 1$

In this section we prove Part 2 of Theorem 4.1:

Proposition 7.1. *Let $i \geq 1$ and $j \geq 2i + 1$ be integers, and let $(G, \mathbf{t})$ be an (i, j)-critical pair. Then $\rho(G, \mathbf{t}) \leq 2i - j$.*

Recall that in this case, $a_{i,j} = 2i + 1$ and $b_{i,j} = i + 1$. By (4), $w_k = 2i + 1 - k$ for every $k \in \{0, 1, \ldots, j + 1\}$. Suppose the proposition does not hold, and $(G, \mathbf{t})$ is a (i, j)-critical pair with potential larger than $w_{j+1}(i, j) = 2i - j$ with the minimum $|V(G)| + |E(G)|$. Let $\mathcal{H} = (L, H)$ be a cover of G such that H does not have an $(i, j, \mathbf{t})$-coloring.

Lemma 7.2. *G contains at most one j-tough vertex.*

Proof. If u and v are two j-tough vertices, then, since $2i + 1 - j \leq 0$,

$$ \rho_{G,\mathbf{t}}(\{u, v\}) \leq \rho_{G,\mathbf{t}}(u) + \rho_{G,\mathbf{t}}(v) = 2w_j(i, j) = 2(2i + 1 - j) \leq 2i + 1 - j, $$

contradicting Lemma 5.1. $\qquad\square$

Lemma 7.3. *Every edge in G is incident to a j-tough vertex.*

Proof. Suppose there is $e \in E_G(x, y)$ such that neither x nor y is j-tough. Let $G' = G - e$ and define $\mathbf{t}'(w) = \mathbf{t}(w)$ for every $w \in V(G') \setminus \{x, y\}$ and $\mathbf{t}'(w) = \mathbf{t}(w) + 1$ for $w \in \{x, y\}$. Let H' be formed from H by deleting $M(e)$. Then $\mathscr{H}' = (L, H')$ is a cover on G'.

Suppose there is $S \subseteq V(G')$ such that $\rho_{G', \mathbf{t}'}(S) \leq 2i - j$. Then $S \cap \{x, y\} \neq \varnothing$, say $x \in S$. If also $y \in S$, then $|E(G[S])| = 1 + |E(G'[S])|$ and hence

$$\rho_{G, \mathbf{t}}(S) \leq \rho_{G', \mathbf{t}'}(S) + 2 - (i + 1) \leq 2i - j + 2 - (i + 1) \leq 2i - j,$$

a contradiction. So let $y \notin S$. Then $\rho_{G, \mathbf{t}}(S) \leq \rho_{G', \mathbf{t}'}(S) + 1 \leq 2i - j + 1$. So by Lemma 5.1, $S = \{x\}$ and x is j-tough, a contradiction. Hence $\rho(G', \mathbf{t}') \geq 2i - j + 1$. By the minimality of G, H' has an $(i, j, \mathbf{t}')$-coloring ϕ. Since adding $M(e)$ back to H' may increase in H'_ϕ only the degrees of $\phi(x)$ and $\phi(y)$ and only by at most 1, ϕ is also an $(i, j, \mathbf{t})$-coloring on H, a contradiction. $\square$

Lemmas 7.2 and 7.3 together imply that G has a j-tough vertex v_0 such that each edge of G is incident with v_0. If for some $v \in V(G) - v_0$, $|E_G(v, v_0)| \geq 2$, then

$$\rho_{G, \mathbf{t}}(\{v, v_0\}) \leq w_j + (2i + 1) - 2(i + 1) = 2i - j = w_{j+1}.$$

So G has no multiple edges, and G is a star.

Again, let $v \in V(G) - v_0$. By Lemma 5.2, $\mathbf{t}(v) \geq i + 1$. Hence

$$\rho_{G, \mathbf{t}}(\{v, v_0\}) \leq w_{i+1} + w_j - (i + 1) = (2i + 1 - i - 1) + (2i + 1 - j) - (i + 1) = 2i - j,$$

a contradiction. This proves Proposition 7.1.

8. Proof of Theorem 4.1 for $i \geq 1$ and $i + 2 \leq j \leq 2i$

In this section we prove Part 3 of Theorem 4.1:

Proposition 8.1. *Let $i \geq 1$ and $i + 2 \leq j \leq 2i$ be integers, and let $(G, \mathbf{t})$ be an (i, j)-critical pair. Then $\rho(G, \mathbf{t}) \leq -2$.*

The proof is very similar to the proof of Proposition 7.1. In this case, $a_{i,j} = 2j$, $b_{i,j} = j+1$, and $w_k = 2j - 2k$ for all k. In particular, $w_j = 0$. Suppose the proposition does not hold, and $(G, \mathbf{t})$ is a (i, j)-critical pair with potential larger than $w_{j+1}(i, j) = -2$ with the minimum $|V(G)| + |E(G)|$. Let $\mathscr{H} = (L, H)$ be a cover of G such that H does not have an $(i, j, \mathbf{t})$-coloring.

Since $w_j = 0$, the following lemmas have the same statements and practically the same simple proofs as Lemmas 7.2 and 7.3 (so, we omit the proofs).

Lemma 8.2. *G contains at most one j-tough vertex.*

Lemma 8.3. *Every edge in G is incident to a j-tough vertex.*

As in Section 7, Lemmas 8.2 and 8.3 together imply that G has a j-tough vertex v_0 such that each edge of G is incident with v_0. If for some $v \in V(G) - v_0$, $|E_G(v, v_0)| \geq 2$, then $\rho_{G, \mathbf{t}}(\{v, v_0\}) \leq w_j + 2j - 2(j + 1)i = -2 = w_{j+1}$. So G has no multiple edges, hence is a star.

Let $v \in V(G) - v_0$. By Lemma 5.2, $\mathbf{t}(v) \geq i + 1$. Hence

$$\rho_{G, \mathbf{t}}(\{v, v_0\}) \leq w_{i+1} + w_j - (j + 1) = (2j - 2(i + 1)) + 0 - (j + 1) = j - 2i - 3.$$

Since $j - 2i \leq 0$, this is a contradiction proving Proposition 8.1.

In this section, we introduce a more flexible toughness function, and will use it to prove a generalization of the lower bound in Part 4 of Theorem 2.1.

9.1. **A more refined model.** We modify the definitions in Section 4 as follows.

For a graph G, a *toughness function on G* is a function $\mathbf{t}$ mapping each $v \in V(G)$ into a pair $(\mathbf{t}_p(v), \mathbf{t}_r(v))$, where $\mathbf{t}_p(v) \in \{0, 1, \ldots, i + 1\}$ and $\mathbf{t}_r(v) \in \{0, 1, \ldots, j + 1\}$. A pair $(G, \mathbf{t})$ where G is a graph and $\mathbf{t}$ is a toughness function will be called a *weighted pair*.

Definition 8 (An $(i, j, \mathbf{t})$-coloring). *Given a weighted pair $(G, \mathbf{t})$, and a cover $\mathscr{H} = (L, H)$ of G, an $(i, j, \mathbf{t})$-coloring of H is a $\mathscr{H}$-map ϕ such that the degree of every poor vertex $p(v)$ in H_ϕ is at most $i - \mathbf{t}_p(v)$ and the degree of every rich vertex $r(v)$ in H_ϕ is at most $j - \mathbf{t}_r(v)$.*
A vertex $v \in V(G)$ is (k_1, k_2)-tough in $(G, \mathbf{t})$ if $\mathbf{t}(v) = (k_1, k_2)$.

Definition 9. Given a weighted pair $(G, \mathbf{t})$ and its cover $\mathscr{H} = (L, H)$, the potential of a vertex $v \in V(G)$ is defined by

$$(8) \qquad \rho_{G,\mathbf{t}}(v) := \begin{cases} 2i^2 + 4i + 1 - (i+1)\mathbf{t}_p(v) - i\mathbf{t}_r(v) & \text{if } \mathbf{t}_p(v) - \mathbf{t}_r(v) \geq 0, \\ 2i^2 + 4i + 1 - i\mathbf{t}_p(v) - (i+1)\mathbf{t}_r(v) & \text{if } \mathbf{t}_p(v) - \mathbf{t}_r(v) < 0. \end{cases}$$

For a subset $S \subset V(G)$, the potential of S is defined by

$$\rho_{G,\mathbf{t}}(S) := \sum_{v \in S} \rho_{G,\mathbf{t}}(v) - (i^2 + 3i + 1)|E(G[S])|.$$

The potential of $(G, \mathbf{t})$ is defined by $\rho(G, \mathbf{t}) := \min_{S \subset V(G)} \rho_{G,\mathbf{t}}(S)$

The definition of critical pairs is the same as in Section 4.
The main result of the section is:

Theorem 9.1. *Let $i \geq 1$ be an integer. If $(G, \mathbf{t})$ is $(i, i+1)$-critical, then $\rho(G, \mathbf{t}) \leq -1$.*

Observe that if we take $\mathbf{t}(v) = (0, 0)$ for every $v \in G$, then Theorem 9.1 yields the lower bound in Part 4 of Theorem 2.1.

9.2. **Lemmas for the proof of Theorem 9.1.** Suppose the theorem does not hold. Then we can choose an $(i, i + 1)$-critical pair $(G, \mathbf{t})$ with $\rho(G, \mathbf{t}) \geq 0$ that has minimum possible $|V(G)| + |E(G)|$ and modulo this — the maximum $\rho(G, \mathbf{t})$.
We start from a useful observation.

Lemma 9.2. *Pair $(G, \mathbf{t})$ has no vertices v with $\mathbf{t}_r(v) \geq i + 2$.*

Proof. Suppose $v \in V(G)$ and $\mathbf{t}_r(v) \geq i + 2$. Since $\rho_{G,\mathbf{t}}(v) \geq 0$, $\mathbf{t}_p(v) \leq i - 1$. For every 2-fold cover H_1 of $(G, \mathbf{t})$, in each $(i, i + 1, \mathbf{t})$-coloring ϕ of H_1,

$$(9) \quad \phi(v) = p(v) \quad and \quad \left| \bigcup_{u \in V(G) - v} \{e \in E_G(v, u) : p(v)\phi(u) \in M_{H_1}(e)\} \right| \leq i - \mathbf{t}_p(v).$$

Let $\mathbf{t}'(v) = (i + 1, \mathbf{t}_p(v))$ and $\mathbf{t}'(w) = \mathbf{t}(w)$ for each $w \in V(G) - v$. For each 2-fold cover H_1 of $(G, \mathbf{t})$, let H_1' be obtained from H_1 by switching the parities of the matchings $M_{H_1}(e)$ for all edges e incident with v. By (9) and the definitions of $\mathbf{t}'$ and H_1', any subgraph of H_1 (including the whole H_1) has an $(i, i+1, \mathbf{t})$-coloring if and only if the corresponding subgraph

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of H_1' has an $(i, i+1, \mathbf{t}')$-coloring. Thus, since $(G, \mathbf{t})$ is $(i, i+1)$-critical, the pair $(G, \mathbf{t}')$ also is $(i, i+1)$-critical. But by (8),

$$\rho_{G,\mathbf{t}'}(v) - \rho_{G,\mathbf{t}}(v) = 2i^2 + 4i + 1 - (i+1)(i+1) - i\mathbf{t}_p(v) - (2i^2 + 4i + 1 - i\mathbf{t}_p(v) - (i+1)\mathbf{t}_r(v))$$

$$= (i+1)(\mathbf{t}_r(v) - (i+1)) > 0,$$

and hence $\rho(G, \mathbf{t}') > \rho(G, \mathbf{t})$, a contradiction to the choice of $(G, \mathbf{t})$. □

We now derive an analog of Lemma 5.1 (with almost the same proof):

Lemma 9.3. *For every $S \subsetneq V(G)$, if $\rho_{G,\mathbf{t}}(S) \le i$, then S is a $(i+1, i+1)$-tough vertex. In particular, $\rho_{G,\mathbf{t}}(S) = i$.*

Proof. Let S be a largest proper subset of $V(G)$ with $\rho_{G,\mathbf{t}}(S) \le i$. If $S = \{v\}$, then to have $0 \le \rho_{G,\mathbf{t}}(S) \le i$, by (8), v is either $(i+1, i+1)$-tough or $(i, i+2)$-tough. But the latter is excluded by Lemma 9.2, thus our lemma holds in this case. So, suppose $|S| \ge 2$.

If there is $u \in \overline{S}$ such that $|E_G(S, u)| \ge 2$, then

$$\rho_{G,\mathbf{t}}(S \cup \{u\}) \le i + 2i^2 + 4i + 1 - 2(i^2 + 3i + 1) = -i - 1,$$

a contradiction. Thus,

(10) $$\text{for each } u \in \overline{S}, \quad |E_G(S, u)| \le 1.$$

Let $\mathscr{H} = (L, H)$ be a cover on G such that H does not have an (i, j)-coloring. Form a pair $(G', \mathbf{t}')$ from $(G, \mathbf{t})$ as follows:
(a) Let $V(G') = V(G) \setminus S \cup \{v^*\}$;
(b) let $\mathbf{t}'(w) = \mathbf{t}(w)$ for every $w \in V(G') \setminus \{v^*\}$, and let $\mathbf{t}'(v^*) = (i+1, i+1)$;
(c) for each edge $e \in E_G(uw)$ with $u \in S$ and $w \in V(G) \setminus S$, add an edge between v^* and w.

If $\rho(G', \mathbf{t}') \le -1$, let $S' \subsetneq V(G')$ be a maximal subset with $\rho_{G',\mathbf{t}'}(S') \le -1$. By construction of $(G', \mathbf{t}')$, $v^* \in S'$. Let $S'' = S' \setminus \{v^*\}$. Then

$$\rho_{G,\mathbf{t}}(S'' \cup S) = \rho_{G,\mathbf{t}}(S) + \rho_{G',\mathbf{t}'}(S') - \rho_{G',\mathbf{t}'}(v^*) \le i + (-1) - i = -1,$$

a contradiction to $\rho(G, \mathbf{t}) > -1$. This yields $\rho(G', \mathbf{t}') \ge 0$.

Since $(G, \mathbf{t})$ is $(i, i+1)$-critical, $H[S]$ has an (i, j)-coloring ϕ.

For every $z \in N_G(S)$ and its neighbor $u \in S$, for each $e \in E_G(u, z)$, denote the neighbor of $\phi(u)$ in $M(e)$ by $z_a(e)$, and the other vertex in $L(z)$ by $z_b(e)$. Let $\mathscr{H}' = (L', H')$ be a cover of G', such that :
1) $L'(v^*) = \{p(v^*), r(v^*)\}$;
2) for every $z \in N(v^*)$ and every edge $e \in E_G(S, z)$, $p(v^*)$ is adjacent to $z_b(e)$ and $r(v^*)$ is adjacent to $z_a(e)$;
3) for every edge $xy \in E(G')$ such that neither x nor y is equal to v^*, $H'[\{x, y\}] = H[\{x, y\}]$.

Then by the minimality of $(G, \mathbf{t})$ and the fact that $\rho(G', \mathbf{t}') \ge 0$, H' has an $(i, i+1, \mathbf{t})$-coloring ψ. Since $\mathbf{t}'(v^*) = (i+1, i+1)$, $\psi(v^*) = r(v^*)$ and $r(v^*)$ has degree 0 in H'. Now we define an $\mathscr{H}$-map σ by $\sigma(z) = \phi(z)$ for every $z \in S$, and $\sigma(z) = \psi(z)$ for every $z \in V(G) \setminus S$. By the construction of G', for every $vu \in E(G)$ such that $v \in S$ and $u \in V(G) \setminus S$, $\sigma(v)$ is not adjacent to $\sigma(u)$. Hence σ is an $(i, i+1, \mathbf{t})$-coloring of H, a contradiction. □

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9.3. **Low sets and vertices.** The following notion is quite useful. A *low set* is a proper subset S of $V(G)$ with $\rho_{G,\mathbf{t}}(S) \leq 2i$.

Lemma 9.4. *If S is a low set with $|S| \geq 2$ and $|E[S, V(G) \setminus S]| \geq 2$, then $|S| = 2$, S is independent, and each $x \in S$ is $(i+1, i+1)$-tough.*

Proof. Suppose lemma is not true. Choose a largest low set S with $|S| \geq 2$ and $|E[S, V(G) \setminus S]| \geq 2$ that is not an independent set of two vertices.

Let $E_G(S, V(G) - S) = \{e_1, \ldots, e_k\}$ where e_h connects $v_h \in S$ with $u_h \in V(G) - S$ for $h = 1, \ldots, k$ (some vertices can coincide). Under the conditions of the lemma, $k \geq 2$. Construct G' by adding to $G - S$ two new vertices x and y and the set of edges $\{e'_1, \ldots, e'_k\}$ where e_h connects x with u_h for $h = 1, \ldots, k-1$ and e_k connects y with u_k.

We claim that

(11) $$|V(G')| + |E(G')| < |V(G)| + |E(G)|.$$

Indeed, $|V(G')| \leq |V(G)|$ since $|S| \geq 2$, and $|E(G')| \leq |E(G)|$ by construction. Moreover, if we have equalities in both inequalities, then $|S| = 2$ and $|E(G[S])| = 0$, a contradiction to the choice of S. This proves (11).

Let $\mathbf{t}'(u) = (i+1, i+1)$ for $u \in \{x, y\}$ and $\mathbf{t}'(u) = \mathbf{t}(u)$ for $u \in V(G) \setminus S$.

If there is $S' \subseteq V(G')$ such that $\rho_{G',\mathbf{t}'}(S') < 0$, then $S' \cap \{x, y\} \neq \emptyset$, say $x \in S'$. If also $y \in S'$, then

$$\rho_{G,\mathbf{t}}((S' - \{x, y\}) \cup S) \leq (-1) - \rho_{G,\mathbf{t}}(\{x, y\}) + \rho_{G,\mathbf{t}}(S) \leq -1,$$

a contradiction. On the other hand, if $x \in S'$ and $y \notin S'$, then $\rho_{G,\mathbf{t}}(G[S' \setminus \{x\} \cup S]) < 0 - i + 2i = i$. Since $|S| \geq 2$, this contradicts Lemma 9.3. Hence $\rho(G', \mathbf{t}') \geq 0$. Denote the subgraph of H induced by $G[S]$ by H_S. Since $(G, \mathbf{t})$ is $(i, i+1)$-critical, H_S has an $(i, i+1, \mathbf{t})$-coloring ϕ_1.

For every $1 \leq h \leq k$, denote the neighbor of $\phi_1(v_h)$ in $M(e_h)$ by $a(u_h)$, and the other vertex in $L(u_h)$ by $b(u_h)$. Let $\mathcal{H}' = (L', H')$ be the cover of G', such that :
1) for every $1 \leq h \leq k-1$, $p(x)$ is adjacent to $b(u_h)$ and $r(x)$ is adjacent to $a(u_h)$;
2) $p(y)$ is adjacent to $b(u_k)$ and $r(y)$ is adjacent to $a(u_k)$;
3) $H'[L'(V(G') \setminus \{x, y\})] = H[L(V(G) \setminus S)]$.

By (11) and the fact that $\rho(G', \mathbf{t}') \geq 0$, H' has an $(i, i+1, \mathbf{t})$-coloring ϕ_2. Since $\mathbf{t}'(x) = \mathbf{t}'(y) = (i+1, i+1)$,

(12) $\phi_2(x) = r(x)$, $\phi_2(y) = r(y)$, *and each of $r(x)$ and $r(y)$ has degree 0 in H'.*

Define a representative map ϕ on H by letting $\phi(v) = \phi_1(v)$ for every $v \in S$ and $\phi(v) = \phi_2(v)$ for every $v \in V(G) \setminus S$. By (12), for each $1 \leq h \leq k$, $\phi(v_h)$ is not adjacent to $\phi(u_h)$. Thus ϕ is an $(i, i+1, \mathbf{t})$-coloring of H, a contradiction. $\qquad\square$

Similarly to a low set, a *low vertex* is a vertex v with $\rho_{G,\mathbf{t}}(v) \leq 2i$.

Lemma 9.5. *Every low set consists of either one low vertex or two $(i+1, i+1)$-tough vertices.*

Proof. Suppose there exists a low set S with $|S| \geq 3$. By Lemma 9.4, $|E_G(S, V(G) - S)| = 1$. So we may assume $E_G(S, V(G) - S) = E_G(x, y) = \{e\}$ where $x \in S$ and $y \notin S$.

Let $G' = G - S$ and $\mathbf{t}'$ be defined by $\mathbf{t}'(y) = (\mathbf{t}_p(y) + 1, \mathbf{t}_r(y) + 1)$ and $\mathbf{t}'(w) = \mathbf{t}(w)$ for all $w \in V(G) - S - y$. By the definition of $\mathbf{t}'$, if $\rho(G', \mathbf{t}') \leq -1$, then there exists a low set $S' \subset V(G) - S$ with $y \in S'$. But in this case, $\rho_{G,\mathbf{t}}(S \cup S') \leq 2i + 2i - (i^2 + 3i + 1) = i - i^2 - 1 \leq -1$, a contradiction to $\rho(G, \mathbf{t}) \geq 0$. Thus $\rho(G', \mathbf{t}') \geq 0$, and by the minimality of G, G' has an $(i, i+1, \mathbf{t}')$-coloring ϕ. Let $a \in \{p, r\}$ be such that $a(x)$ is the neighbor of $\phi(y)$ in H.

Case 1: Vertex x is not $(i+1, i+1)$-tough. Let $\mathbf{t}''$ differ from $\mathbf{t}$ on S only in that $\mathbf{t}''_a(x) = \mathbf{t}_a(x) + 1$. By Lemma 9.3, $\rho(G[S], \mathbf{t}'') \geq 0$. So by the minimality of G, $G[S]$ has an $(i, i+1, \mathbf{t}'')$-coloring ψ. We claim that $\phi \cup \psi$ is an $(i, i+1, \mathbf{t})$-coloring of G. Indeed, if $\psi(x) \neq a(x)$, this is trivial, and if $\psi(x) = a(x)$, this follows from the definitions of $\mathbf{t}'$ and $\mathbf{t}''$.

Case 2: Vertex x is $(i+1, i+1)$-tough. Let $G_3 = G[S]$ and $G_4 = G - (S - x)$. By the minimality of G, G_3 has an $(i, i+1, \mathbf{t})$-coloring ϕ_3 and G_4 has an $(i, i+1, \mathbf{t})$-coloring ϕ_4. Since x is $(i+1, i+1)$-tough, $\phi_3(x) = \phi_4(x) = r(x)$, and $r(x)$ has neighbors neither in H_{ϕ_3} nor in H_{ϕ_4}. But then $\phi_3 \cup \phi_4$ is an $(i, i+1, \mathbf{t})$-coloring of G, a contradiction. $\square$

Lemma 9.6. *For every $v \in V(G)$, at most one edge connects v with a low vertex.*

Proof. Suppose for $h \in [2]$, $e_h \in E_G(v, u_h)$ and u_h is low (possibly, $u_1 = u_2$). Then

$$\rho_{G,\mathbf{t}}(\{v, u_1, u_2\}) \leq 2(2i) + (2i^2 + 4i + 1) - 2(i^2 + 3i + 1) = 2i - 1.$$

Then by Lemma 9.5, $V(G) = \{v, u_1, u_2\}$. Furthermore, in order to have $\rho(G, \mathbf{t}) \geq 0$, we need $E(G) = \{e_1, e_2\}$ and $\max\{\rho_{G,\mathbf{t}}(u_1), \rho_{G,\mathbf{t}}(u_2)\}\} > i$, say, $\rho_{G,\mathbf{t}}(u_1) > i$. Then either $\mathbf{t}_p(u_1) \leq i - 1$ or $\mathbf{t}_r(u_1) \leq i$, say $\mathbf{t}_r(u_1) \leq i$. In this case, we let $\phi(u_1) = r(u_1)$, let $\phi(u_2)$ be any color $\alpha \in \{p(u_2), r(u_2)\}$ of nonnegative capacity, and choose $\phi(v) \in \{p(v), r(v)\}$ not adjacent to α. By construction, the only possibility that ϕ is not an $(i, i+1, \mathbf{t})$-coloring of G is that $\phi(v)r(u_1) \in E(H)$ and either $\phi(v) = p(v)$ and $\mathbf{t}_p(v) \geq i$ or $\phi(v) = r(v)$ and $\mathbf{t}_r(v) \geq i+1$. Since $\rho_{G,\mathbf{t}}(v) \leq 2i^2 + 4i + 1 - (i+1)\max\{\mathbf{t}_p(v), \mathbf{t}_r(v)\}$, in order to have $\rho(G, \mathbf{t}) \geq 0$, we need $\max\{\mathbf{t}_p(v), \mathbf{t}_r(v)\} \leq 1$, which yields $i = 1$ and so $\rho_{G,\mathbf{t}}(v) \leq 2(1^2) + 4(1) + 1 - (1+1)1 = 5$. Hence $\rho_{G,\mathbf{t}}(\{v, u_1, u_2\}) \leq 2(2 \times 1) + 5 - 2(1^2 + 3(1) + 1) = 9 - 2(5) = -1$, a contradiction. $\square$

9.4. Potentials of the vertices of small degree.

Lemma 9.7. *If $v \in V(G)$ is a leaf, then $\rho_{G,\mathbf{t}}(v) \leq 2i$.*

Proof. Suppose $\rho_{G,\mathbf{t}}(v) \geq 2i + 1$, and $N(v) = \{u\}$. Then either $\mathbf{t}_p(v) = i+1$ or $\mathbf{t}_r(v) = i+2$, since otherwise we can extend to v any $(i, i+1, \mathbf{t})$-coloring of $G - v$. Moreover, by Lemma 9.2, the latter cannot hold. Thus $\mathbf{t}_p(v) = i+1$ and

$$(13) \qquad \rho_{G,\mathbf{t}}(v) \leq 2i^2 + 4i + 1 - (i+1)(i+1) = i^2 + 2i.$$

On the other hand, since $\rho_{G,\mathbf{t}}(v) \geq 2i + 1$,

$$(14) \qquad \mathbf{t}_r(v) \leq i.$$

Let $\beta(u) \in \{p(u), r(u)\}$ be the neighbor of $p(v)$ in H and $\overline{\beta}(u) \in L(u) - \beta(u)$. Let $G' = G - v$ and let $\mathbf{t}'$ differ from $\mathbf{t}$ on $V(G')$ only for $\mathbf{t}'(u)$, where the toughness of $\overline{\beta}(u)$ increases by 1. Since the potential of each subset of $V(G')$ decreases by at most $i + 1$, the only possibility that $\rho(G', \mathbf{t}') \leq -1$ is that $\rho_{G,\mathbf{t}}(u) \leq i$. But in this case by (13), $\rho_{G,\mathbf{t}}(\{u, v\}) \leq i + (i^2 + 2i) - (i^2 + 3i + 1) = -1$, a contradiction. Thus $\rho(G', \mathbf{t}') \geq 0$ and hence by the minimality of G, G' has an $(i, i+1, \mathbf{t}')$-coloring ϕ. Extend ϕ to v by letting $\phi(v) = r(v)$. If

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$\phi(u) = \beta(u)$, then we do not create conflicts, and if $\phi(u) = \overline{\beta}(u)$, then ϕ is an $(i, i+1, \mathbf{t})$-coloring of G because of (14) and the definition of $\mathbf{t}'$. $\qquad\square$

Lemma 9.8. *If $v \in V(G)$ and $d_G(v) = 2$, then $\rho_{G,\mathbf{t}}(v) \leq i^2 + 3i + 1$.*

Proof. Suppose $\rho_{G,\mathbf{t}}(v) \geq i^2 + 3i + 2$ and $E_G(v, V(G) - v) = \{e_1, e_2\}$ where $e_h = vu_h$ for $h \in [2]$. Then by (8), $\mathbf{t}_p(v) \leq i - 1$ and $\mathbf{t}_r(v) \leq i - 1$. By Lemma 9.6, we may assume u_1 is not low. Let $G' = G - v$ and let $\mathbf{t}'$ differ from $\mathbf{t}$ on $V(G')$ only in that $\mathbf{t}'(u_1) = (\mathbf{t}_p(u_1) + 1, \mathbf{t}_r(u_1) + 1)$. We claim that $\rho(G', \mathbf{t}') \geq 0$. Indeed, suppose $\rho_{G', \mathbf{t}'}(S) \leq -1$ for some $S \subseteq V(G')$. By the definition of $\mathbf{t}'$ this implies that $u_1 \in S$ and $\rho_{G,\mathbf{t}}(S) \leq (2i + 1) + \rho_{G', \mathbf{t}'}(S) \leq 2i$. Since u_1 is not low, this contradicts Lemma 9.5.

Thus $\rho(G', \mathbf{t}') \geq 0$, and by the minimality of G, G' has an $(i, i+1, \mathbf{t}')$-coloring ϕ. Extend ϕ to v by letting $\phi(v)$ be the color $\alpha \in L(v)$ not adjacent to $\phi(u_2)$. If α is not adjacent to $\phi(u_1)$, then $d_{H_\phi}(\alpha) = 0$, but even if $\alpha\phi(u_1) \in E(H)$, then by the choice of $\mathbf{t}'$ and the fact that $\mathbf{t}_p(v) \leq i - 1$ and $\mathbf{t}_r(v) \leq i - 1$, ϕ is an $(i, i+1, \mathbf{t})$-coloring of G. $\qquad\square$

Lemma 9.9. *If $v \in V(G)$ and $d_G(v) = 3$, then $\rho_{G,\mathbf{t}}(v) \leq i^2 + 4i + 2$.*

Proof. Suppose $\rho_{G,\mathbf{t}}(v) \geq i^2 + 4i + 3$ and $E_G(v, V(G) - v) = \{e_1, e_2, e_3\}$ where $e_h = vu_h$ for $h \in [3]$ (some u_h can coincide). By Lemma 9.6, we may assume that u_1 and u_2 are not low. Let $G' = G - v$. Define $\mathbf{t}'(x) = \mathbf{t}(x)$ for all $x \in V(G') - \{u_1, u_2\}$ and $\mathbf{t}'(u_h) = (\mathbf{t}_p(u_h) + 1, \mathbf{t}_r(u_h) + 1)$ for $h \in [2]$ (if $u_1 = u_2$, then $\mathbf{t}'(u_1) = (\mathbf{t}_p(u_1) + 2, \mathbf{t}_r(u_1) + 2)$).

Suppose there is $S \subseteq V(G')$ with $\rho_{G', \mathbf{t}'}(S) \leq -1$. Since u_1 and u_2 are not low, by Lemma 9.5, $\{u_1, u_2\} \subseteq S$, and $\rho_{G,\mathbf{t}}(S) = 2(2i + 1) + \rho_{G', \mathbf{t}'}(S) \leq 4i + 1$. Then

$$(15) \qquad \rho_{G,\mathbf{t}}(S \cup \{v\}) \leq (4i + 1) + \rho_{G,\mathbf{t}}(v) - 2(i^2 + 3i + 1) \leq 2i - 1.$$

Since v is not low, Lemma 9.5 and (15) yield that $S \cup \{v\} = V(G)$. But if $S \cup \{v\} = V(G)$, then in (15) we did not take e_3 into account. So, instead of (15), we have

$$\rho_{G,\mathbf{t}}(S \cup \{v\}) \leq (4i + 1) + \rho_{G,\mathbf{t}}(v) - 3(i^2 + 3i + 1) \leq -i^2 - i - 1,$$

a contradiction. $\qquad\square$

9.5. **Discharging.** At the start, each vertex v has charge $ch(v) = \rho_{G,\mathbf{t}}(v)$ and each edge e has charge $ch(e) = -(i^2 + 3i + 1)$. Then

$$\sum_{x \in V(G) \cup E(G)} ch(x) = \sum_{v \in V(G)} \rho_{G,\mathbf{t}}(v) - (i^2 + 3i + 1)|E(G)| = \rho_{G,\mathbf{t}}(V(G)) \geq 0.$$

In the discharging, every edge e gives charge $-0.5(i^2 + 3i + 1)$ to each of its ends. Denoting the resulting charge of an $x \in V(G) \cup E(G)$ by $\mu(x)$, we obtain that $\mu(e) = 0$ for each $e \in E(G)$, so that

$$(16) \qquad \sum_{v \in V(G)} \mu(v) = \sum_{x \in V(G) \cup E(G)} \mu(x) = \sum_{x \in V(G) \cup E(G)} ch(x) = \rho_{G,\mathbf{t}}(V(G)) \geq 0.$$

On the other hand, for each $v \in V(G)$, $\mu(v) = \rho_{G,\mathbf{t}}(v) - d(v)\frac{i^2 + 3i + 1}{2}$, and so

- If $d(v) \geq 4$, then $\mu(v) \leq (2i^2 + 4i + 1) - 2(i^2 + 3i + 1) < 0$;
- If $d(v) = 3$, then by Lemma 9.9, $\rho_{G,\mathbf{t}}(v) \leq i^2 + 4i + 2$, then $\mu(v) \leq (i^2 + 4i + 2) - 3(i^2 + 3i + 1)/2 < 0$;
- If $d(v) = 2$, then by Lemma 9.8, $\mu(v) \leq (i^2 + 3i + 1) - (i^2 + 3i + 1) = 0$;

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- If $d(v) = 1$, then by Lemma 9.7, $\mu(v) \le 2i - (i^2 + 3i + 1)/2 < 0$.

By (16), this implies that $\sum_{v \in V(G)} \mu(v) = \rho(G, \mathbf{t}) \le 0$ with equality only if each vertex has degree 2 and potential exactly $i^2 + 3i + 1$. Since G is connected, it must be a cycle. Furthermore, if $v \in V(G)$ with potential $i^2 + 3i + 1$ has $\mathbf{t}_p(v) \le i - 1$ and $\mathbf{t}_r(v) \le i$, then the proof of Lemma 9.8 still goes through. Therefore, for each $v \in V(G)$, $\mathbf{t}(v) = (i, 0)$. But then we color every $v \in V(G)$ with $r(v)$. Since $i + 1 \ge 2$, this is an $(i, i+1, \mathbf{t})$-coloring of G. This finishes the proof of Theorem 9.1.

10. Constructions

Given a multigraph G, for every $v \in V(G)$ we say v is a *d-vertex* if $d(v) = d$.

Definition 10 (flags). Given a vertex v in a multigraph G, a *flag at v* is a 2-cycle vuv such that u has degree 2, i.e., there are two edges connecting v with u, and no other edges incident to u. See Figure 1, v is the *base vertex* of the flags, and $u_1, \ldots, u_k$ are *flag vertices*.

Definition 11 (weak flags). Given a vertex v in a multigraph G, a *weak flag of weight i at v* is subgraph of G with vertex set $\{u_1, \ldots, u_i, x, y\}$ such that $u_1, \ldots, u_i$ are flags at x and y is a 2-vertex adjacent to x and v; see Figure 2.
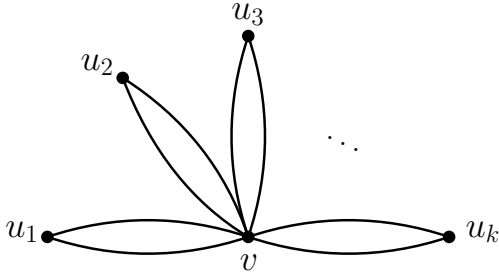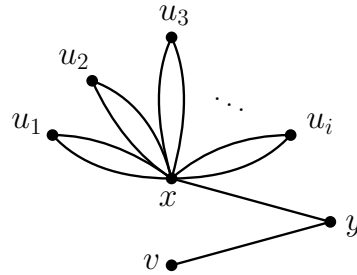

FIGURE 1. A vertex v with k flags.

FIGURE 2. A vertex v with one weak flag of weight i.

Call a vertex $v \in V(G)$ a *d^--vertex* if $d_G(v) \le d$, and a *d^+-vertex* in G if $d_G(v) \ge d$. We will use the following simple properties of (i, j)-critical graphs.

Lemma 10.1. *Let $0 \le i \le j$ and G be an (i, j)-critical graph. If v is a vertex of G with $N(v) = \{u\}$, and $\mathcal{H} = (L, H)$ is a cover of G that does not have any (i, j)-coloring, then some matching in $L(v)L(u)$ is even and some matching is odd. In particular, $\delta(G) \ge 2$, and for each flag vertex v with the neighbor u, one matching in $L(v)L(u)$ is even and one is odd.*

Proof. Suppose all the matchings in $L(v)L(u)$ have the same parity. Let $G' = G - v$ and $H' = H - L(v)$. By the minimality of G, H' admits an (i, j)-coloring ϕ. Then we can choose $\phi(v) \in L(v)$ not adjacent to $\phi(u)$, a contradiction. $\square$

Lemma 10.2. *Let $0 \le i \le j - 1$ and G be an (i, j)-critical graph. If v is a vertex of G with at most one edge connecting v with a 3^+-vertex, then $|N_G(v)| \ge i + 2$.*

Proof. Suppose $N_G(v) = \{u_0, \ldots, u_s\}$, where $s \le i$ and all vertices apart from u_0 are 2^--vertices. Then by Lemma 10.1, $d_G(u_1) = \ldots = d_G(u_s) = 2$. If all $u_0, \ldots, u_s$ are flag vertices,

then $V(G) = \{u_0, \ldots, u_s, v\}$. In this case, we let $\phi(v) = r(v)$, and for $0 \le h \le s$, choose $\phi(u_h) = r(u_h)$. By Lemma 10.1, $r(v)$ has in H_ϕ at most $s + 1 \le i + 1 \le j$ neighbors, and each $\phi(u_h)$ has at most one neighbor, contradicting the choice of G. Thus, not all $u_0, \ldots, u_s$ are flag vertices, and we may assume that u_0 is not a flag vertex. Then under the conditions of the lemma,

$$(17) \qquad \qquad \textit{only one edge connects } v \textit{ with } u_0.$$

Let $\mathscr{H} = (L, H)$ be a cover of G that does not have any (i, j)-coloring. Since G is (i, j)-critical, the cover $\mathscr{H}_1 = \mathscr{H} - L(v)$ of $G_1 = G - v$ has an (i, j)-coloring ϕ. By (17), we can choose $\phi(v) \in L(v)$ not adjacent to $\phi(u_0)$. Then for each $1 \le h \le s$, we do the following: If $d_{H_\phi}(\phi(u_h)) \le 1$, then leave the color unchanged, and if $d_{H_\phi}(\phi(u_h)) = 2$, then recolor u_h with the other color, and the degree of the new color will be 0. This way, $d_{H_\phi}(\phi(v)) \le s \le i$ and $d_{H_\phi}(\phi(u_h)) \le 1 \le i$ for each $1 \le h \le s$. Thus we obtained an (i, j)-coloring of $\mathscr{H}$, a contradiction to its choice. $\qquad \square$

10.1. **Examples of** $(0, j)$-**critical graphs.** We construct G_m as follows. Start from the cycle $v_0 v_1 \ldots v_m v_0$, and then for $h = 1, \ldots, j$, add a 3-cycle $x_h y_h u_h$ where u_h is adjacent to v_0, see Figure 3. By construction, for every integer m, $|V(G_m)| = 3j + m + 1$ and $|E(G_m)| = |V(G_m)| + j$.
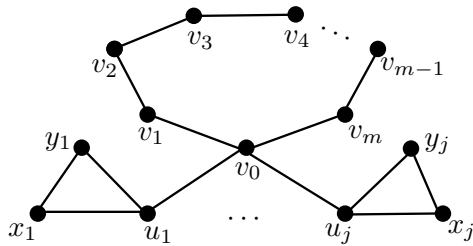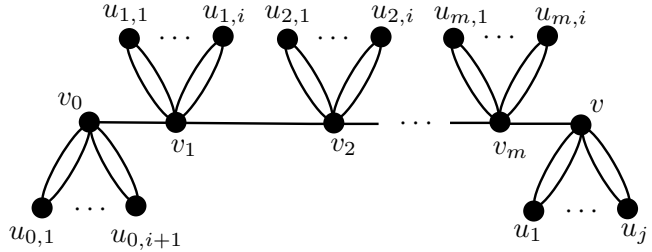

FIGURE 3. Critical graphs for $(0, j)$-coloring.

FIGURE 4. Critical graphs for (i, j)-colorings when j is large.

Proposition 10.3. G_m *is* $(0, j)$-*critical for every* m.

Proof. First, we construct a cover $\mathscr{H} = (L, H)$ of G_m, such that there is no $(0, j)$-coloring of H. For all $h = 1, \ldots, j$, we let the matchings $L(u_i)L(x_i)$ and $L(u_i)L(y_i)$ be odd, and the matching $L(x_i)L(y_i)$ be even. The matchings $L(v_g)L(v_{g+1})$ for $0 \le g \le m - 1$ are odd, all remaining matchings in H are even.

Suppose H admits a $(0, j)$-coloring ϕ. If for some $h \in [j]$, $\phi(u_h) = p(u_h)$, then since $L(u_i)L(x_i)$ and $L(u_i)L(y_i)$ are odd, we also have $\phi(x_h) = p(x_h)$ and $\phi(y_h) = p(y_h)$. But $L(x_i)L(y_i)$ is even, a contradiction. Thus, $\phi(u_h) = r(u_h)$ for all $h \in [j]$, and if $\phi(v_0) = r(v_0)$, then (since $L(v_0)L(v_m)$ is even) $\phi(v_m) = p(v_m)$. Since $L(v_{m-1})L(v_m)$ is odd, this yields $\phi(v_{m-1}) = p(v_{m-1})$. Similarly, we get $\phi(v_{m-2}) = p(v_{m-2})$, and so on. Finally, $\phi(v_1) = p(v_1)$, which means that $\phi(v_0)$ has $j + 1$ neighbors in H_ϕ, a contradiction. Hence we may assume $\phi(v_0) = p(v_0)$. Then symmetrically, $\phi(v_m) = r(v_m)$ and consecutively for $g = m - 1, m - 2, \ldots, 1$ we obtain $\phi(v_g) = r(v_g)$. This means that $\phi(v_0)$ has neighbor $\phi(v_1)$ in H_ϕ, a contradiction. So, G_m is not $(0, j)$-colorable.

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Now, let G' be any proper subgraph of G. Since every block of G is a cycle or an edge, G' has fewer than $j + 1$ cycles or is disconnected. In both cases, $|E(G')| - |V(G')| \leq j - 1$. So by Proposition 6.1, no proper subgraph of G is $(0, j)$-critical. Hence each proper subgraph of G is $(0, j)$-colorable. $\qquad\square$

10.2. **Examples of (i, j)-critical graphs for $i \geq 1$ and $j \geq 2i + 1$.** Let G_m be obtained from the path $v_0 v_1 \ldots v_m v$ by adding j flags with base v, $i + 1$ flags with base v_0 and for $h = 1, \ldots, m$, i flags with base v_h, see Fig. 4. By construction, for every $m \geq 0$, we have $|V(G_m)| = (i+1)(m+1) + 2 + j$ and $|E(G_m)| = (2i+1)(m+1) + 2 + 2j = \frac{(2i+1)|V(G_m)| + j - 2i}{i+1}$.

Proposition 10.4. *Suppose $i, j \geq 1$ and $j \geq 2i + 1$ be integers. Then G_m is (i, j)-critical for every m.*

Proof. We construct a cover $\mathscr{H} = (L, H)$ of G_m as follows. To each digon in G_m correspond one even and one odd matching in H. To each edge in the path $v_0 v_1 \ldots v_m$ corresponds an odd matching in H, and the matching corresponding to $v v_m$ is even.

Suppose H admits an (i, j)-coloring ϕ. Since $\phi(v_0)$ is adjacent to the $i + 1$ vertices in $\{\phi(u_{0,1}), \ldots, \phi(u_{0,i+1})\}$ (see Fig. 4), $\phi(v_0) = r(v_0)$. Since the matching corresponding to $v_0 v_1$ is odd and $\phi(v_1)$ is adjacent to the i vertices in $\{\phi(u_{1,1}), \ldots, \phi(u_{1,i})\}$, we also have $\phi(v_1) = r(v_1)$. Similarly, for each $2 \leq h \leq m$ we conclude that $\phi(v_h) = r(v_h)$. Finally, $\phi(v)$ is adjacent to the j vertices in $\{\phi(u_1), \ldots, \phi(u_j)\}$. Hence $\phi(v) = r(v)$, and $\phi(v_m)$ cannot be adjacent to $\phi(v)$. But $\phi(v_m) = r(v_m)$ and the matching corresponding to $v v_m$ is even, a contradiction.

Thus if G_m is not (i, j)-critical, then it contains a proper (i, j)-critical subgraph G'. Suppose the cover $\mathscr{H}' = (L', H')$ of G' has no (i, j)-coloring. Let $q = |V(G_m)| - |V(G')|$. By Proposition 7.1, $q \geq 1$. Also, $\delta(G') \geq 2$. Try to color G' as follows: for each $w \in V(G') \cap \{v_0, \ldots, v_m, v\}$, let $\psi(w) = r(w)$, and then for every remaining vertex u (which is a flag vertex), choose $\psi(u) \in L'(u)$ with at most one edge connecting $\psi(u)$ with $\psi(w)$ where w is the neighbor of u in G'. Since $\mathscr{H}'$ has no (i, j)-coloring, some vertex H'_ψ has more than j neighbors. By the definition of G, the only such vertex is $\psi(v)$. Thus $v \in V(G')$ and also

(18) $$\{u_1, \ldots, u_j, v_m\} \subset V(G')$$

If not every v_h belongs to $V(G')$, then let g be the largest index such that $v_g \notin V(G')$. By (18), $g \leq m - 1$. Then v_{g+1} has at most one incident non-flag edge and is the base of at most i flags, contradicting Lemma 10.2. Thus $\{u_1, \ldots, u_j, v, v_m, v_{m-1}, \ldots, v_0\} \subseteq V(G')$. Hence, G' is obtained from G by deleting q flag vertices and maybe some edges. But then, since $q \geq 1$,

$$|E(G')| \leq |E(G_m)| - 2q = \frac{(2i+1)(|V(G')| + q) + j - 2i}{i+1} - 2q < \frac{(2i+1)|V(G')| + j - 2i}{i+1},$$

contradicting Proposition 7.1. $\qquad\square$

10.3. **Examples of (i, j)-critical graphs for $i + 2 \leq j \leq 2i$.** Let $m \geq 1$. Let G_m be obtained from the path $P = v_0 v_1 \ldots v_{2m}$ by adding $j - 1$ flags at each of $v_2, v_4, \ldots, v_{2m-2}$ and adding j flags at v_0 and v_{2m}. Then $|V(G_m)| = (j+1)m + 2 + j$ and $|E(G_m)| = 2jm + 2j + 2 = \frac{2j|V(G_m)| + 2}{j+1}$.

Proposition 10.5. *Suppose $i + 2 \leq j \leq 2i$. Then G_m is (i, j)-critical for every $m \geq 1$.*

Proof. Since $i + 2 \leq 2i$, we have $i \geq 2$ and $j \geq 4$. We construct H as follows. To each digon in G_m correspond two disjoint matchings in H. For each $1 \leq h \leq m$, the matching between $L(v_{2h-2})$ and $L(v_{2h-1})$ is even, and the matching between $L(v_{2h-1})$ and $L(v_{2h})$ is odd.

Suppose H admits an (i, j)-coloring ϕ. Then for $h = 0, \ldots, m$, $\phi(v_{2h})$ has at least $j - 1 \geq i + 1$ neighbors in H_ϕ, hence $\phi(v_{2h}) = r(v_{2h})$. Furthermore, $\phi(v_0)$ has j neighbors in the flags based at v_0, so $\phi(v_1)$ is not adjacent to $\phi(v_0)$. Since the matching corresponding to $v_0 v_1$ is even, this means $\phi(v_1) = p(v_1)$. Since the matching corresponding to $v_1 v_2$ is odd, it follows that $\phi(v_2) = r(v_2)$ is adjacent to $\phi(v_1)$ and to $j - 1$ neighbors in the flags based at v_2. So, similarly to the situation with v_0, $\phi(v_3)$ is not adjacent to $\phi(v_2)$ and is adjacent to $\phi(v_4)$. Repeating the argument, we get that $\phi(v_5)$ is adjacent to $\phi(v_6)$, and so on. Finally, we get that $\phi(v_{2m-1})$ is adjacent to $\phi(v_{2m})$, and hence $\phi(v_{2m})$ has $j + 1$ neighbors in H_ϕ, a contradiction.

Thus, if G_m is not (i, j)-critical, then it contains a proper (i, j)-critical subgraph G'. Suppose the cover $\mathscr{H}' = (L', H')$ of G' has no (i, j)-coloring. Choose the smallest k such that $v_k \in V(G')$ and suppose $k > 0$. Since G' is (i, j)-critical, $H' - L(v_k)$ has an (i, j)-coloring ϕ. Since $k > 0$, v_k has at most j neighbors in G' and hence by Lemma 10.1, we can extend ϕ to v_k as follows: If k is even, then we let $\phi(v_k) = r(v_k)$, and if k is odd, then we choose $\phi(v_k) \in L(v_k)$ not adjacent to $\phi(v_{k+1})$ (if v_{k+1} is not in $V(G')$, then no restrictions). In both cases, we get an (i, j)-coloring of H', a contradiction.

Thus, $v_0 \in V(G')$. Symmetrically, $v_{2m} \in V(G')$. Since every (i, j)-critical multigraph is connected, this means $\{v_0, \ldots, v_{2m}\} \subseteq V(G')$. If G' has exactly s flag vertices, then $|V(G')| = 2m + 1 + s$ and $|E(G')| \leq 2m + 2s$. So by Proposition 8.1, $s \geq 2 + (m+1)(j-1)$, which means $V(G') = V(G)$. Since the minimum degree of each (i, j)-critical multigraph is at least 2, this yields $G' = G$, a contradiction. $\qquad\square$

10.4. **Examples of $(i, i+1)$-critical graphs for $i \geq 1$.** Let $v_0, \ldots, v_{m+1}$ be the vertices of P_{m+2}, where v_0 and v_{m+1} are end vertices. Define G_m by adding $i + 1$ weak flags of weight $i + 1$ to v_0, adding i weak flags of weight $i + 1$ to $v_1, \ldots, v_m$, and by adding $i + 1$ flags to v_{m+1}, see Fig. 5. Then for every m, $|V(G_m)| = (m+1)i^2 + (3m+4)i + i + m + 6$, $E(G_m) = 2(m+1)i^2 + 4(m+2)i + m + 7$, and thus

$$|E(G_m)| = \frac{(2i^2 + 4i + 1)|V(G_m)| + 1}{i^2 + 3i + 1}.$$

Proposition 10.6. *Let $i \geq 2$ be an integer. Then G_m is $(i, i+1)$-critical for every m.*

Proof. Let $\mathscr{H} = (L, H)$ be a cover of G_m, we define H as follows. Each digon in G_m represents two disjoint matchings in H, and the edges in each weak flag but not in any digons represent even matchings. The matchings between $L(v_t)$ and $L(v_{t+1})$ are odd for every $t \leq m - 1$, and the matching between $L(v_m)$ and $L(v_{m+1})$ is even.

Suppose H admits an $(i, i+1)$-coloring ϕ. Then $\phi(x_{\alpha,\beta})$, $\phi(v_{m+1})$ (See Fig. 5) are rich for every possible (α, β), and thus $\phi(y_{\alpha,\beta})$ are poor for all (α, β). Hence $\phi(v_i)$ are rich for all $0 \leq i \leq m$. But then $\phi(v_{m+1})$ has $i + 2$ neighbors, a contradiction.

If G_m is not $(i, i+1)$-critical, then it contains a proper critical subgraph G'. Suppose the cover $\mathscr{H}' = (L', H')$ of G' has no $(i, i+1)$-coloring. By Proposition 9.1, $|V(G_m)| - |V(G')| \geq$
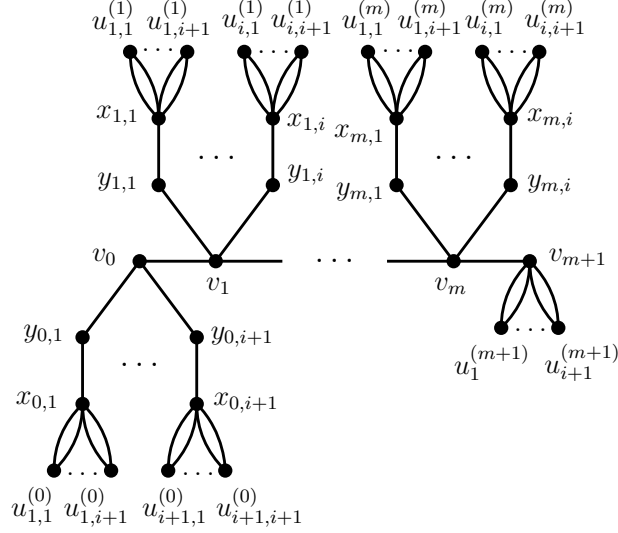
FIGURE 5. Critical graphs for $(i, i+1)$-colorings.

1, and $\delta(G') \geq 2$. By Lemma 10.2, $v_t \in V(G')$ for every $t = 0, \ldots, m+1$, and $u_s^{(m+1)} \in V(G')$ for $s = 1, \ldots, i+1$. We try to color G' as follows:

Suppose one of the vertices of a weak flag of v_t is not in G', where $0 \leq t \leq m$. We may assume $y_{t,1} \notin V(G')$ (By Lemma 10.1 if $y_{t,1} \in V(G')$ then $x_{t,1} \in V(G')$, and by Lemma 10.2 if $x_{t,1} \in V(G')$ then all the flag vertices of $x_{t,1}$ are in $V(G')$). For every $w \in V(G') \cap (\bigcup \{x_{\alpha,\beta}\} \cup \{v_{m+1}\})$, let $\Phi(w) = r(w)$. Let $\Phi(y_{a,b}) \in L'(y_{a,b})$ such that it is not adjacent to $\Phi(x_{a,b})$. For every $w \in \{v_0, \ldots, v_{t-1}\}$, let $\Phi(w) = r(w)$. For every $t \leq s \leq m$, let $\Phi(v_s) \in L'(v_s)$ such that it is not adjacent to $\Phi(v_{s+1})$. Since v_t has only at most $i+1$ neighbors in G', Φ is an $(i, i+1)$-coloring of $\mathscr{H}'$ of G', a contradiction. $\square$

10.5. **Examples of (i, i)-critical graphs for $i \geq 1$.** Let G_m be obtained from the $2m$-cycle $C = v_0 v_1 \ldots v_{2m-1} v_0$ by adding i flags (with flag vertices $u_{2h,1}, \ldots, u_{2h,i}$) at vertex v_{2h} for each $0 \leq h \leq m-1$. Then $|V(G_m)| = (i+2)m$ and $|E(G_m)| = 2m + 2im = \frac{(2i+2)|V(G_m)|}{i+2}$ for every positive integer m.

Proposition 10.7. *Let $i \geq 1$ be an integer. Then G_m is (i, i)-critical for every positive integer m.*

Proof. Let $\mathscr{H}_m = (L, H_m)$ be a cover of G_m, such that

(19) $\qquad\qquad$ *the matching between $L(v_{2m-1})$ and $L(v_0)$ is odd*

and

(20) $\qquad$ *for each $0 \leq h \leq 2m-2$, the matching between $L(v_h)$ and $L(v_{h+1})$ is even.*

Also, one of the matchings between $L(v_{2h})$ and $L(u_{2h,q})$ is odd and the other is even for each $0 \leq h \leq m-1$ and each $1 \leq q \leq i$. Suppose H_m has an (i, i)-coloring ϕ. Then for each $0 \leq h \leq m-1$, vertex $\phi(v_{2h})$ has i neighbors in the set $\{\phi(u_{2h,1}), \ldots, \phi(u_{2h,i})\}$. Therefore, $\phi(v_{2h}) \not\sim \phi(v_{2h-1})$ and $\phi(v_{2h}) \not\sim \phi(v_{2h+1})$. By (20), this yields that the parity of $\phi(v_{2h+1})$ differs from the parities of $\phi(v_{2h})$ and $\phi(v_{2h+2})$ for each $0 \leq h \leq m-1$. It follows that the

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parities of $\phi(v_0), \phi(v_2), \ldots, \phi(v_{2(m-1)})$ are the same, and the parity of $\phi(v_{2m-1})$ is different from them. But this contradicts (19).

Thus, if G_m is not (i,i)-critical, then it contains a proper (i,i)-critical subgraph G'. Suppose the cover $\mathcal{H}' = (L', H')$ of G' has no (i,i)-coloring. If G' does not contain a vertex v_{2h} for some $0 \le h \le m-1$, then by Lemma 10.1, also v_{2h-1} and v_{2h+1} are not in G'. But then by Lemma 10.2, also v_{2h-2} and v_{2h+2} are not in G', and so on. Thus, all vertices of C are in G'. Then again by Lemma 10.2, all vertices of G are in G'. It follows that G' is obtained from G by deleting some edges. This contradicts Proposition 3.1. $\qquad\square$

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