ON THE PILA-WILKIE THEOREM

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Abstract. In this expository paper we give an account of the Pila-Wilkie counting theorem and some of its extensions and generalizations. We use semialgebraic cell decomposition to simplify part of the original proof. We include a full treatment of a result due to Pila and Bombieri, and of a variant of the Yomdin-Gromov theorem that are used in this proof.

1. Introduction and some notations

In these notes we prove the Pila-Wilkie theorem following the original paper [6], but exploiting cell decomposition more thoroughly to simplify the deduction from its main ingredients. Apart from assuming some knowledge of o-minimality, we make this self-contained by including proofs of these ingredients.

We also obtain two generalizations due to Pila [5], one where instead of rational points we count points with coordinates in a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ with a finite bound on its dimension, and one where instead we count points with coordinates that are algebraic of at most a given degree over $\mathbb{Q}$. The general approach is as in [5], but the technical details seem to us a bit simpler.

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Throughout, $d, e, k, l, m, n \in \mathbb{N}$, and $\varepsilon, c, K \in \mathbb{R}^{>}$. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ we set $|\alpha| := \alpha_1 + \cdots + \alpha_m$, and given a field $\boldsymbol{k}$ (often $\boldsymbol{k} = \mathbb{R}$) we set $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ for the usual coordinate functions $x_1, \ldots, x_m$ on $\boldsymbol{k}^m$, and likewise $a^\alpha := a_1^{\alpha_1} \cdots a_m^{\alpha_m}$ for any point $a = (a_1, \ldots, a_m) \in \boldsymbol{k}^m$. Let $U \subseteq \mathbb{R}^m$ be open. For a function $f : U \to \mathbb{R}$ of class C^k and $\alpha \in \mathbb{N}^m$, $|\alpha| \leqslant k$, we let

$$f^{(\alpha)} \;:=\; \frac{\partial^{|\alpha|}}{\partial x^\alpha} f$$

denote the corresponding partial derivative of order α. We extend the above to C^k-maps $f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$, where

$$f^{(\alpha)} \;:=\; (f_1^{(\alpha)}, \ldots, f_n^{(\alpha)}) : U \to \mathbb{R}^n$$

for α as before. This includes the case $m = 0$, where $\mathbb{R}^0$ has just one point and any map $f : U \to \mathbb{R}^n$ is of class C^k for all k, with $f^{(\alpha)} = f$ for the unique $\alpha \in \mathbb{N}^0$. For $a_1, \ldots, a_n \in \mathbb{R}^{\geqslant}$ the number $\max\{a_1, \ldots, a_n\} \in \mathbb{R}^{\geqslant}$ equals 0 by convention if $n = 0$. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ we set $|a| := \max\{|a_1|, \ldots, |a_n|\} \in \mathbb{R}^{\geqslant}$; this conflicts with our notation $|\alpha|$ for $\alpha \in \mathbb{N}^n$, but in practice no confusion will arise. We also use these notational conventions when instead of $\mathbb{R}$ we have any o-minimal field with U and f definable in it, where an o-minimal field is by convention an o-minimal expansion of an ordered (necessarily real closed) field.

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The Pila-Wilkie theorem and two ingredients of the proof. First some notation needed to state the theorem. We define the *multiplicative height function* $\mathrm{H}:\mathbb{Q}\to\mathbb{R}$ by $\mathrm{H}(\frac{a}{b}) := \max(|a|,\ |b|)\in\mathbb{N}^{\geqslant 1}$ for coprime $a,b\in\mathbb{Z}$, $b\neq 0$. Thus $\mathrm{H}(0)=\mathrm{H}(1)=\mathrm{H}(-1)=1$, and for $q\in\mathbb{Q}$ we have $\mathrm{H}(q)\geqslant 2$ if $q\notin\{0,1,-1\}$, $\mathrm{H}(q)=\mathrm{H}(-q)$, and $\mathrm{H}(q^{-1})=\mathrm{H}(q)$ for $q\neq 0$. For $a=(a_1,...,a_n)\in\mathbb{Q}^n$,

$$\mathrm{H}(a) := \max\{\mathrm{H}(a_1),\dots,\mathrm{H}(a_n)\}\in\mathbb{N}.$$

Let $X\subseteq\mathbb{R}^n$. We set $X(\mathbb{Q})=X\cap\mathbb{Q}^n$. Throughout T ranges over real numbers $\geqslant 1$, and we set $X(\mathbb{Q},T):=\{a\in X(\mathbb{Q}):\ \mathrm{H}(a)\leqslant T\}$ be the (finite) set of rational points of X of height $\leqslant T$, and set $\mathrm{N}(X,T):=\#X(\mathbb{Q},T)\in\mathbb{N}$. The *algebraic part* of X, denoted by X^{alg}, is the union of the connected infinite semialgebraic subsets of X. So for $n\geqslant 1$, the interior of X is part of X^{alg}.

Example. Set $X:=\{(x,y,z)\in\mathbb{R}^3:\ 1<x,y<2,\ z=x^y\}$, so the set X is definable in $\mathbb{R}_{\exp}$. For rational $q\in(1,2)$, we have a semialgebraic curve

$$X_q := \{x,q,z):\ z=x^q\} \subseteq X.$$

One can show that X^{alg} is the union of those X_q.

We also set

$$X^{\mathrm{tr}} := X\setminus X^{\mathrm{alg}} \qquad (\textit{the transcendental part of } X).$$

We can now state the Pila-Wilkie theorem, also called the *Counting Theorem*:

Theorem 1.1. *Let $X\subseteq\mathbb{R}^n$ with $n\geqslant 1$ be definable in some o-minimal expansion of the real field. Then for all ε there is a c such that for all T,*

$$\mathrm{N}(X^{\mathrm{tr}},T) \leqslant cT^{\varepsilon}.$$

Roughly speaking, it says there are few rational points on the transcendental part of a set definable in an o-minimal expansion of the real field: the number of such points grows slower than any power T^{ε} with T bounding their height. To apply the counting theorem one needs to describe X^{alg} in some useful way. This typically involves *Ax-Schanuel* type transcendence results.

Note that $X^{\mathrm{tr}}(\mathbb{Q})=\emptyset$ in the example above, so the theorem is trivial for this X. We shall include a refinement, Theorem 2.5, which is nontrivial for this X.

The proof of Theorem 1.1 depends on two intermediate results. The first of these has nothing to do with o-minimality. To state it we define for $k,n\geqslant 1$ and $X\subseteq\mathbb{R}^n$ a *strong k-parametrization of X* to be a C^k-map $f:(0,1)^m\to\mathbb{R}^n$, $m<n$, with image X, such that $|f^{(\alpha)}(a)|\leqslant 1$ for all $\alpha\in\mathbb{N}^m$ with $|\alpha|\leqslant k$ and all $a\in(0,1)^m$. We also define a *hypersurface in $\mathbb{R}^n$ of degree $\leqslant e$* to be the zeroset in $\mathbb{R}^n$ of a nonzero polynomial in $x=(x_1,\dots,x_n)$ over $\mathbb{R}$ of (total) degree $\leqslant e$.

Theorem 1.2. *Let $n\geqslant 1$ be given. Then for any $e\geqslant 1$ there are $k=k(n,\ e)\geqslant 1$, $\varepsilon=\varepsilon(n,\ e)$, and $c=c(n,\ e)$, such that if $X\subseteq\mathbb{R}^n$ has a strong k-parametrization, then for all T at most cT^{ε} many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover $X(\mathbb{Q},T)$, with $\varepsilon(n,\ e)\to 0$ as $e\to\infty$.*

Let now R be any o-minimal field, and let $X\subseteq R^n$ be definable, $n\geqslant 1$. Then we introduce the notion of a *definable strong k-parametrization of X* as before, with R and the interval $(0,1)_R$ in R instead of $\mathbb{R}$ and the real interval $(0,1)$, and where f is definable. The second intermediate result in the proof of the Pila-Wilkie theorem is about decomposing a definable set in an o-minimal field into finitely many definable subsets that admit such a parametrization:

Theorem 1.3. *Given an o-minimal field R, every definable set $X \subseteq [-1,1]^n_R$ with empty interior and $n \geqslant 1$ is for every $k \geqslant 1$ a finite union of definable subsets, each having a definable strong k-parametrization.*

We use Theorem 1.3 not just when R is an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, even though Theorem 1.1 is only about definable sets in such expansions. This is because by model theory we obtain from Theorem 1.3 a uniform version of the corresponding result for any o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field. Here 'uniform' means that if instead of a single definable $X \subseteq \mathbb{R}^n$ we have a definable family $(X_b)_{b\in B}$ of such sets, then the decomposition of X_b into definable subsets and their k-parametrizations can also be taken to depend definably on $b \in B$.

2. Proof of the Counting Theorem from the two ingredients

Throughout this section we assume $n \geqslant 1$. We begin by stating some elementary facts about X^{alg} and X^{tr} for $X \subseteq \mathbb{R}^n$. The first is obvious:

Lemma 2.1. *If $X = X_1 \cup \cdots \cup X_m$, then $X^{\text{alg}} \supseteq X_1^{\text{alg}} \cup \cdots \cup X_m^{\text{alg}}$, and thus*

$$X^{\text{tr}} \ \subseteq \ X_1^{\text{tr}} \cup \cdots \cup X_m^{\text{tr}}.$$

Note also that if X is open in $\mathbb{R}^n$, then $X^{\text{tr}} = \emptyset$.

Lemma 2.2. *Suppose $S \subseteq \mathbb{R}^n$ is semialgebraic, $f : S \to \mathbb{R}^m$ is semialgebraic and injective, and f maps the set $X \subseteq S$ homeomorphically onto $Y = f(X) \subseteq \mathbb{R}^m$. Then $f(X^{\text{alg}}) = Y^{\text{alg}}$ and thus $f(X^{\text{tr}}) = Y^{\text{tr}}$. (We allow $m = 0$ for later inductions.)*

Proof. It is clear that $f(X^{\text{alg}}) \subseteq Y^{\text{alg}}$. Also, for any connected infinite semialgebraic set $C \subseteq Y$, the set $f^{-1}(C) \subseteq S$ is semialgebraic (since C and f are), contained in X (since f is injective), hence connected and infinite, and thus $f^{-1}(C) \subseteq X^{\text{alg}}$. This shows $f^{-1}(Y^{\text{alg}}) \subseteq X^{\text{alg}}$, and thus $f(X^{\text{alg}}) = Y^{\text{alg}}$. □

In order to apply Theorem 1.3 we need to reduce to the case of subsets of $[-1,1]^n$. This is done as follows. For $X \subseteq \mathbb{R}^n$ and $I \subseteq \{1,\dots,n\}$, set

$$X_I \ := \ \{a \in X : \ |a_i| > 1 \text{ for all } i \in I, \ |a_i| \leqslant 1 \text{ for all } i \notin I\}$$

and define the semialgebraic map $f_I : \mathbb{R}^n_I \to \mathbb{R}^n$ by $f_I(a) = b$ where $b_i := a_i^{-1}$ for $i \in I$ and $b_i := a_i$ for $i \notin I$. Thus f_I maps $\mathbb{R}^n_I$ homeomorphically onto its image, a subset of $[-1,1]^n$. If $I = \emptyset$, then f_I is the inclusion map $\mathbb{R}^n_I = [-1,1]^n \to \mathbb{R}^n$. Note that for $a \in \mathbb{Q}^n$ we have $f_I(a) \in \mathbb{Q}^n$ and $\mathrm{H}(a) = \mathrm{H}\big(f_I(a)\big)$. Moreover, X is the disjoint union of the sets X_I, and for $Y_I = f_I(X_I)$ we have $Y_I \subseteq [-1,1]^n$, $Y_I^{\text{tr}} = f_I(X_I^{\text{tr}})$ by Lemma 2.2, so $\mathrm{N}(Y_I^{\text{tr}}, T) = \mathrm{N}(X_I^{\text{tr}}, T)$ for all T.

The sketch below actually proves the Counting Theorem, modulo a uniformity assumption that arises at the end of the sketch. This motivates a stronger "definable family" version of the theorem, which we then prove as in the sketch. In the rest of this section we fix an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, and *definable* is with respect to $\widetilde{\mathbb{R}}$, and so are cells.

Sketch of the proof of Theorem 1.1 from Theorems 1.2 and 1.3. Let $X \subseteq \mathbb{R}^n$ be definable. We need to show that there are 'few' rational points on X outside X^{alg}. We proceed by induction on n. By Lemma 2.1 and the remark following it we can remove the interior of X in $\mathbb{R}^n$ from X and arrange that X has empty interior. As indicated just before this sketch we also arrange $X \subseteq [-1,1]^n$.

Let ε be given, and take $e \geqslant 1$ so large that $\varepsilon(n,e) \leqslant \varepsilon/2$ in Theorem 1.2, and take $k = k(n,e)$. By Theorem 1.3 for $\widetilde{\mathbb{R}}$, $X = X_1 \cup \cdots \cup X_M$, $M \in \mathbb{N}$, where each $X_i \subseteq \mathbb{R}^n$ is definable and admits a strong k-parametrization.

By Theorem 1.2, $X(\mathbb{Q},T) \subseteq \bigcup_{i=1}^M \bigcup_{j=1}^J H_{ij}$, where the H_{ij} are hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$, and $J \in \mathbb{N}$, $J \leqslant cT^{\varepsilon/2}$, $c = c(n,e)$ as in that theorem. If $a \in X^{\mathrm{tr}}(\mathbb{Q},T)$ and $a \in H_{ij}$, then clearly $a \in (X \cap H_{ij})^{\mathrm{tr}}$. Thus

$$X^{\mathrm{tr}}(\mathbb{Q},T) \ \subseteq\ \bigcup_{i=1}^{M}\bigcup_{j=1}^{J}(X \cap H_{ij})^{\mathrm{tr}}(\mathbb{Q},T).$$

Let H be any hypersurface in $\mathbb{R}^n$ of degree $\leqslant e$. We aim for an upper bound on $\mathrm{N}\big((X \cap H)^{\mathrm{tr}},T\big)$ of the form $c_1T^{\varepsilon/2}$ with $c_1 \in \mathbb{R}^{>}$ independent of H and T. (If we achieve this, then applying this to the hypersurfaces H_{ij} we obtain

$$\mathrm{N}(X^{\mathrm{tr}},T) \ \leqslant\ MJc_1T^{\varepsilon/2} \ \leqslant\ M \cdot cT^{\varepsilon/2} \cdot c_1T^{\varepsilon/2} \ =\ Mcc_1 \cdot T^{\varepsilon},$$

and we are done.) Take semialgebraic cells $C_1,\dots,C_L$ in $\mathbb{R}^n$, $L \in \mathbb{N}$, such that

$$H \ =\ C_1 \cup \cdots \cup C_L.$$

Suppose $C = C_l$ is one of those cells. Then by [2, (III, 2.7)] we have a semialgebraic homeomorphism $p = p_C : C \to p(C) = p(C_l)$ onto an open cell $p(C_l)$ in $\mathbb{R}^{n_l}$ with $n_l < n$, and so p maps $X \cap C_l$ homeomorphically onto its image $Y_l \subseteq p(C_l) \subseteq \mathbb{R}^{n_l}$. Now p is given by omitting $n - n_l$ of the coordinates, so for $a \in C_l(\mathbb{Q})$ we have $p(a) \in \mathbb{Q}^{n_l}$ and $\mathrm{H}\big(p(a)\big) \leqslant \mathrm{H}(a)$. The hypersurfaces of degree $\leqslant e$ in $\mathbb{R}^n$ belong to a single semialgebraic family, hence by [2, (III, 3.6)] we can (and do) take here $L \leqslant L(e,n)$, with $L(e,n) \in \mathbb{N}^{\geqslant 1}$ depending only on e,n. By Lemma 2.1,

$$(X \cap H)^{\mathrm{tr}} \ \subseteq\ (X \cap C_1)^{\mathrm{tr}} \cup \cdots \cup (X \cap C_L)^{\mathrm{tr}}.$$

Since the $n_l < n$ we can (and do) assume inductively that for all T,

$$\mathrm{N}(Y_l^{\mathrm{tr}},T) \ \leqslant\ B_lT^{\varepsilon/2}, \quad l = 1,\dots,L$$

with $B_l \in \mathbb{R}^{>}$ independent of T. Hence for all T,

$$\mathrm{N}\big((X \cap C_l)^{\mathrm{tr}}\big),T) \ \leqslant\ B_lT^{\varepsilon/2}, \quad l = 1,\dots,L$$

by Lemma 2.2 applied to the maps $p = p_{C_l}$, and thus

$$\mathrm{N}\big((X \cap H)^{\mathrm{tr}},T\big) \ \leqslant\ (B_1 + \cdots + B_L)T^{\varepsilon/2}.$$

Assume we can take $B_1,\dots,B_L \leqslant B$ with $B \in \mathbb{R}^{>}$ depending only on X,ε, not on $H, Y_1,\dots,Y_L$. Then $c_1 := L(e,n)B$ is a positive real number as aimed for.

The above sketch is a proof, modulo the assumption at the end. The hypersurfaces H in the sketch belong fortunately to a single semialgebraic family, a fact we already used, and so the sets Y_l as H varies can be taken to belong to a single definable family, depending on X. The inductive hypothesis should accordingly include this uniformity, and so the full proof should be carried out not just for one set X, but uniformly for all sets from a definable family, with constants depending only on the family. This is why we need Theorem 1.3 not just for $\widetilde{\mathbb{R}}$ but also for its elementary extensions, though in the above sketch we only used it for $\widetilde{\mathbb{R}}$. (As to the M introduced at the beginning of the sketch, Theorem 1.3 also provides an M that works for all members of the family.) Below we carry out the details.

Remarks on definable families. Let $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$. For $s \in E$, set

$$X(s) \;:=\; \{a \in \mathbb{R}^n :\; (s,a) \in X\} \quad \text{(a } \textit{section} \text{ of } X\text{)}$$

We consider E, X as describing the family $\big(X(s)\big)_{s\in E}$ of sections $X(s) \subseteq \mathbb{R}^n$; these sets $X(s)$ are called the members of the family. If E and X are definable, we call this a *definable family*, and then its members are definable subsets of $\mathbb{R}^n$. (In case $\widetilde{\mathbb{R}}$ is the ordered field of real numbers, we also write *semialgebraic family* instead of *definable family*.) We often divide the family given by E, X into subfamilies given by a covering $E = E_1 \cup \cdots \cup E_N$, where E_ν is typically the set of $s \in E$ for which $X(s)$ satisfies a certain condition e_ν. Then $X = X_1 \cup \cdots \cup X_N$ with $X_\nu := X \cap (E_\nu \times \mathbb{R}^n)$, so that $X_\nu(s)$ satisfies e_ν for all $s \in E_\nu$.

For the next lemma, a routine consequence of [2, III, Section 3], we recall from [2, III, Section 2] that for $\boldsymbol{i} = (i_1, \dots, i_n) \in \{0,1\}^n$ we have a projection map $p_{\boldsymbol{i}} : \mathbb{R}^n \to \mathbb{R}^d$, $d := i_1 + \cdots + i_n$, that maps every $\boldsymbol{i}$-cell homeomorphically onto its image, an open cell in $\mathbb{R}^d$.

Lemma 2.3. *Let $e \geqslant 1$ and set $D := \binom{e+n}{n}$, the dimension of the $\mathbb{R}$-linear space of polynomials over $\mathbb{R}$ in n variables and of degree $\leqslant e$. Then there are $L \in \mathbb{N}^{\geqslant 1}$ and semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$, $F := \mathbb{R}^D \setminus \{0\}$, such that*

$$\{\mathcal{H}(t) :\; t \in F\} = \text{ set of hypersurfaces in } \mathbb{R}^n \text{ of degree} \leqslant e,$$

$\mathcal{H}(t) = \mathcal{C}_1(t) \cup \cdots \cup \mathcal{C}_L(t)$ for all $t \in F$, and for each $l \in \{1, \dots, L\}$ there is an $\boldsymbol{i} = (i_1, \dots, i_n) \in \{0,1\}^n$, $\boldsymbol{i} \neq (1, \dots, 1)$, with the property that every $\mathcal{C}_l(t)$ with $t \in F$ is a semialgebraic $\boldsymbol{i}$-cell in $\mathbb{R}^n$ or empty.

Two family versions of the counting theorem. In this subsection we assume that $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ are definable.

Theorem 2.4. *Let any ε be given. Then there is a constant $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T we have $\mathrm{N}(X(s)^{\mathrm{tr}}, T) \leqslant cT^\varepsilon$.*

Proof. We proceed by induction on n. As in the sketch we reduce to the case where $X(s)$ is for every $s \in E$ a subset of $[-1,1]^n$ with empty interior. Take $e \geqslant 1$ so large that $\varepsilon(n,e) \leqslant \varepsilon/2$ in Theorem 1.2, and set $k = k(n,e)$. So for every $Z \subseteq \mathbb{R}^n$ with a strong k-parameterization we can cover $Z(\mathbb{Q}, T)$ with at most $cT^{\varepsilon/2}$ hypersurfaces of degree $\leqslant e$ where $c = c(n,e)$ is as in Theorem 1.2. From Theorem 1.3 we obtain definable sets $X_1, \dots, X_M \subseteq E \times \mathbb{R}^n$, $M \in \mathbb{N}$, such that for all $s \in E$, $X(s) = X_1(s) \cup \cdots \cup X_M(s)$ and each $X_i(s)$ is empty or has a strong k-parametrization. Let $s \in E$, and let H be a hypersurface of degree $\leqslant e$. As in the sketch we see that by our choice of k, e it is enough to show:

$$\mathrm{N}((X(s) \cap H)^{\mathrm{tr}}, T) \;\leqslant\; c_1 T^{\varepsilon/2}, \text{ for all } T,$$

where $c_1 \in \mathbb{R}^{>}$ depends only on X, ε, not on s, H, T. Below we provide such c_1.

With the present values of e and n, set $D := \binom{e+n}{n}$, $F := \mathbb{R}^D \setminus \{0\}$, and let $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$ be as in Lemma 2.3. For $l = 1, \dots, L$, take $\boldsymbol{i}^l = (i^l_1, \dots, i^l_n)$

in $\{0,1\}^n$, not equal to $(1,\dots,1)$, such that for all $t \in F$ the subset $\mathcal{C}_l(t)$ of $\mathbb{R}^n$ is a semialgebraic $\boldsymbol{i}^l$-cell or empty, so

$$p_{\boldsymbol{i}^l} : \mathbb{R}^n \to \mathbb{R}^{n_l}, \quad n_l := i_1^l + \cdots + i_n^l < n,$$

maps $\mathcal{C}_l(t)$ homeomorphically onto its image. Then we have for $l = 1,\dots,L$ a definable set $Y_l \subseteq (E \times F) \times \mathbb{R}^{n_l}$ such that for all $(s,t) \in E \times F$,

$$Y_l(s,t) \;=\; p_{\boldsymbol{i}^l}\big(X(s) \cap \mathcal{C}_l(t)\big).$$

Since all $n_l < n$ we can assume inductively that for all $(s,t) \in E \times F$ and all T,

$$\mathrm{N}(Y_l(s,t)^{\mathrm{tr}}, T) \;\leqslant\; B_l T^{\varepsilon/2}, \quad l = 1,\dots,L$$

with $B_l = B_l(Y_l,\varepsilon) \in \mathbb{R}^{>}$ independent of s,t,T. Since $H = \mathcal{H}(t)$ for some $t \in F$,

$$\mathrm{N}\big((X(s) \cap H)^{\mathrm{tr}}, T\big) \;\leqslant\; (B_1 + \cdots + B_L) T^{\varepsilon/2},$$

as in the sketch. Thus $c_1 := B_1 + \cdots + B_L$ is as promised. □

Next a variant of Theorem 2.4 where we remove from the sets $X(s)$ only a definable part $V(s)$ of $X(s)^{\mathrm{alg}}$ instead of all of it. The example preceding the statement of Theorem 1.1 shows that this variant is strictly stronger than Theorem 2.4.

Theorem 2.5. *Let any ε be given. Then there is a definable set $V = V(X,\varepsilon) \subseteq X$ and a constant $c = c(X,\varepsilon)$ such that for all $s \in E$ and all T,*

$$V(s) \subseteq X(s)^{\mathrm{alg}} \quad \textit{and} \quad \mathrm{N}\big(X(s) \setminus V(s), T\big) \leqslant cT^{\varepsilon}.$$

Proof. By induction on n. We follow closely the proof of Theorem 2.4. Let $V_0 \subseteq X$ be given by $V_0(s) =$ interior of $X(s)$ in $\mathbb{R}^n$ for $s \in E$. This definable set V_0 will be part of a V as required. Replacing X by $X \setminus V_0$ we arrange that $X(s)$ has empty interior for all $s \in E$. We arrange in addition that $X(s) \subseteq [-1,1]^n$ for all $s \in E$. Now take e and $k = k(n,e)$ as in the proof of Theorem 2.4. It will be enough to find a definable $V \subseteq X$ and a constant $c_1 \in \mathbb{R}^{>}$ such that for all $s \in E$, every hypersurface H of degree $\leqslant e$ in $\mathbb{R}^n$, and all T we have

$$V(s) \;\subseteq\; X(s)^{\mathrm{alg}}, \qquad \mathrm{N}\big((X(s) \cap H) \setminus V(s), T\big) \;\leqslant\; c_1 T^{\varepsilon/2}.$$

We take the semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$ and the definable sets $Y_l \subseteq E \times F \times \mathbb{R}^{n_l}$ for $l = 1,\dots,L$ as in the proof of Theorem 2.4. For such l we have $n_l < n$, so we can assume inductively that we have a definable set $W_l \subseteq Y_l$ and a number $B_l = B_l(Y_l,\varepsilon) \in \mathbb{R}^{>}$ such that for all $s \in E$, $t \in F$, and T we have

$$W_l(s,t) \;\subseteq\; Y_l(s,t)^{\mathrm{alg}} \quad \text{and} \quad N(Y_l(s,t) \setminus W_l(s,t), T) \;\leqslant\; B_l T^{\varepsilon/2}.$$

It is now easy to check that the definable set $V \subseteq X$ such that for all $s \in E$,

$$V(s) \;=\; \bigcup_{l=1}^{L} \bigcup_{t \in F} \mathcal{C}_l(t) \cap p_{\boldsymbol{i}^l}^{-1}\big(W_l(s,t)\big)$$

has the desired property. □

In the next sections we establish the results used in the proofs above, namely Theorems 1.2 and 1.3. In Section 8 we strengthen and extend Theorem 2.5 in several ways without changing the basic inductive set-up of its proof.

3. Proof of Theorem 1.2

We begin with introducing a key determinant. Let $\boldsymbol{k}$ be a field and set

$$D(n,\ e)\ :=\ \binom{e+n}{n}\ =\ \#\{\alpha\in\mathbb{N}^n:\ |\alpha|\ \leqslant\ e\}\in\mathbb{N}^{\geqslant 1},$$

the dimension of the $\boldsymbol{k}$-linear space of n-variable polynomials over $\boldsymbol{k}$ of (total) degree at most e. Thus $D(n,\ 0)=1$, $D(n,e)=\frac{e^n}{n!}\big(1+o(1)\big)$ as $e\to\infty$, and if $n\geqslant 1$, then $D(n,e)$ is strictly increasing as a function of e.

For now we fix n and e, set $D:=D(n,e)$ and let α range over $\mathbb{N}^n$. By a *hypersurface in* $\boldsymbol{k}^n$ *of degree* $\leqslant e$ we mean the set of zeros in $\boldsymbol{k}^n$ of a nonzero n-variable polynomial of degree $\leqslant e$ with coefficients in $\boldsymbol{k}$.

Lemma 3.1. *Let $a_1,\dots,a_D\in\boldsymbol{k}^n$. Then $a_1,\dots,a_D$ lie on a common hypersurface in $\boldsymbol{k}^n$ of degree at most e if and only if* $\det(a_i^\alpha)_{|\alpha|\leqslant e,i=1,\dots,D}=0$.

Proof. Let $f=\sum_{|\alpha|\leqslant e}c_\alpha x^\alpha$ be a nonzero polynomial in $x=(x_1,\dots,x_n)$ of degree at most e with coefficients $c_\alpha\in\boldsymbol{k}$ such that $f(a_1)=\cdots=f(a_D)=0$. Then

$$\sum_{|\alpha|\leqslant e}c_\alpha\big(a_1^\alpha,\dots,a_D^\alpha\big)\ =\ 0\ \text{ in }\ \boldsymbol{k}^D,$$

so the D vectors $(a_1^\alpha,\dots,a_D^\alpha)$ ($|\alpha|\leqslant e$) in the $\boldsymbol{k}$-linear space $\boldsymbol{k}^D$ are linearly dependent, which gives the desired conclusion.

Conversely, suppose $\det(a_i^\alpha)_{|\alpha|\leqslant e,i=1,\dots,D}=0$. Then we can reverse the argument above: the D vectors above are linearly dependent, and this provides coefficients c_α of a polynomial f as required. □

Next we introduce some numbers related to $D=D(n,e)$:

$$E(n,\ e)\ :=\ \binom{e+n-1}{n-1}\ =\ \#\{\alpha:\ |\alpha|=e\},$$

the dimension of the $\boldsymbol{k}$-linear space of homogeneous n-variable polynomials of degree e over $\boldsymbol{k}$. (Here $\binom{-1}{-1}:=1$ and $\binom{k}{-1}:=0$.) So $D(n,e)=\sum_{i=0}^e E(n,i)$. Next, we set $V(n,e):=\sum_{i=0}^e iE(n,i)$. Now for $i\geqslant 1$,

$$iE(n,i)\ =\ i\binom{i+n-1}{n-1}\ =\ n\binom{i+n-1}{n}\ =\ nE(n+1,i-1),\ \text{ so}$$

$$V(n,e)\ =\ n\sum_{i=1}^{e}E(n+1,i-1)\ =\ nD(n+1,e-1)\text{ for }e\geqslant 1,\quad V(n,0)=0,$$

and thus for fixed n we have $V(n,e)=\frac{ne^{n+1}}{(n+1)!}\big(1+o(1)\big)$ as $e\to\infty$.

Let $e,m,n\geqslant 1$ below and define $b=b(m,\ n,\ e)\in\mathbb{N}$ by requiring

$$D(m,\ b)\ \leqslant\ D(n,\ e)\ <\ D(m,\ b+1)\ .$$

Next, we set for $b = b(m,n,e)$:

$$\begin{aligned} B(m,\ n,\ e)\ &:=\ \sum_{i=0}^{b} iE(m,i) + (b+1)\cdot\big(D(n,\ e) - \sum_{i=0}^{b} E(m,i)\big) \\ &=\ V(m,b) + (b+1)\cdot\big(D(n,e) - D(m,b)\big) \in \mathbb{N}^{\geqslant 1}, \\ \varepsilon(m,\ n,\ e)\ &:=\ \frac{mneD(n,e)}{B(m,n,e)}. \end{aligned}$$

Lemma 3.2. *With fixed $m,n \geqslant 1$ and $e \to \infty$, we have:*

(1) $b(m,\ n,\ e)\ =\ \big(\frac{m!e^n}{n!}\big)^{1/m}\big(1+o(1)\big)$;
(2) $B(m,\ n,\ e)\ =\ \frac{m}{(m+1)!}\big(\frac{m!}{n!}\big)^{(m+1)/m} e^{n(m+1)/m}\big(1+o(1)\big)$;
(3) *if $m < n$, then $\varepsilon(m,\ n,\ e) \to 0$.*

Proof. As to (1), for $e \to \infty$ we have $b = b(m,n,e) \to \infty$, so

$$D(m,b)\ =\ \frac{b^m}{m!}\big(1+o(1)\big)\ \leqslant\ \frac{e^n}{n!}\big(1+o(1)\big)\ \leqslant\ \frac{(b+1)^m}{m!}\big(1+o(1)\big),$$

but the last term here is also $\frac{b^m}{m!}\big(1+o(1)\big)$, like the first term, and this easily yields the asymptotics claimed for b. For (2), substituting the result of (1) in the asymptotics for $D(m,b)$ as $b \to \infty$ leads to $(b+1)\cdot\big(D(n,e)-D(m,b)\big) = o\big(e^{n(m+1)/m}\big)$, and then in the asymptotics for $V(m,b)$ yields the asymptotics claimed for $B(m,n,e)$. Now (3) is an easy consequence of (2). □

In the proof of Proposition 3.4 below we need a reasonable bound on the absolute value of the determinant of a certain $(D \times D)$-matrix of the form $\big(a_i^\alpha\big)_{|\alpha| \leqslant e, i=1,\dots,D}$. We achieve this by expressing the matrix as a sum of simpler matrices. In this connection we need a useful expression for the determinant of a sum of matrices.

Turning to this, let $N \in \mathbb{N}$ and consider an $(N \times N)$-matrix $a = (a_{\mu\nu})_{1 \leqslant \mu,\nu \leqslant N}$ over a field $\boldsymbol{k}$. The determinant of an $(N \times N)$-matrix over $\boldsymbol{k}$ is an alternating multilinear function of its columns. The columns of a are $a_1, \dots, a_N \in \boldsymbol{k}^N$ where $a_\nu = (a_{1\nu}, \dots, a_{N\nu})^t \in \boldsymbol{k}^N$ is the νth column of a. Thus

$$a\ =\ (a_1, \dots, a_N) \in \boldsymbol{k}^N \times \cdots \times \boldsymbol{k}^N \text{ (with } N \text{ factors } \boldsymbol{k}^N).$$

Next, let $a = a^1 + \cdots + a^r$ with $r \in \mathbb{N}$ and $a^1, \dots, a^r$ also $(N \times N)$-matrices over $\boldsymbol{k}$, with a^j having ν^{th} column a_ν^j. Then

$$\begin{aligned} \det a\ &=\ \det\big(a_1, \dots, a_N\big)\ =\ \det\big(\sum_{j=1}^{r} a_1^j, \dots, \sum_{j=1}^{r} a_N^j\big) \\ &= \sum_{\boldsymbol{j}} \det\big(a_1^{j_1}, \dots, a_N^{j_N}\big) \end{aligned}$$

where $\boldsymbol{j} = (j_1, \dots, j_N)$ ranges here and below over elements of $\{1, \dots, r\}^N$. Let $\boldsymbol{j}$ be given. If for some j in $\{1, \dots, r\}$ the number of $\nu \in \{1, \dots, N\}$ with $j_\nu = j$ is more than $\operatorname{rank} a^j$, then the column vectors $a_1^{j_1}, \dots, a_N^{j_N}$ are $\boldsymbol{k}$-linearly dependent, so $\det\big(a_1^{j_1}, \dots, a_N^{j_N}\big) = 0$. Thus if $J \subseteq \{1, \dots, r\}^N$ contains all $\boldsymbol{j}$ such that

$$\#\{\nu \in \{1, \dots, N\} :\ j_\nu = j\}\ \leqslant\ \operatorname{rank} a^j,\ \text{for } j = 1, \dots, r,$$

then

$$(*) \qquad \det a \ = \ \sum_{\boldsymbol{j}\in J} \det\left(a_1^{j_1},\dots,a_N^{J_N}\right) \ = \ \sum_{\boldsymbol{j}\in J} \det\left(a_{\mu\nu}^{j_\nu}\right)_{1\leqslant \mu,\nu\leqslant N} .$$

We shall also use the following observation:

Lemma 3.3. *Let A be a set and V a finite-dimensional subspace of the $\boldsymbol{k}$-linear space $\boldsymbol{k}^A$. Then for any $N\in\mathbb{N}$, functions $f_1,\dots,f_N\in V$, and points $a_1,\dots,a_N$ in A, the rank of the $(N\times N)$-matrix $\left(f_\mu(a_\nu)\right)_{1\leqslant\mu,\nu\leqslant N}$ over $\boldsymbol{k}$ is $\leqslant \dim V$.*

Proof. The map $f\mapsto \left(f(a_1),\dots,f(a_N)\right) : V\to \boldsymbol{k}^N$ is $\boldsymbol{k}$-linear, so the image of this map is a subspace of the $\boldsymbol{k}$-linear space $\boldsymbol{k}^N$ of dimension $\leqslant \dim V$. $\square$

Recall our norm $|(t_1,\dots,t_m)| := \max\{|t_1|,\dots,|t_m|\}$ on $\mathbb{R}^m$, $m\geqslant 1$.

Proposition 3.4. *Let $e,m,n\geqslant 1$, $m<n$, and $k:=b(m,n,e)+1$. Then there is a constant $K=K(m,n,e)$ with the following property: if $f:(0,1)^m\to\mathbb{R}^n$ is a strong k-parametrization, $0<r\leqslant 1$, and $a_0,\dots,a_D\in(0,1)^m$ with $D=D(n,e)$ are such that $|a_i-a_0|\leqslant r$ for $i=1,\dots,D$, then*

$$|\det\left(f(a_i)^\alpha\right)_{|\alpha|\leqslant e,\ i=1,\dots,D}| \ < \ Kr^{B(m,n,e)}.$$

Proof. Let $f=(f_1,\dots,f_n)$ with $f_j:(0,1)^m\to\mathbb{R}$. Taylor expansion around a_0 gives for $i=1,\dots,D$ and $j=1,\dots,n$, and with $b:=b(m,n,e)$:

$$f_j(a_i)=P_j(a_i-a_0)+R_{ij}(a_i-a_0)$$

where $P_j\in\mathbb{R}[x_1,\dots,x_m]$ has degree $\leqslant b$, the remainder is given by a homogeneous polynomial $R_{ij}\in\mathbb{R}[x_1,\dots,x_m]$ of degree $k=b+1$, and all coefficients of P_j and R_{ij} are bounded in absolute value by 1. Hence for $|\alpha|\leqslant e$,

$$\begin{aligned} f(a_i)^\alpha \ &= \ \prod_{j=1}^n f_j(a_i)^{\alpha_j} \ = \ \prod_{j=1}^n \left(P_j(a_i-a_0)+R_{ij}(a_i-a_0)\right)^{\alpha_j} \\ &= \ P_\alpha(a_i-a_0)+R_{i\alpha}(a_i-a_0) \end{aligned}$$

with $P_\alpha\in\mathbb{R}[x_1,\dots,x_m]$ of degree $\leqslant b$, the remainder $R_{i\alpha}\in\mathbb{R}[x_1,\dots,x_m]$ has only monomials of degree $>b$, and every coefficient of P_α and $R_{i\alpha}$ is bounded in absolute value by $D(m,k)^{|\alpha|}$, the latter because $\prod_{j=1}^n f_j(a_i)^{\alpha_j}$ is a product of $|\alpha|$ factors of the form $\sum c_\beta(a_i-a_0)^\beta$, with the summation over the $\beta\in\mathbb{N}^m$ with $|\beta|\leqslant k$, and real coefficients c_β with $|c_\beta|\leqslant 1$. Note that $D(m,k)^{|\alpha|}\leqslant D(m,k)^e\leqslant C$ for a positive constant $c=c(m,n,e)$ depending only on m,n,e. Hence for $|\alpha|\leqslant e$ we have $P_\alpha=\sum_{j=0}^b P_\alpha^j$ where $P_\alpha^j\in\mathbb{R}[x_1,\dots,x_m]$ is homogeneous of degree j. In the matrix algebra $\mathbb{R}^{D\times D}$ this yields the sum decomposition

$$\begin{aligned} \left(f(a_i)^\alpha\right)_{\alpha,i} \ &= \ \sum_{j=0}^b \left(P_\alpha^j(a_i-a_0)\right)_{\alpha,i}+\left(R_{i\alpha}(a_i-a_0)\right)_{\alpha,i} \\ &= \sum_{j=0}^k \left(P_{i\alpha}^j(a_i-a_0)\right)_{\alpha,i} \end{aligned}$$

where $P_{i\alpha}^j := P_\alpha^j$ for $j=0,\dots,b$ and $P_{i\alpha}^k:=R_{i\alpha}$. For $j=0,\dots,b$ the rank of the matrix $\left(P_{i\alpha}^j(a_i-a_0)\right)_{\alpha,i}=\left(P_\alpha^j(a_i-a_0)\right)_{\alpha,i}$ is at most $E(m,j)$ by Lemma 3.3, so expression $(*)$ for the determinant of such a sum gives

$$\det\big(f(a_i)^\alpha\big)_{\alpha,i} \;=\; \sum_{\boldsymbol{j}\in J}\det\big(P_{i\alpha}^{j_i}(a_i-a_0)\big)_{\alpha,i}$$

where J is the set of all $\boldsymbol{j}=(j_1,\dots,j_D)\in\{0,\dots,b+1\}^D$ such that

$$\#\{\nu\in\{1,\dots,D\}:\ j_\nu=j\}\ \leqslant\ E(m,j),\qquad \text{for } j=0,\dots,b.$$

Let $\boldsymbol{j}\in J$. Then $|\det\big(P_{i\alpha}^{j_i}(a_i-a_0)\big)_{\alpha,i}|\leqslant D!c^D r^{|\boldsymbol{j}|}$. As to the exponent $|\boldsymbol{j}|$, let $d_j\in\mathbb{N}$ for $j=0,\dots,b$ be such that

$$\#\{\nu\in\{1,\dots,D\}:\ j_\nu=j\}\ =\ E(m,j)-d_j,$$

and set $f:=\#\{\nu\in\{1,\dots,D\}:\ j_\nu=b+1\}$. Then

$$D\ =\ D(n,e)\ =\ \sum_{j=0}^{b}(E(m,j)-d_j)+f\ =\ D(m,b)-\sum_{j=0}^{b}d_j+f,$$

so $f\ =\ D(n,e)-D(m,b)+d$ with $d:=\sum_{j=0}^{b}d_j$. Hence

$$\begin{aligned}
|\boldsymbol{j}|\ &=\ \sum_{\nu=1}^{b+1}j_\nu\ =\ \sum_{j=0}^{b}j\big(E(m,j)-d_j\big)+(b+1)f\\
&=\ V(m,b)-\sum_{j=0}^{b}jd_j+(b+1)\big(D(n,e)-D(m,b)+d\big)\\
&=\ V(m,b)+(b+1)\big(D(n,e)-D(m,b)\big)+\sum_{j=0}^{b}(b+1-j)d_j\\
&\geqslant\ B(m,n,e).
\end{aligned}$$

Therefore, $|\det\big(f(a_i)^\alpha\big)_{|\alpha|\leqslant e,\ i=1,\dots,D}|\ \leqslant\ \#J\cdot D!c^D r^{B(m,n,e)}$, which gives a constant $K=K(m,n,e)$ as claimed. □

We need one more simple observation:

Lemma 3.5. *Let points $b_1,\dots,b_D\in\mathbb{Q}^n$ with $D=D(n,e)$ be given such that $\mathrm{H}(b_1),\dots,\mathrm{H}(b_D)\leqslant t$, where $t\geqslant 1$. Then*

$$\det\big(b_i^\alpha\big)_{|\alpha|\leqslant e,i}\ \in\ \frac{\mathbb{Z}}{s}\qquad \textit{with } s\in\mathbb{N}^{\geqslant 1},\ s\leqslant t^{neD}.$$

Proof. For $i=1,\dots,D$ we have $b_i=(b_{i1},\dots,b_{in})$ with $b_{ij}=c_{ij}/s_{ij}$, $c_{ij},s_{ij}\in\mathbb{Z}$, $1\leqslant s_{ij}\leqslant t$, so

$$b_i^\alpha=\prod_{j=1}^{n}c_{ij}^{\alpha_j}\Big/\prod_{j=1}^{n}s_{ij}^{\alpha_j}\ \in\ \frac{\mathbb{Z}}{s_{i\alpha}},\qquad s_{i\alpha}\ :=\ \prod_{j=1}^{n}s_{ij}^{\alpha_j}.$$

Let $\{\alpha:\ |\alpha|\leqslant e\}=\{\alpha_1,\dots,\alpha_D\}$. Then $\det\big(b_i^\alpha\big)_{|\alpha|\leqslant e,i}$ is a sum of terms of the form $\pm\prod_{i=1}^{D}b_i^{\alpha_{\sigma(i)}}$ where σ is a permutation of $\{1,\dots,D\}$. Now the term $\pm\prod_{i=1}^{D}b_i^{\alpha_{\sigma(i)}}$ corresponding to σ lies in $\frac{\mathbb{Z}}{s_\sigma}$ with

$$s_\sigma\ :=\ \prod_{i=1}^{D}s_{i\alpha_{\sigma(i)}}\ =\ \prod_{i=1}^{D}\prod_{j=1}^{n}s_{ij}^{\alpha_{\sigma(i)j}}$$

and clearly $s := \prod_{i=1}^{D} \prod_{j=1}^{n} s_{ij}^e$ is a common integer multiple of the integers s_σ with $1 \leqslant s \leqslant t^{neD}$, so s has the desired property. $\square$

The following is Theorem 3 with more explicit values of k and ε.

Theorem 3.6. *Let $e, m, n \geqslant 1$, $m < n$; set $k := b(m,n,e)+1$, $\varepsilon := \varepsilon(m,n,e)$. Let $X \subseteq \mathbb{R}^n$ have a strong k-parametrization $f : (0,1)^m \to \mathbb{R}^n$. Then for all T at most cT^ε hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover $X(\mathbb{Q}, T)$, where $c = c(m,n,e)$ depends only on m, n, e.*

Proof. Let $K = K(m,n,e)$ be as in Proposition 3.4, and let T be given. With $D = D(n,e)$, let $a_1, \dots, a_D \in (0,1)^m$ be such that $f(a_1), \dots, f(a_D) \in X(\mathbb{Q}, T)$. Then Lemma 3.5 gives $s \in \mathbb{N}^{\geqslant 1}$ with $s \leqslant T^{neD}$ (so $T^{-neD} \leqslant 1/s$) such that

$$\det \left(f(a_i)^\alpha \right)_{|\alpha| \leqslant e, i=1,\dots D} \in \frac{\mathbb{Z}}{s}.$$

Assume also that $0 < r \leqslant 1$ and $a_0 \in (0,1)^m$ are such that $|a_i - a_0| \leqslant r$ for $i = 1, \dots, D$. Can we guarantee that $f(a_1), \dots, f(a_D)$ lie on a common hypersurface in $\mathbb{R}^n$ of degree $\leqslant e$ if r is small enough? Proposition 3.4 gives

$$|\det \left(f(a_i)^\alpha \right)_{|\alpha| \leqslant e,\ i=1,\dots,D}| < Kr^B, \quad B = B(m,n,e).$$

So by Lemma 3.1 the answer to the question is *yes*: it is enough that $Kr^B \leqslant T^{-neD}$, that is, $r \leqslant \left(K^{-1}T^{-neD}\right)^{1/B}$. Next, considering closed balls of radius r with respect to the norm $|\cdot|$, centered at a point in $(0,1)^m$, how many are enough to cover $(0,1)^m$? For $m = 1$, the interval $(0,1)$ is covered by e segments $[a-r, a+r]$ with $0 < a < 1$, for any natural number e with $2re \geqslant 1$, and there is clearly such an e with $e \leqslant r^{-1}$. Hence at most r^{-m} closed balls of radius r centered at points in $(0,1)^m$ are enough to cover $(0,1)^m$. Taking $r = \left(K^{-1}T^{-neD}\right)^{1/B}$ it follows that at most $K^{m/B}T^{mneD/B} = K^{m/B}T^\varepsilon$ hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $X(\mathbb{Q}, T)$. So the theorem holds with $c = K^{m/B}$. $\square$

4. Parametrization

Throughout R is an o-minimal field. As usual we identify $\mathbb{Q}$ with the prime subfield of R. We drop the subscript R in expressions like $(0,1)_R$.

Let $X \subseteq R^m$ be definable. Call X *strongly bounded* if $X \subseteq [-N,N]^m$ for some N in $\mathbb{N}$. Call a definable map $f : X \to R^n$ strongly bounded if its graph $\Gamma(f) \subseteq R^{m+n}$ is strongly bounded; equivalently, $X \subseteq R^m$ and $f(X) \subseteq R^n$ are strongly bounded.

A *partial k-parametrization of X* is a definable C^k-map $f : (0,1)^l \to R^m$ such that $l = \dim X$ (so $X \neq \emptyset$), the image of f is contained in X, and $f^{(\beta)}$ is strongly bounded for all $\beta \in \mathbb{N}^l$ with $|\beta| \leqslant k$. A *k-parametrization of X* is a finite set of partial k-parametrizations of X whose images cover X; note that then X is strongly bounded. As a trivial example, if X is finite and strongly bounded, then X has the k-parametrization $\{\phi_a : a \in X\}$, where $\phi_a : (0,1)^0 \to R^m$ takes the value a.

The basic ideas for the proofs of the next two parametrization theorems stem from Yomdin [8] and Gromov [3]. They considered the semialgebraic case over $\mathbb{R}$. For us it is convenient to work in an arbitrary o-minimal field.

Theorem 4.1. *Any strongly bounded definable set $X \subseteq R^m$ has for every $k \geqslant 1$ a k-parametrization.*

In the inductive proof of this theorem we also need a version for definable maps. A *k-reparametrization* of a definable map $f : X \to R^n$ is a k-parametrization Φ of its domain X such that for every $\phi : (0,1)^l \to R^m$ in Φ, $f \circ \phi$ is of class C^k and $(f \circ \phi)^{(\beta)}$ is strongly bounded for all $\beta \in \mathbb{N}^l$ with $|\beta| \leqslant k$; note that then $\{f \circ \phi : \ \phi \in S\}$ is a k-parametrization of $f(X)$, provided $\dim X = \dim f(X)$.

Theorem 4.2. *Any strongly bounded definable map $f : X \to R^n$, $X \subseteq R^m$ has for any $k \geqslant 1$ a k-reparametrization.*

The next three sections are devoted to the proof of Theorems 4.1 and 4.2. When convenient we can assume there that R is $\aleph_0$-saturated, and thus non-archimedean. This can always be arranged by taking a suitable elementary extension and noting that the statements of 4.1 and 4.2 pull back to the original structure.

We often use the following facts, proved by repeated use of the Chain Rule:

Lemma 4.3. *Let $f : U \to R,\ g : V \to R$ be definable of class C^k, $k \geqslant 1$, with U, V (definable) open subsets of R. Then $f \circ g : V \cap g^{-1}(U) \to R$ is of class C^k with*

$$(f \circ g)^{(k)} \ = \ \sum_{i=1}^{k} (f^{(i)} \circ g) \cdot p_{ik}\big(g^{(1)}, \dots, g^{(k-i+1)}\big)$$

where the $p_{ik} \in \mathbb{Z}[x_1, \dots, x_{k-i+1}]$ have constant term 0 and $p_{kk} = x_1^k$.

Lemma 4.4. *With $U \subseteq R^l$, $V \subseteq R^m$, let $f : U \to R^m$, $g : V \to R^n$ be definable of class C^k such that $f(U) \subseteq V$ and $f^{(\alpha)}$ and $g^{(\beta)}$ are strongly bounded for all $\alpha \in \mathbb{N}^l$ and $\beta \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ and $|\beta| \leqslant k$. Then the definable map $g \circ f : U \to R^n$ is of class C^k with strongly bounded $(g \circ f)^{(\alpha)}$ for all $\alpha \in \mathbb{N}^l$ with $|\alpha| \leqslant k$.*

Some analytic facts about definable families. Here R is an o-minimal field. For a definable map $f : X \to R^n$, $X \subseteq R^m$, we define

$$\|f\| \ := \ \sup_{a \in X} |f(a)| \in [0, +\infty].$$

Note that if X is nonempty and closed and bounded in R^m and f is continuous, then this supremum is a maximum.

Let $m, n \geqslant 1$, $c \in R^{>0}$, X a nonempty definable subset of R^m, and $(f_s)_{0<s<1}$ a definable family of maps $f_s : X \to [-c, c]^n$. Then we have the definable (pointwise) limit map $f_0 : X \to [-c, c]^n$ given by $f_0(a) = \lim_{s \downarrow 0} f_s(a)$. Throughout this subsection s ranges over the elements of R with $0 < s < 1$.

Lemma 4.5. *Suppose the family (f_s) has a Lipschitz constant $\ell \in R^{\geqslant}$, that is, $|f_s(a) - f_s(b)| \leqslant \ell |a - b|$ for all s and all $a, b \in X$; in particular, the f_s are continuous. Then f_0 has Lipschitz constant ℓ, and is thus continuous. If in addition X is closed and bounded in R^m, then $\|f_s - f_0\| \to 0$ as $s \downarrow 0$.*

Proof. Given $a, b \in X$ and taking the limit of $|f_s(a) - f_s(b)|$ as $s \downarrow 0$ we see that f_0 has Lipschitz constant ℓ. Suppose X is closed and bounded. Definable Selection gives a definable 'curve' $\gamma : (0,1) \to X$ such that $|f_s(\gamma(s)) - f_0(\gamma(s))| = \|f_s - f_0\|$ for all s. Suppose $\|f_s - f_0\|$ does not tend to 0 as $s \downarrow 0$. Then we have $\delta, \varepsilon > 0$ with $\|f_s - f_0\| \geqslant \varepsilon$ for all $s \leqslant \delta$, and thus $|f_s(\gamma(s)) - f_0(\gamma(s))| \geqslant \varepsilon$ for all $s \leqslant \delta$. Now

$\gamma(s) \to a \in X$ as $s \downarrow 0$. Then for $s < \delta$,

$$\begin{aligned}\varepsilon &\leqslant |f_s(\gamma(s)) - f_0(\gamma(s))| \\ &\leqslant |f_s(\gamma(s)) - f_s(a)| + |f_s(a) - f_0(a)| + |f_0(a) - f_0(\gamma(s))| \\ &\leqslant \ell|\gamma(s) - a| + |f_s(a) - f_0(a)| + |f_0(a) - f_0(\gamma(s))|\end{aligned}$$

but each of the last three terms tends to 0 as $s \downarrow 0$, a contradiction. □

Lemma 4.6. *Suppose X is open in R^m, $k \geqslant 1$, and the maps f_s are of class C^k such that $\|f_s^{(\alpha)}\| \leqslant c$ for all s and all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$. Then $f_0 : X \to R^n$ is of class C^{k-1}, and $f_s^{(\alpha)} \to f_0^{(\alpha)}$ pointwise as $s \downarrow 0$, for all $\alpha \in \mathbb{N}^m$ with $|\alpha| < k$.*

Proof. Let $\alpha \in \mathbb{N}^m$, $|\alpha| < k$, and $a \in X$. Take $\varepsilon > 0$ such that the closed ball

$$B \ := \ [a_1 - \varepsilon, a_1 + \varepsilon] \times \cdots \times [a_m - \varepsilon, a_m + \varepsilon]$$

centered at a with radius ε is contained in X. By MVT (the Mean Value Theorem) the definable family $(f_s^{(\alpha)})$ has Lipschitz constant c on B, so by Lemma 4.5 converges uniformly on B as $s \downarrow 0$ to a continuous definable limit map $B \to [-c, c]^n$; since a is arbitrary, this gives a continuous definable map $f_{0,\alpha} : X \to [-c, c]^n$ such that $f_s^{(\alpha)} \to f_{0,\alpha}$ pointwise as $s \downarrow 0$ (but uniformly on B). Note that $f_{0,\alpha} = f_0$ for $\alpha = (0, \ldots, 0)$. To prove the rest we arrange $n = 1$ by considering the n component functions of f_0 separately; to simplify notation we also assume $m = 1$. (For general m the derivatives are instead appropriate partial derivatives, where only one of the m components varies.) So let $i < k - 1$, and let h range over the elements of R with $|h| \leqslant \varepsilon$ our job is to show that then

$$(*) \qquad \lim_{h \to 0} \frac{f_{0,i}(a + h) - f_{0,i}(a)}{h} \ = \ f_{0,i+1}(a).$$

MVT gives

$$\frac{f_s^{(i)}(a + h) - f_s^{(i)}(a)}{h} \ = \ f_s^{(i+1)}\big(a(s, h)\big)$$

with $a(s, h)$ between a and $a + h$. By Definable Selection we can take $a(s, h)$ definable as a function of (s, h). Then $a(s, h) \to a(h)$ as $s \downarrow 0$ for a definable function $a(h)$ of h. Since $i + 1 < k$ we have by MVT

$$|f_s^{(i+1)}\big(a(s, h)\big) - f_s^{(i+1)}\big(a(h)\big)| \ \leqslant \ c|a(s, h) - a(h)| \ \leqslant \ c|h|.$$

Let $\delta(s) = \max_{b \in B} |f_s^{(i+1)}(b) - f_{0,i+1}(b)|$. Then $\delta(s) \to 0$ as $s \downarrow 0$, by Lemma 4.5, and $f_s^{(i+1)}\big(a(s, h)\big) = f_{0,i+1}\big(a(h)\big) + \delta(s, h)$ with $|\delta(s, h)| \leqslant c|h| + \delta(s)$, and thus

$$\frac{f_s^{(i)}(a + h) - f_s^{(i)}(a)}{h} \ = \ f_{0,i+1}\big(a(h)\big) + \delta(s, h).$$

Fixing h and taking limits as $s \downarrow 0$ we obtain

$$\frac{f_{0,i}(a + h) - f_{0,i}(a)}{h} \ = \ f_{0,i+1}(a(h)) + \varepsilon(h), \qquad |\varepsilon(h)| \leqslant c|h|.$$

Now $a(h)$ lies between a and $a + h$, endpoints a, $a + h$ included, which gives $(*)$. □

The above properly belongs to the topic of function spaces over o-minimal fields, cf. M. Thomas [7].

5. Reparametrizing unary functions

Much in this section is just bookkeeping, but we begin with a key analytic fact:

Lemma 5.1. *Let $f:(0,1)\to R$ be a definable C^k-function, $k\geqslant 2$, with strongly bounded $f^{(j)}$ for $0\leqslant j\leqslant k-1$ and decreasing $|f^{(k)}|$. Define $g:(0,1)\to R$ by $g(t)=f(t^2)$. Then $g^{(j)}$ is strongly bounded for $0\leqslant j\leqslant k$.*

Proof. Let t range over $(0,1)$. Lemma 4.3 gives

$$g^{(j)}(t) \;=\; \sum_{i=0}^{j}\rho_{ij}(t).f^{(i)}(t^2), \qquad j=0,\dots,k$$

where each function ρ_{ij} is given by a 1-variable polynomial with integer coefficients, of degree $\leqslant i$, and with $\rho_{jj}(t)=2^jt^j$. All summands here are strongly bounded except possibly the one with $i=j=k$, which is $2^kt^kf^{(k)}(t^2)$. So it suffices that $t^kf^{(k)}(t^2)$ is strongly bounded. Let $c\in\mathbb{Q}^{>0}$ be a strong bound for $f^{(k-1)}$. We claim that then $|f^{(k)}(t)|\leqslant 4c/t$ for all t. Suppose towards a contradiction that $t_0\in(0,1)$ is a counterexample, that is, $|f^{(k)}(t_0)|>4c/t_0$. Then the Mean Value Theorem provides a $\xi\in[t_0/2,t_0]$ such that

$$f^{(k-1)}(t_0)-f^{(k-1)}(t_0/2) \;=\; f^{(k)}(\xi).(t_0-t_0/2) \;=\; f^{(k)}(\xi)\cdot t_0/2.$$

Since $|f^{(k)}|$ is decreasing by assumption, $|f^{(k)}(\xi)| \;\geqslant\; |f^{(k)}(t_0)|>4c/t_0$. Hence

$$2c \;\geqslant\; |f^{(k-1)}(t_0)-f^{(k-1)}(t_0/2)| \;>\; (4c/t_0)\cdot(t_0/2) \;=\; 2c.$$

This contradiction proves our claim. Then for all t,

$$|t^kf^{(k)}(t^2)| \;\leqslant\; t^k\cdot(4c/t^2) \;=\; 4ct^{k-2} \;\leqslant\; 4c$$

using $k\geqslant 2$ for the last inequality. $\square$

The lemma fails for $k=1$, with $t\mapsto t^{1/3}$ as a counterexample.

Lemma 5.2. *Let $f:(0,1)\to R$ be definable and strongly bounded. Then f has a 1-reparametrization Φ such that for every $\phi\in\Phi$, ϕ or $f\circ\phi$ is given by a 1-variable polynomial with strongly bounded coefficients in R.*

Proof. Take elements $a_0=0<a_1<\cdots<a_n<a_{n+1}=1$ in R such that, for $i=0,1,\dots,n$, f is of class C^1 on (a_i,a_{i+1}), and either $|f'|\leqslant 1$ on (a_i,a_{i+1}), or $|f'|>1$ on (a_i,a_{i+1}). Let $i\in\{0,\dots,n\}$. If $|f'|\leqslant 1$ on (a_i,a_{i+1}), define

$$\phi_i\;:\;(0,1)\to R, \qquad \phi_i(t)\;:=\;a_i+(a_{i+1}-a_i)t.$$

If $|f'|>1$ on (a_i,a_{i+1}), set

$$b_i\;:=\;\lim_{t\downarrow a_i}f(t), \qquad b_{i+1}\;:=\;\lim_{t\uparrow a_{i+1}}f(t)$$

and as in this case f is continuous and strictly monotone on (a_i,a_{i+1}) we can define $\phi_i:(0,1)\to R$ by $\phi_i(t)=f^{-1}\big(b_i+(b_{i+1}-b_i)t\big)$, where f^{-1} denotes the compositional inverse of the restriction of f to (a_i,a_{i+1}), where f^{-1} has domain (b_i,b_{i+1}) if $b_i<b_{i+1}$, and domain (b_{i+1},b_i) if $b_i>b_{i+1}$.

In either case, ϕ_i maps $(0,1)$ onto (a_i,a_{i+1}) and both ϕ_i and $f\circ\phi_i$ are of class C^1 with strongly bounded derivative. Moreover, ϕ_i or $f\circ\phi_i$ is given by a univariate polynomial of degree 1 with strongly bounded coefficients in R. Thus

$$\Phi\;:=\;\{\phi_0,\dots,\phi_n,\widehat{a}_1,\dots,\widehat{a}_n\}$$

is a 1-reparametrization of f as required, where $\widehat{a}_i$ denotes the constant function on $(0,1)$ with value a_i. $\square$

Lemma 5.3. *Let $k \geqslant 1$ and suppose $f : (0,1) \to R$ is definable and strongly bounded. Then f has a k-reparametrization Φ such that for all $\phi \in \Phi$, ϕ or $f \circ \phi$ is given by a 1-variable polynomial with strongly bounded coefficients in R.*

Proof. By induction on k. The case $k = 1$ is Lemma 5.2. Suppose $k \geqslant 2$ and Φ is a $(k-1)$-reparametrization of f with the additional property. Let $\phi \in \Phi$. Then $\{\phi, f \circ \phi\} = \{g, h\}$ where g is given by a univariate polynomial with strongly bounded coefficients in R. Thus g is of class C^∞, and $g^{(i)}$ is strongly bounded for all $i \in \mathbb{N}$, and h is of class C^{k-1} with strongly bounded $h^{(j)}$ for $j = 0, \ldots, k-1$. In order to apply Lemma 5.2 we use o-minimality: take elements

$$a_0 \;=\; 0 \;<\; a_1 \;<\; \ldots \;<\; a_{n_\phi} \;<\; a_{n_\phi+1} \;=\; 1$$

in R such that for $i = 0, \ldots, n_\phi$, the function h is of class C^k on (a_i, a_{i+1}) and $|h^{(k)}|$ is monotone on (a_i, a_{i+1}). Define $\theta_{\phi,i} : (0,1) \to R$ as $t \mapsto a_i + (a_{i+1} - a_i)t$, if $|h^{(k)}|$ is decreasing, and as $t \mapsto a_{i+1} + (a_i - a_{i+1})t$, otherwise; so $\theta_{\phi,i}$ has image (a_i, a_{i+1}). Then $h \circ \theta_{\phi,i} : (0,1) \to R$ is of class C^k, $(h \circ \theta_{\phi,i})^{(j)}$ is strongly bounded for $j = 0, \ldots, k-1$, and $|(h \circ \theta_{\phi,i}^{(k)}|$ is decreasing. Let $\rho : (0,1) \to (0,1)$ be the C^∞-bijection sending t to t^2. By Lemma 5.2, the definable C^k-function $h \circ \theta_{\phi,i} \circ \rho : (0,1) \to R$ has strongly bounded jth derivative for $j = 0, \ldots, k$. The function $g \circ \theta_{\phi,i} \circ \rho$ is still given by a 1-variable polynomial with strongly bounded coefficients in R, and $\{g \circ \theta_{\phi,i} \circ \rho, h \circ \theta_{\phi,i} \circ \rho\} = \{\phi \circ \theta_{\phi,i} \circ \rho, f \circ (\phi \circ \theta_{\phi,i} \circ \rho)\}$. The images of the functions $\phi \circ \theta_{\phi,i} \circ \rho$ with $i \in \{0, \ldots, n_\phi\}$ cover the image of ϕ apart from finitely many points. So adding finitely many constant functions with domain $(0,1)$ and values in $(0,1)$ to the set $\{\phi \circ \theta_{\phi,i} \circ \rho : \; \phi \in S, \; i = 0, \ldots, n_\phi\}$ we obtain a k-reparametrization of f as claimed in the statement of the lemma. $\square$

Corollary 5.4. *Let $f : X \to R$ be definable and strongly bounded with $X \subseteq R$. Then f has a k-reparametrization, for every $k \geqslant 1$.*

Proof. The case that X is finite is obvious. Suppose X is infinite, and let $k \geqslant 1$. Since X is a finite union of strongly bounded intervals and points, it has a k-parametrization Φ by constant and linear functions. Now Lemma 5.3 provides for every $\phi : (0,1) \to R$ in Φ a k-reparametrization Ψ_ϕ of $f \circ \phi : (0,1) \to R$, and then $\{\phi \circ \psi : \; \phi \in \Phi, \; \psi \in \Psi_\phi\}$ is a k-reparametrization of f. $\square$

Next one might reparametrize "curves" $(0,1) \to R^n$ with $n \geqslant 2$, but there is nothing special about the univariate case here, so we do the general case:

Lemma 5.5. *Let $k, m \geqslant 1$, and suppose that every strongly bounded definable function $X \to R$ with $X \subseteq R^l$, $l \leqslant m$, has a k-reparameterization. Then every strongly bounded definable map $X \to R^n$ with $X \subseteq R^l$, $l \leqslant m$ and $n \geqslant 1$ has a k-reparametrization.*

Proof. Let $n \geqslant 1$, and suppose $F : X \to R^n$ and $f : X \to R$ with $X \subseteq R^m$ are definable, strongly bounded, and F has a k-reparametrization. It is enough to show that then the strongly bounded definable map $(F, f) : X \to R^{n+1}$ has a k-reparametrization. The case of finite X being trivial, assume X is infinite. Let Φ be a k-reparametrization of F and let $\phi \in \Phi$, $\phi : (0,1)^l \to R^m$, $l = \dim X \leqslant m$. Applying the hypothesis of the lemma to the map $f \circ \phi : (0,1)^l \to R$ we obtain a

k-reparametrization Ψ_ϕ of it. Then using Lemma 4.4, $\{\phi\circ\psi : \phi\in\Phi,\ \psi\in\Psi_\phi\}$ is a k-reparametrization of (F,f). □

Remark. At one point we need a slight variant of this lemma, with the same proof: *Let $k,m \geqslant 1$, and suppose that every strongly bounded definable function $(0,1)^l \to R$ with $l \leqslant m$ has a k-reparametrization. Then every strongly bounded definable map $(0,1)^l \to R^n$ with $l \leqslant m$ and $n \geqslant 1$ has a k-reparametrization.*

Corollary 5.6. *Let $n \geqslant 1$ and suppose $f : X \to R^n$ is definable and strongly bounded, with $X \subseteq R$. Then f has a k-reparametrization, for every $k \geqslant 1$.*

Proof. Immediate from Corollary 5.4 and the case $m = 1$ of Lemma 5.5. □

6. Convergence

In this section we assume that our ambient o-minimal field R is $\aleph_0$-saturated.

Let $k, N \in \mathbb{N}^{\geqslant 1}$ and let s, t range over $(0,1)$. Let $\big(F_s\big)$ be a definable family of maps

$$F_s\ :\ (0,1)\to(0,1)^N$$

of class C^k with strongly bounded derivatives $F_s^{(i)}$ for $i = 0,\dots,k$. As R is $\aleph_0$-saturated, we have a uniform bound $c\in\mathbb{N}^{\geqslant 1}$ with $|F_s^{(i)}(t)| \leqslant c$ for $i=0,\dots,k$ and all s,t. Then o-minimality gives a definable limit map

$$F_0\ :\ (0,1)\to[0,1]^N,\quad F_0(t)\ :=\ \lim_{s\downarrow 0} F_s(t),$$

and F_0 is of class C^{k-1}, with $F_0^{(i)}(t) = \lim_{s\downarrow 0} F_s^{(i)}(t)$ for $i = 0,\dots,k-1$, by Lemma 4.6. We have $F_s = (F_{s1},\dots,F_{sN})$ and set $\Phi_s := \{F_{s1},\dots,F_{sN}\}$, the set of component functions of F_s. Suppose $\bigcup_{\phi\in\Phi_s}\operatorname{image}(\phi) = (0,1)$ for all s (so Φ_s is a k-parametrization of $(0,1)$ for all s). Now $F_0 = (F_{01},\dots,F_{0N})$ and we let Φ_0 be the set of functions $\phi|_{\phi^{-1}(0,1)}$ with $\phi\in\{F_{01},\dots,F_{0N}\}$.

Lemma 6.1. *The set Φ_0 has the following properties:*
- (A) $\bigcup_{\psi\in\Phi_0}\operatorname{image}(\psi)$ *is a cofinite subset of* $(0,1)$.
- (B) *each function $\psi\in\Phi_0$ has as its domain an open subset of $(0,1)$ and is of class C^{k-1} with strongly bounded $\psi^{(i)}$ for $i=0,\dots,k-1$.*

Proof. Suppose (A) fails. Then o-minimality gives $a < b$ in $(0,1)$ such that $[a,b]$ is disjoint from image(ψ) for every $\psi\in\Phi_0$ and thus disjoint from image(F_{0i}) for $i = 1,\dots,N$. Let s be given. It follows from $\bigcup_{i=1}^N\operatorname{image}(F_{si}) \supseteq [a,b]$ and o-minimality that for some $i\in\{1,\dots,N\}$, the image of F_{si} contains a segment $[a_s,b_s]$ with $a\leqslant a_s<b_s\leqslant b$ and $b_s-a_s\geqslant(b-a)/(N+1)$. By o-minimality we have a fixed $i\in\{1,\dots,N\}$ and an $\varepsilon\in(0,1)$ such that for all $s<\varepsilon$ the image of F_{si} contains a segment $[a_s,b_s]$ with $a\leqslant a_s<b_s\leqslant b$ and $b_s-a_s\geqslant(b-a)/(N+1)$. Take $\delta\in(0,1/2)$ so small that $2c\delta<(b-a)/(N+1)$ and let $s<\varepsilon$. Now $|F'_{si}|\leqslant c$, so F_{si} has Lipschitz constant c, and thus the F_{si}-images of the intervals $(0,\delta)$ and $(1-\delta,1)$ cannot cover a segment $[a_s,b_s]$ as above. Therefore, we have a point $t_s\in[\delta,1-\delta]$ such that $F_{s,i}(t_s)\in[a,b]$. (We do not need the a_s,b_s any longer.) By Definable Selection we can take t_s as a definable function of $s\in(0,\varepsilon)$. Then for $t_0:=\lim_{s\downarrow 0}t_s$ we have $1-\delta\leqslant t_0\leqslant 1+\delta$. Now for $s<\varepsilon$ we have

$$|F_{0i}(t_0)-F_{si}(t_s)|\ \leqslant\ |F_{0i}(t_0)-F_{0i}(t_s)|+|F_{0i}(t_s)-F_{si}(t_s)|.$$

The first summand on the right tends to 0 as $s \downarrow 0$ because F_{0i} is continuous, and the second does so because $F_{si} \to F_{0i}$ unformly on $[\delta, 1-\delta]$ as $s \downarrow 0$, by Lemma 4.5. Hence $F_{0i}(t_0) \in [a,b]$, contradicting the defining property of $[a,b]$. This finishes the proof of (A). As to (B), just note that F_0 is of class C^{k-1} with $\|F_0^{(i)}\| \leqslant c$ for $i = 0, \dots, k-1$ by Lemma 4.6. □

We now apply this lemma to set up the inductive process for proving Theorems 4.1 and 4.2. For the rest of this section we fix an $m \geqslant 1$.

Notation. For definable open $U \subseteq R^{m+1}$, $V \Subset U$ means that V is a definable open subset of R^{m+1} with $V \subseteq U$ and $\dim(U \setminus V) \leqslant m$.

Here is some notation about "changing the last variable": For $\phi : (0,1) \to R$, set

$$I_\phi \ : \ (0,1)^{m+1} \to R^{m+1}, \qquad (t_1, \dots, t_m, t_{m+1}) \mapsto \big(t_1, \dots, t_m, \phi(t_{m+1})\big),$$

and for $f : X \to R^n$, $X \subseteq R^{m+1}$ we set

$$f_\phi \ := \ f \circ I_\phi \ : \ (I_\phi)^{-1}(X) \to R^n, \quad (t_1, \dots, t_m, t_{m+1}) \mapsto f\big(t_1, \dots, t_m, \phi(t_{m+1})\big).$$

Lemma 6.2. *Let $k \geqslant 2$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R$ be a strongly bounded definable C^1-function. Suppose also that $\partial f/\partial x_i$ is strongly bounded for $i = 1, \dots, m$. Then there is a $(k-1)$-parametrization Φ of a cofinite subset of $(0,1)$ and a set $V \Subset U$ such that for every $\phi \in \Phi$: $I_\phi(V) \subseteq U$, f_ϕ is of class C^1 on V, and $\partial f_\phi/\partial x_i$ is strongly bounded on V, for $i = 1, \dots, m+1$.*

Proof. We construct Φ from the limit set Φ_0 of a suitable family $(\Phi_s)_{0<s<1}$ as described above. (Lemma 6.1 almost gives that Φ_0 is a $(k-1)$-parametrization.) O-minimality gives $W \Subset U$ such that f is of class C^1 on W. For $s, t \in (0,1)$, let $W_s(t)$ be the set of those $a \in [0,1]^m$ such that the open ball in R^{m+1} centered at (a,t) with radius s is entirely contained in W; note that $W_s(t) \times \{t\} \subseteq W$, in particular, $W_s(t) \subseteq (0,1)^m$, and $W_s(t)$ is closed in R^m, not just in $(0,1)^m$. Thus for $0 < s, t < 1$ we have a definable continuous function

$$a \mapsto |\frac{\partial f}{\partial x_{m+1}}(a,t)| \ : \ W_s(t) \to R,$$

which achieves its maximum value at some point $a_s(t) \in W_s(t)$, provided $W_s(t)$ is nonempty. By Definable Selection we may take $(s,t) \mapsto a_s(t)$ to be definable, taking by convention the value $(1/2, \dots, 1/2) \in (0,1)^m$ if $W_s(t) = \emptyset$. Then for all $s, t \in (0,1)$ and $a \in W_s(t)$ we have

$$(*) \qquad (a_s(t), t) \in W, \qquad |\frac{\partial f}{\partial x_{m+1}}(a_s(t), t)| \ \geqslant \ |\frac{\partial f}{\partial x_{m+1}}(a,t)|.$$

Now consider the definable family $(g_s)_{0<s<1}$ of maps

$$g_s : (0,1) \to (0,1)^m \times R, \qquad g_s(t) := \big(a_s(t), f(a_s(t), t)\big),$$

where for convenience we set $f\big(a_s(t), t\big) := 0$ if $\big(a_s(t), t\big) \notin U$. By Corollary 5.6 there is for all $s \in (0,1)$ a k-reparametrization of g_s. Now R is $\aleph_0$-saturated, and together with Definable Selection this yields an $N \in \mathbb{N}^{\geqslant 1}$ and a definable family $(F_s)_{0<s<1}$ of maps $F_s : (0,1) \to (0,1)^N$ such that $\Phi_s := \{F_{s1}, \dots, F_{sN}\}$ is a k-reparametrization of g_s for $0 < s < 1$.

Let Φ_0 be the limit, as $s \downarrow 0$, of this family as described in Lemma 6.1. By partitioning the domains of the functions in Φ_0 and restricting these functions accordingly we obtain a finite collection Φ of functions taking values in $(0,1)$ whose domains are subsets of $(0,1)$ and are either singletons or subintervals of $(0,1)$, and such that any function in Φ whose domain is a subinterval of $(0,1)$ is either constant or strictly monotone. By throwing away the constant functions in Φ (which include those whose domain is a singleton) and composing each remaining function with a suitable injective linear function with coefficients in $[0,1]$, we arrange that Φ is a $(k-1)$-parametrization of a cofinite subset of $(0,1)$. Now set

$$V \ := \ U \setminus \bigcup_{\phi \in \Phi} I_\phi^{-1}[(0,1)^{m+1} \setminus W].$$

The injectivity (and continuity) of the $\phi \in \Phi$ gives $V \Subset U$. For $\phi \in \Phi$ we have $I_\phi(V) \subseteq W \subseteq U$ and so, using $k \geqslant 2$, the function f_ϕ is of class C^1 on V. Let $\phi \in \Phi$; it only remains to show that then $\partial f_\phi/\partial x_i$ is strongly bounded on V for $i = 1, \dots, m+1$. Since R is $\aleph_0$-saturated, it is enough to show, given any point $(a_0, t_0) \in V$, that $\partial f_\phi/\partial x_i$ is strongly bounded just at this point, for $i = 1, \dots, m+1$.

Since $(a_0, \phi(t_0)) \in W \subseteq U$, this is certainly the case for $i = 1, \dots, m$. For the remaining case $i = m+1$, note first that we have a linear function $\lambda : R \to R$ with coefficients in $[0,1]$ and a function $\psi \in \Phi_0$ such that λ maps the interval $(0,1)$ into the domain of ψ and $\phi(t) = \psi(\lambda(t))$ for all $t \in (0,1)$. So it is enough to show for t_1 in the domain of ψ with $\big(a_0, \psi(t_1)\big) \in W$ that $\psi'(t_1) \cdot (\partial f/\partial x_{m+1})\big(a_0, \psi(t_1)\big)$ is strongly bounded. Let such a t_1 be given. By definition of Φ_0 we have a definable family $(\phi_s)_{0<s<1}$ of functions $\phi_s \in \Phi_s$ such that $\lim_{s \downarrow 0} \phi_s(t_1) = \psi(t_1)$ and, as $k \geqslant 2$, $\lim_{s\downarrow 0} \phi_s'(t_1) = \psi'(t_1)$. Hence for all small enough $s \in (0,1)$:

(i) $\big(a_0, \phi_s(t_1)\big) \in W$, $|(\partial f/\partial x_{m+1})\big(a_0, \psi(t_1)\big) - (\partial f/\partial x_{m+1})\big(a_0, \phi_s(t_1)\big)| \leqslant 1$, by the continuity of $\partial f/\partial x_{m+1}$ on W;
(ii) $|\phi_s'(t_1) - \psi'(t_1)| \cdot |(\partial f/\partial x_{m+1})(a_0, \psi(t_1))| \leqslant 1$;
(iii) $a_0 \in W_s\big(\phi_s(t_1)\big)$: use that $\big(a_0, \psi(t_1)\big) \in W$, that W is open in R^{m+1}, and that $\phi_s(t_1) \to \psi(t_1)$ as $s \downarrow 0$.

Take $s \in (0,1)$ such that (i), (ii), (iii) hold. Then

$$\begin{aligned} |\psi'(t_1) \cdot \frac{\partial f}{\partial x_{m+1}}\big(a_0, \psi(t_1)\big)| \ &\leqslant \ |\phi_s'(t_1)| \cdot |\frac{\partial f}{\partial x_{m+1}}\big(a_0, \psi(t_1)\big)| + 1, \text{ by (ii)}, \\ &\leqslant \ |\phi_s'(t_1)| \cdot |\frac{\partial f}{\partial x_{m+1}}\big(a_0, \phi_s(t_1)\big)| + |\phi_s'(t_1)| + 1, \text{ by (i)}, \\ &\leqslant \ |\phi_s'(t_1)| \cdot |\frac{\partial f}{\partial x_{m+1}}(b)| + |\phi_s'(t_1)| + 1, \end{aligned}$$

by (iii) and $(*)$, where $b \ := \ \big(a_s(\phi_s(t_1)), \phi_s(t_1)\big) \in W$.

Now $|\phi_s'(t_1)|$ is strongly bounded, as $\phi_s \in \Phi_s$, so it suffices to show that

$$\phi_s'(t_1) \cdot \frac{\partial f}{\partial x_{m+1}}(b)$$

is strongly bounded. Since Φ_s is a k-reparametrization of g_s, we have:

(iv) $(a_s \circ \phi_s)'(t_1)$ is strongly bounded, and
(v) $(d/dt)|_{t=t_1} f\big(a_s(\phi_s(t)), \phi_s(t)\big)$ is strongly bounded.

By the Chain Rule, the quantity in (v) equals

$$\sum_{i=1}^m (a_{si} \circ \phi_s)'(t_1) \cdot \frac{\partial f}{\partial x_i}(b) \ + \ \phi_s'(t_1) \cdot \frac{\partial f}{\partial x_{m+1}}(b)$$

The left hand sum here is strongly bounded by (iv) and the strong boundedness of the functions $\partial f/\partial x_i$ for $i = 1, \ldots, m$. Hence the right hand term is strongly bounded, which we already showed to be enough. $\square$

Corollary 6.3. *Let $k \geqslant 2$, $n \geqslant 1$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R^n$ be a strongly bounded definable C^1-map. Suppose also that $\partial f/\partial x_i$ is strongly bounded for $i = 1, \ldots, m$. Then there is a $(k-1)$-parametrization Φ of a cofinite subset of $(0,1)$ and a set $V \Subset U$ such that for every $\phi \in \Phi$: $I_\phi(V) \subseteq U$, f_ϕ is of class C^1 on V, and $\partial f_\phi/\partial x_i$ is strongly bounded on V for $i = 1, \ldots, m+1$.*

Proof. For $n = 1$ this is Lemma 6.2. As an inductive assumption, let $f : U \to R^n$ be as in the hypothesis of the corollary and Φ and V as in its conclusion. Let $g : U \to R$ be a strongly bounded definable C^1-function such that $\partial g/\partial x_i$ is strongly bounded for $i = 1, \ldots, m$. Then the strongly bounded definable C^1-map $(f, g) : U \to R^{n+1}$ has strongly bounded partial $\partial(f,g)/\partial x_i = (\partial f/\partial x_i, \partial g/\partial x_i)$ for $i = 1, \ldots, m$. It now suffices to show that there is a $(k-1)$-parametrization Θ of a cofinite subset of $(0,1)$ and a set $W \Subset U$ such that for all $\theta \in \Theta$: $I_\theta(W) \subseteq U$, $(f,g)_\theta$ is of class C^1 on W, and $\partial(f,g)_\theta/\partial x_i$ is strongly bounded on W for $i = 1, \ldots, m+1$. To construct Θ and W, let $\phi \in \Phi$. Then applying Lemma 6.2 to the function $g_\phi : V \to R$ gives a $(k-1)$-parametrization Ψ_ϕ of a cofinite subset of $(0,1)$ and a set $V_\phi \Subset V$ such that for all $\psi \in \Psi_\phi$: $I_\psi(V_\phi) \subseteq V$ and $(g_\phi)_\psi = g_{\phi\circ\theta}$ is of class C^1 on V_ϕ, and $\partial g_{\phi,\psi}/\partial x_i$ is strongly bounded on V_ϕ. Now we set

$$\Theta := \{\phi \circ \psi : \ \phi \in \Phi, \ \psi \in \Psi_\phi\}, \quad W := \bigcap_{\phi\in\Phi} V_\phi.$$

It follows easily from Lemma 4.4 that Θ and W have the desired properties. $\square$

To state the next corollary, let U be a definable open subset of R^{m+1}. Then we have for $t \in R$ the definable open subset U^t of R^m given by

$$U^t \ = \ \{(t_1, \ldots, t_m) \in R^m : \ (t_1, \ldots, t_m, t) \in U\}.$$

We call a definable map $f : U \to R^n$ *of class C^k in the first m variables* if for every $t \in R$ the (definable) map

$$f^t : U^t \to R^n, \qquad (t_1, \ldots, t_m) \mapsto f(t_1, \ldots, t_m, t)$$

is of class C^k. In that case $f^{(\alpha)}$ for $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ denotes the definable map

$$(t_1, \ldots, t_m, t) \mapsto (f^t)^{(\alpha)}(t_1, \ldots, t_m) \ : \ U \to R^n,$$

which for fixed t is continuous as a function of $(t_1, \ldots, t_m)$.

Corollary 6.4. *Let $k, n \geqslant 1$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R^n$ be a strongly bounded definable map that is of class C^k in the first m variables, such that $f^{(\alpha)}$ is strongly bounded for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$. Then for every $l \leqslant k$ there is a $V_l \Subset U$ and a k-parametrization Φ_l of a cofinite subset of $(0,1)$ such that for all $\phi \in \Phi_l$: $I_\phi(V_l) \subseteq U$, f_ϕ is of class C^k on V_l and $f_\phi^{(\alpha)} := \big(f_\phi\big)^{(\alpha)}$ is strongly bounded on V_l for all $\alpha \in \mathbb{N}^{m+1}$ with $|\alpha| \leqslant k$, $\alpha_{m+1} \leqslant l$.*

Proof. O-minimality gives $V_0 \Subset U$ such that f is of class C^k on V_0. Then V_0 and $\Phi_0 = \{\mathrm{id}|_{(0,1)}\}$ have the desired properties for $l = 0$. Suppose, inductively, that $l < k$ and V_l and Φ_l are as stated in the Corollary. Let

$$\Delta \ := \ \{\alpha \in \mathbb{N}^{m+1} \ : \ |\alpha| \leqslant k-1, \ \alpha_{m+1} \leqslant l\},$$

set $\widetilde{n} := \#\Delta \cdot \#\Phi_l$, and let $F_1, \dots, F_{\widetilde{n}} : V_l \to R^n$ enumerate the set of C^1-maps

$$\{f_\phi^{(\alpha)} :\ V_l \to R^n :\ \alpha \in \Delta,\ \phi \in \Phi_l\}.$$

Then we can apply Corollary 6.3 to $F := (F_1, \dots, F_{\widetilde{n}}) : V_l \to R^{\widetilde{n}\cdot n}$ in the role of f, and V_l, $\widetilde{n}\cdot n$, $k+1$ instead of U, n, k. This gives a k-parametrization Ψ of a cofinite subset of $(0,1)$ and a set $V_{l+1} \Subset V_l$ such that for all $\psi \in \Psi$: $I_\psi(V_{l+1}) \subseteq V_l$, F_ψ is of class C^1 on V_{l+1}, and $\partial F_\psi / \partial x_i$ is strongly bounded on V_{l+1} for $i = 1, \dots, m+1$. Next we set

$$\Phi_{l+1} \ :=\ \{\phi \circ \psi :\ \phi \in \Phi_l,\ \psi \in \Psi\}.$$

Then Φ_{l+1} is a k-parametrization of a cofinite subset of $(0,1)$ and $I_\theta(V_{l+1}) \subseteq U$, with f_θ of class C^k for all $\theta \in \Phi_{l+1}$.

Let $\theta = \phi \circ \psi$ with $\phi \in \Phi_l, \psi \in \Psi$ and let $\alpha \in \mathbb{N}^{m+1}$, $|\alpha| \leqslant k$, $\alpha_{m+1} \leqslant l+1$; it remains to show that then $f_\theta^{(\alpha)}$ is strongly bounded on V_{l+1}. If $\alpha_{m+1} = 0$, then this holds because $f_\theta^{(\alpha)} = \big(f_\phi^{(\alpha)}\big)_\psi$ and $f_\phi^{(\alpha)}$ is strongly bounded on V_l. Suppose that $\alpha_{m+1} > 0$. Then $\alpha = \beta + (0, \dots, 0, j)$ with $\beta_{m+1} = 0$ and $j = \alpha_{m+1} \geqslant 1$, so for $a = (a_1, \dots, a_m, a_{m+1}) \in V_{l+1}$ we have

$$\begin{aligned} f_\theta^{(\alpha)}(a) \ &=\ \frac{\partial^j f_\theta^{(\beta)}}{\partial x_{m+1}^j}(a) \ =\ \frac{\partial^j \big(f_\phi^{(\beta)}\big)_\psi}{\partial x_{m+1}^j}(a) \\ &=\ \sum_{i=1}^{j} \frac{\partial^i f_\phi^{(\beta)}}{\partial x_{m+1}^i}\big(a_1, \dots, a_m, \psi(a_{m+1})\big) \cdot p_{ij}\big(\psi^{(1)}(a_{m+1}), \dots, \psi^{(j-i+1)}(a_{m+1})\big) \end{aligned}$$

using Lemma 4.3 and the polynomials p_{ij} from that lemma for the last equality. Since we assumed inductively that the $\frac{\partial^i f_\phi^{(\beta)}}{\partial x_{m+1}^i}$ are strongly bounded on V_l and $\psi^{(1)}, \dots, \psi^{(k)}$ are strongly bounded on $(0,1)$, $f_\theta^{(\alpha)}$ is strongly bounded on V_{l+1}. $\square$

7. Finishing the proofs of the parametrization theorems

We continue to work in an ambient $\aleph_0$-saturated o-minimal field R. We consider the following statements depending on m:

- $(\mathrm{I})_{\mathrm{m}}$ For all $k, n \geqslant 1$, every strongly bounded definable map $f : (0,1)^m \to R^n$ has a k-reparametrization.
- $(\mathrm{II})_{\mathrm{m}}$ For all $k \geqslant 1$, every strongly bounded definable set $X \subseteq R^{m+1}$ has a k-parametrization.

It is clear that $(\mathrm{I})_0$ and $(\mathrm{II})_0$ hold; $(\mathrm{I})_1$ holds by Corollary 5.6. We proceed by induction to show that $(\mathrm{I})_{\mathrm{m}}$ and $(\mathrm{II})_{\mathrm{m}}$ hold for all m. So let $m \geqslant 1$ and suppose that $(\mathrm{I})_{\mathrm{l}}$ holds for all $l \leqslant m$ and that $(\mathrm{II})_{\mathrm{l}}$ holds for all $l < m$. We show that then $(\mathrm{II})_{\mathrm{m}}$ holds and next that $(\mathrm{I})_{\mathrm{m+1}}$ holds. For $(\mathrm{II})_{\mathrm{m}}$, let $k \geqslant 1$ and let $X \subseteq R^{m+1}$ be definable and strongly bounded. In order to show that X has a k-parametrization we can reduce to the case that X is a cell in R^{m+1}; we do the more difficult of the two cases, namely $X = (f,g)_Y$ where Y is a (strongly bounded) cell in R^m, and $f, g : Y \to R$ are strongly bounded continuous definable functions with $f(y) < g(y)$ for all $y \in Y$; the other case, where X is the graph of such a function $Y \to R$, is left to the reader.

Using $(\mathrm{II})_{\mathrm{m-1}}$ we have a k-parametrization Φ of Y. Set $l := \dim Y$. Let $\phi \in \Phi$ be given. Then $\phi : (0,1)^l \to Y$ and $(\mathrm{I})_{\mathrm{l}}$ gives a k-reparametrization Ψ_ϕ of the map

$(f\circ\phi, g\circ\phi) : (0,1)^l \to R^2$. For $\psi\in\Psi_\phi$ we have $\psi : (0,1)^l \to (0,1)^l$, and we define $\theta_{\phi,\psi} : (0,1)^{l+1} \to X$ by

$$\theta_{\phi,\psi}(s,t) \ := \ \big((\phi\circ\psi)(s),\ (1-t)\cdot(f\circ\phi\circ\psi)(s) + t\cdot(g\circ\phi\circ\psi)(s)\big)$$

where $(s,t) = (s_1,\dots,s_l,t) \in (0,1)^{l+1}$. Then the set $\{\theta_{\phi,\psi} :\ \phi\in\Phi,\ \psi\in\Psi_\phi\}$ is readily seen to be a k-parametrization of X, and we have established $(\mathrm{II})_{\mathrm{m}}$.

For $(\mathrm{I})_{\mathrm{m+1}}$ we need only do the case $n = 1$ by the remark following the proof of Lemma 5.5. So let $k \geqslant 1$ and let $f : (0,1)^{m+1} \to R$ be a strongly bounded definable function; our job is to show that f has a k-reparametrization.

In the rest of this proof t ranges over the interval $(0,1)$. By $(\mathrm{I})_{\mathrm{m}}$ there is for all t a k-reparametrization of the function $f^t : (0,1)^m \to R$ given by $f^t(s) = f(s,t)$. Using a saturation and definable selection argument as in the proof of Lemma 6.2 gives an $N\in\mathbb{N}^{\geqslant 1}$ and definable families $(\phi_1^t),\dots,(\phi_N^t)$ of maps

$$\phi_j^t \ :\ (0,1)^m \to (0,1)^m \qquad (j = 1,\dots,N)$$

such that $\Phi^t := \{\phi_1^t,\dots,\phi_N^t\}$ is for every t a k-reparametrization of f^t.

Now, for $j = 1,\dots,N$ we define the function $f_j : (0,1)^{m+1} \to R$ by

$$f_j(s,t) \ := \ f\big(\phi_j(s,t),t\big),$$

where $\phi_j : (0,1)^{m+1} \to (0,1)^m$ is given by $\phi_j(s,t) := \phi_j^t(s)$. Consider the map

$$F \ := \ \big(\phi_1,\dots,\phi_N, f_1,\dots,f_N\big) \ :\ (0,1)^{m+1} \to R^{Nm+N}.$$

Then the hypotheses of Corollary 6.4 are satisfied for $\boldsymbol{k}$ and $(0,1)^{m+1}$ in the role of f and U, and $Nm+N$ for n: this is just restating that Φ^t is a k-reparametrization of f^t, uniformly in t. The conclusion of that corollary for $l = k$ gives a set $V \Subset (0,1)^{m+1}$ and a k-parametrization Ψ of a cofinite subset of $(0,1)$ such that for all $\psi\in\Psi$ the map $F_\psi : (0,1)^{m+1} \to R^{Nm+N}$ is of class C^k on V with strongly bounded $F_\psi^{(\alpha)}$ on V for all $\alpha\in\mathbb{N}^{m+1}$ with $|\alpha| \leqslant k$.

For $j = 1,\dots,N$ and $\psi\in\Psi$, let $\phi_j * \psi : (0,1)^{m+1} \to (0,1)^{m+1}$ be given by

$$(\phi_j*\psi)(s,t) \ := \ \big(\phi_j(s,\psi(t)),\psi(t)\big) \ = \ \big(\phi_j^{\psi(t)}(s),\psi(t)\big).$$

The images of the $\psi\in\Psi$ cover a set $(0,1)\setminus\{t_1,\dots,t_d\}$ and for every t the images of $\phi_1^t,\dots,\phi_N^t$ cover $(0,1)^m$, and thus the images of the above $\phi_j*\psi$ cover $(0,1)^{m+1}$ apart from finitely many hyperplanes $x_{m+1} = t_i$. Setting

$$W \ := \ \bigcup_{1\leqslant j\leqslant N,\ \psi\in\Psi} (\phi_j*\psi)(V)$$

it follows that the definable set $(0,1)^{m+1}\setminus W$ has dimension $\leqslant m$. Using the now established $(\mathrm{II})_{\mathrm{m}}$, let Θ_1 be a k-parametrization of V and Θ_2 a k-parametrization of $(0,1)^{m+1}\setminus W$. For $\theta\in\Theta_2$ we have $\theta : (0,1)^l \to (0,1)^{m+1}$ with $l\leqslant m$ and then $(\mathrm{I})_{\mathrm{l}}$ yields a k-reparametrization Λ_θ of the function $f\circ\theta : (0,1)^l \to R$. The required k-reparametrization of f is now given by

$$\{(\phi_j*\psi)\circ\chi :\ j = 1,\dots,\ \psi\in\Psi, \chi\in\Theta_1\} \cup \{\theta\circ\widehat{\lambda} :\ \theta\in\Theta_2, \lambda\in\Lambda_\theta\}$$

where $\widehat{\lambda} : (0,1)^{m+1} \to (0,1)^l$ (for $l\leqslant m$ as above) is given by $\widehat{\lambda}(t_1,\dots,t_{m+1}) := \lambda(t_1,\dots,t_l)$. This finishes the proof of $(\mathrm{I})_{\mathrm{m+1}}$, and the induction is complete. In particular, Theorem 4.1 is now established. Theorem 4.2 requires one more easy step and we leave this to the reader.

Corollary 7.1. *Let $k, n \geqslant 1$; suppose $X \subseteq [-1,1]^n$ is definable, $d := \dim X \geqslant 0$. Then there exists a finite set Φ of definable C^k-maps $\phi : (0,1)^d \to R^n$ such that*

(i) $\bigcup_{\phi\in\Phi} \operatorname{image}(\phi) = X$;
(ii) $|\phi^{(\alpha)}(s)| \leqslant 1$ *for all* $\phi \in \Phi$ *and* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leqslant k$ *and all* $s \in (0,1)^d$.

Proof. Let Φ^* be a k-parametrization of X. Then (i) holds for Φ^* instead of Φ and (ii) holds for Φ^* instead of Φ, with a certain $c \in \mathbb{N}^{\geqslant 1}$ in place of 1. Cover $(0,1)^d$ with $(c+1)^d$ translates of the 'box' $(0, \frac{1}{c})^d$ and for each such translate B, let $\lambda_B : (0,1)^d \to B$ be the obvious linear (that is, affine) bijection. Then the set of maps $\phi \circ \lambda_B$ as ϕ varies over Φ^* and B over the above translates is the required Φ, since $(\phi\circ\lambda_B)^{(\alpha)} = c^{-|\alpha|} \cdot \big(\phi^{(\alpha)} \circ \lambda_B\big)$ for such ϕ and B and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. $\square$

Definable Selection and $\aleph_0$-saturation lead to a uniform version:

Corollary 7.2. *Let d, k, m, n be given with $k, n \geqslant 1$ and suppose $E \subseteq R^m$ and*

$$Z \ \subseteq\ E \times [-1,1]^n \ \subseteq\ R^{m+n}$$

are definable with $\dim Z(s) = d$ for all $s \in E$. Then there are $N \in \mathbb{N}^{\geqslant 1}$ and a definable set $F \subseteq E \times R^d \times R^{Nn}$ such that for all $s \in E$, $F(s) \subseteq R^d \times R^{Nn}$ is the graph of a C^k-map $(\phi_1, \dots, \phi_N) : (0,1)^d \to (R^n)^N = R^{Nn}$ such that:

(i) $\bigcup_{j=1}^N \operatorname{image}(\phi_j) = Z(s)$;
(ii) $|\phi_j^{(\alpha)}(t)| \leqslant 1$ *for* $j = 1, \dots, N$, $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leqslant k$, *and* $t \in (0,1)^d$.

The implicit proof of Corollary 7.2 uses that R is $\aleph_0$-saturated, but this corollary goes through without this assumption, since we can always pass to an $\aleph_0$-saturated elementary extension. Thus it applies to o-minimal expansions of the real field, and this in turn can be combined with Theorem 3.6 to give:

Corollary 7.3. *Let $n \geqslant 1$ and let an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field be given. Suppose $E \subseteq \mathbb{R}^m$ and $Z \subseteq E \times [-1,1]^n \subseteq \mathbb{R}^{m+n}$ are definable. Then there is for every $\varepsilon > 0$ an $e = e(\varepsilon, n)$ and a K with the following property: for all $s \in E$ with $\dim Z(s) < n$ and all T, at most KT^ε many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $Z(s)(\mathbb{Q}, T)$.*

The expression "$e = e(\varepsilon, n)$" means: e can be chosen to depend only on ε and n. The proof below uses the numbers $\varepsilon(d,n,e) := \frac{dneD(n,e)}{B(d,n,e)}$ from Section 3.

Proof. Replacing E by finitely many definable subsets over each of which $\dim Z(s)$ takes a given value, we arrange that for a certain $d < n$ we have $\dim Z(s) = d$ for all $s \in E$. If $d = 0$, then we have $K \in \mathbb{N}^{\geqslant 1}$ such that $\#Z(s) \leqslant K$ for all $s \in E$, and so at most K hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant 1$ are enough to cover $Z(s)$. Assume $d \geqslant 1$. Take $e \geqslant 1$ such that $\varepsilon(d,n,e) \leqslant \varepsilon$ and set $k := b(d,n,e) + 1$ as in Theorem 3.6. Corollary 7.2 gives an $N \in N^{\geqslant 1}$ and for every $s \in E$ maps $\phi_1, \dots, \phi_N : (0,1)^d \to R^n$ of class C^k such that $Z(s) = \bigcup_{j=1}^N \operatorname{image}(\phi_j)$ and $|\phi_j^{(\alpha)}(t)| \leqslant 1$ for $j = 1, \dots, N$ and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ and all $t \in (0,1)^d$. Applying Theorem 3.6 to each map ϕ_j separately we obtain that for $K := N \cdot C(d,n,e)$ at most KT^ε many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $Z(s)(\mathbb{Q}, T)$. $\square$

8. Strengthening and Extending the Counting Theorem

In this section we fix an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, and *definable* is with respect to $\widetilde{\mathbb{R}}$. Throughout $n \geqslant 1$ and $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ are definable.

A closer look at the proof of Theorem 2.5 gives useful extra information about the definable subsets $V(s)$ of $X(s)^{\mathrm{alg}}$: Theorem 8.4. To express this information efficiently requires the notion of a *block family*, which is here simpler than in [5] and well suited to the inductive set-up of Section 2.

A block family version of the Counting Theorem. Let $d \leqslant n$. A *block in* $\mathbb{R}^n$ *of dimension* d is a definable connected open subset of a semialgebraic set $A \subseteq \mathbb{R}^n$ for which $\dim_a A = d$ for all $a \in A$. Thus the empty subset of $\mathbb{R}^n$ counts as a block in $\mathbb{R}^n$ of dimension d, but if B is a nonempty block in $\mathbb{R}^n$ of dimension d, then $\dim B = d$. Also, a nonempty block of dimension 0 in $\mathbb{R}^n$ consists just of one point. A *block family in* $\mathbb{R}^n$ *of dimension* d is a definable set $V \subseteq E \times \mathbb{R}^n$, all whose sections $V(s)$ are blocks in $\mathbb{R}^n$ of dimension d. Here are two easy lemmas:

Lemma 8.1. *Suppose $U \subseteq \mathbb{R}^m$ is open and semialgebraic, $m \geqslant 1$, and $f : U \to \mathbb{R}^n$ is semialgebraic and maps U homeomorphically onto $f(U)$. Then f maps any block $B \subseteq U$ in $\mathbb{R}^m$ of dimension $d \leqslant m$ onto a block $f(B)$ in $\mathbb{R}^n$ of dimension d.*

In the proof of Theorem 8.4 we apply Lemma 8.1 for every $I \subseteq \{1, \ldots, n\}$ to the map $a \mapsto b : \{a \in \mathbb{R}^n : a_i \neq 0 \text{ for } i \in I\} \to \mathbb{R}^n$ with $b_i = a_i^{-1}$ for $i \in I$ and $b_i = a_i$ for $i \notin I$; these maps extend the maps f_I from Section 2.

Lemma 8.2. *Let B be a block in $\mathbb{R}^n$ of dimension $d \leqslant n$. Then B is a union of connected semialgebraic subsets of dimension d.*

Proof. Take semialgebraic $A \subseteq \mathbb{R}^n$ such that $\dim_a A = d$ for all $a \in A$, and B is an open subset of A. For $b \in B$, take a semialgebraic open neighborhood U of b in A such that $U \subseteq B$. Now use that the connected components of U are open in A, by [2, (III, 2.18)], and thus of dimension d. □

Corollary 8.3. *Let $Y \subseteq \mathbb{R}^n$ and $1 \leqslant d \leqslant n$.*

(i) *if $B \subseteq Y$ and B is a block in $\mathbb{R}^n$ of dimension d, then $B \subseteq Y^{\mathrm{alg}}$;*
(ii) *if V is a block family in $\mathbb{R}^n$ of dimension d, then the union of the sections of V that are contained in Y is contained in Y^{alg}.*

For the inductive proof below we also define a *block family in* $\mathbb{R}^0$ *of dimension* 0 to be a definable set $V \subseteq E \times \mathbb{R}^0$, with $E \times \mathbb{R}^0$ identified with E in the obvious way.

Theorem 8.4. *Let ε be given. Then there are a natural number $N = N(X, \varepsilon) \geqslant 1$, a block family $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ in $\mathbb{R}^n$ of dimension $d_j \leqslant n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \ldots, N$, and a constant $c = c(X, \varepsilon)$, such that:*

(i) *$V_j(s,t) \subseteq X(s)$ for $j = 1, \ldots, N$ and $(s,t) \in E \times F_j$;*
(ii) *for all T and all $s \in E$, $X(s)(\mathbb{Q}, T)$ is covered by at most cT^{ε} blocks $V_j(s,t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.*

This yields an improved Theorem 2.5 as follows. Let $V_1, \ldots, V_N$ and c be as in Theorem 8.4. Then for all $s \in E$ the definable set $V(s) \subseteq \mathbb{R}^n$ given by

$$V(s) \ := \bigcup_{d_j \geqslant 1, t \in F_j} V_j(s,t)$$

is contained in $X(s)^{\mathrm{alg}}$ and $\mathrm{N}(X(s) \setminus V(s), T) \leqslant cT^{\varepsilon}$ for all T.

Proof. If Theorem 8.4 holds for definable sets $X_1, \dots, X_\nu \subseteq E \times \mathbb{R}^n$, $\nu \in \mathbb{N}$, then also for $X = X_1 \cup \dots \cup X_\nu$. We shall tacitly use this below.

We proceed by induction on n, and follow the proof of Theorem 2.5 closely. Set $V_0(s) :=$ interior of $X(s)$. Then [2, (III, 3.6)] gives $M \in \mathbb{N}^{\geqslant 1}$ such that for all $s \in E$,

$$\#\{\text{connected components of } V_0(s)\} \ \leqslant\ M.$$

Definable Selection and the lexicographic ordering on $\mathbb{R}^n$ give definable subsets $V_1, \dots, V_M$ of $E \times \mathbb{R}^n$ such that for all $s \in E$ the sets $V_1(s), \dots, V_M(s)$ are connected (possibly empty), open in $V_0(s)$, pairwise disjoint, with $V(s) = \bigcup_{i=1}^M V_i(s)$. So $V_1, \dots, V_M$ are block families in $\mathbb{R}^n$ of dimension n; we make them the first M of the $V_1, \dots, V_N$ to be constructed. Now replacing X with $X \setminus V_0$ we arrange that $X(s)$ has empty interior for all $s \in E$. Applying Lemma 8.1 to the natural extensions of the maps f_I, $I \subseteq \{1, \dots, n\}$, we arrange also that $X(s) \subseteq [-1,1]^n$ for all $s \in E$.

Next, take e and $k = k(n,e)$ as in the proof of Theorem 2.4. So we have $C = C(X, \varepsilon) \in \mathbb{R}^>$ such that for any $s \in E$, $X(s)(\mathbb{Q}, T)$ is covered by at most $CT^{\varepsilon/2}$ many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$. Therefore it suffices to find $V_1, \dots, V_N$ and c as in the theorem but with (ii) replaced by

(ii)* for all T, all $s \in E$, and all hypersurfaces H of degree $\leqslant e$, $(X(s) \cap H)(\mathbb{Q}, T)$ is covered by at most $\frac{c}{C} T^{\varepsilon/2}$ blocks $V_j(s,t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$;

We use again the semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$, and the definable sets $Y_l \subseteq E \times F \times \mathbb{R}^{n_l}$, $l = 1, \dots, L$, as in the proof of Theorem 2.4. Since $n_l < n$, the induction assumption gives a natural number $N_l = N(Y_l, \varepsilon) \geqslant 1$, a block family

$$W_{l,i} \ \subseteq\ \big((E \times F) \times G_{l,i}\big) \times \mathbb{R}^{n_l}$$

in $\mathbb{R}^{n_l}$ of dimension $d_{l,i} \leqslant n_l$ with definable $G_{l,i} \subseteq \mathbb{R}^{m_{l,i}}$, for $i = 1, \dots, N_l$, and $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}^>$, such that

(i)$'$ $W_{l,i}(s,t,g) \subseteq Y_l(s,t)$ for $i = 1, \dots, N_l$, $(s,t,g) \in (E \times F) \times G_{l,i}$;

(ii)$'$ for all T and all $(s,t) \in E \times F$, $Y_l(s,t)(\mathbb{Q}, T)$ is covered by at most $B_l T^{\varepsilon/2}$ blocks $W_{l,i}(s,t,g)$, $(1 \leqslant i \leqslant N_l,\ g \in G_{l,i})$.

Set $N := N_1 + \dots + N_L$, and for $l = 1, \dots, L$, $1 \leqslant i \leqslant N_l$ and $j = N_1 + \dots + N_{l-1} + i$, set $F_j := F \times G_{l,i}$, and let $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ be the definable set given by

$$V_j\big(s,(t,g)\big) = \mathcal{C}_l(t) \cap p_{\boldsymbol{i}^l}^{-1}\big(W_{l,i}(s,t,g)\big), \qquad (s \in E,\ t \in F,\ g \in G_{l,i}),$$

so V_j is a block family in $\mathbb{R}^n$ of dimension $d_{l,i} < n$, by Lemma 8.1. It is easy to check that $V_1, \dots, V_N$ and $c := C(B_1 + \dots + B_L)$ are as desired. $\square$

A generalization. In this subsection we fix $d \geqslant 1$. Instead of rational points we now allow points with coordinates in a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ of dimension $\leqslant d$. Let $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$, and set $\mathbb{Q}\lambda := \mathbb{Q}\lambda_1 + \dots + \mathbb{Q}\lambda_d \subseteq \mathbb{R}$. For $a \in \mathbb{Q}\lambda$ we set

$$\mathrm{H}_\lambda(a) \ :=\ \min\{\mathrm{H}(q) :\ q \in \mathbb{Q}^d,\ q \cdot \lambda = a\} \in \mathbb{N}^{\geqslant 1}.$$

Here $q \cdot \lambda := q_1\lambda_1 + \dots + q_d\lambda_d$. We define a height function H_λ on $(\mathbb{Q}\lambda)^n \subseteq \mathbb{R}^n$ by

$$\mathrm{H}_\lambda(a) \ =\ \max\{\mathrm{H}_\lambda(a_1), \dots, \mathrm{H}_\lambda(a_n)\} \ \text{ for } a = (a_1, \dots, a_n) \in (\mathbb{Q}\lambda)^n.$$

For $Y \subseteq \mathbb{R}^n$ we introduce its finite subsets $Y_\lambda(T)$ and their cardinalities:

$$Y_\lambda(T) \ :=\ \{a \in Y \cap (\mathbb{Q}\lambda)^n :\ \mathrm{H}_\lambda(a) \leqslant T\}, \qquad \mathrm{N}_\lambda(Y,T) \ :=\ \#Y_\lambda(T).$$

Theorem 8.5. *Let any definable $Y \subseteq \mathbb{R}^n$ and any ε be given. Then there is a constant $c = c(Y, d, \varepsilon) \in \mathbb{R}^>$ such that for all T and all $\lambda \in \mathbb{R}^d$,*

$$\mathrm{N}_\lambda(Y^{\mathrm{tr}}, T) \;\leqslant\; cT^\varepsilon.$$

Proof of Theorem 8.5. First a couple of useful lemmas about blocks:

Lemma 8.6. *If B is a block in $\mathbb{R}^n$ (of some dimension) and $p, q \in B$, then $\gamma(0) = p$ and $\gamma(1) = q$ for some continuous semialgebraic path $\gamma : [0,1] \to B$.*

Proof. Even better, let B be a connected open subset of a semialgebraic set $A \subseteq \mathbb{R}^n$, and let $p \in B$. We claim: there is for every $q \in B$ a continuous semialgebraic path $\gamma : [0,1] \to A$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma([0,1]) \subseteq B$. To see this, let $B(p)$ be the set of all $q \in B$ for which there is such a path. The sets $B(p)$ as p ranges over B form a partition of B, so it is enough to show that the $B(p)$ are open in B, which reduces to showing that $B(p)$ is a neighborhood of p in B. The latter follows using that B is open in the semialgebraic set A, and using [2, (III, 2.18)]. $\square$

Corollary 8.7. *If B is a block in $\mathbb{R}^m$ (of some dimension), A is a semialgebraic subset of $\mathbb{R}^m$ with $B \subseteq A$, and $\phi : A \to \mathbb{R}^n$ is a continuous semialgebraic map such that $\phi(B)$ has more than one point, then $\phi(B) = \phi(B)^{\mathrm{alg}}$.*

Proof. Use that the ϕ-image of a path γ as in Lemma 8.6 is a connected semialgebraic subset of $\phi(B)$. $\square$

The next result is basically a consequence of Theorem 8.4, as the proof will show.

Theorem 8.8. *Given ε, there are a natural number $N = N(X, d, \varepsilon) \geqslant 1$, a definable set $V_j \subseteq (E \times \mathbb{R}^d \times F_j) \times \mathbb{R}^n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X, d, \varepsilon)$, such that for $j = 1, \dots, N$ and all $(s, \lambda, t) \in E \times \mathbb{R}^d \times F_j$:*

(i) *$V_j(s, \lambda, t) \subseteq X(s)$ and $V_j(s, \lambda, t)$ is connected;*
(ii) *if $\dim V_j(s, \lambda, t) \geqslant 1$, then $V_j(s, \lambda, t) \subseteq X(s)^{\mathrm{alg}}$,*

and such that for all T and $(s, \lambda) \in E \times \mathbb{R}^d$, the set $X(s)_\lambda(T)$ is covered by at most cT^ε sections $V_j(s, \lambda, t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.

This yields a family version of Theorem 8.5 as follows. Let $V_1, \dots, V_N$ and c be as in Theorem 8.8. Then for all $s \in E$ the definable set $V(s) \subseteq \mathbb{R}^n$ given by

$$V(s) \;:=\; \bigcup \{V_j(s, \lambda, t) :\ 1 \leqslant j \leqslant N,\ (\lambda, t) \in \mathbb{R}^d \times F_j,\ \dim V_j(s, \lambda, t) \geqslant 1\}$$

is contained in $X(s)^{\mathrm{alg}}$ and $\mathrm{N}_\lambda(X(s) \setminus V(s), T) \leqslant cT^\varepsilon$ for all T.

Proof. Let $\pi : \mathbb{R}^d \times (\mathbb{R}^d)^n \to \mathbb{R}^n$ be given by $\pi(\lambda, a_1, \dots, a_n) = (\lambda \cdot a_1, \dots, \lambda \cdot a_n)$, where $a_1, \dots, a_n \in \mathbb{R}^d$. Set

$$X^* \;:=\; \{(s, \lambda, a_1, \dots, a_n) \in (E \times \mathbb{R}^d) \times (\mathbb{R}^d)^n :\ \big(s, \pi(\lambda, a_1, \dots, a_n)\big) \in X\},$$

viewed as a definable family of subsets of $(\mathbb{R}^d)^n$. Note that for $s \in E$ and $\lambda \in \mathbb{R}^d$,

$$(*) \qquad \pi\big(\{\lambda\} \times X^*(s, \lambda)\big) \;\subseteq\; X(s), \qquad \pi\big(\{\lambda\} \times X^*(s, \lambda)(\mathbb{Q}, T)\big) \;=\; X(s)_\lambda(T).$$

We apply Theorem 8.4 to X^* in the role of X. It gives $N = N(X^*, \varepsilon) \geqslant 1$, a block family $V_j^* \subseteq (E \times \mathbb{R}^d \times F_j) \times (\mathbb{R}^d)^n$ in $(\mathbb{R}^d)^n = \mathbb{R}^{dn}$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X^*, \varepsilon)$ such that:

(i)* $V_j^*(s, \lambda, t) \subseteq X^*(s, \lambda)$ for $j = 1, \dots, N$ and (s, λ, t) in $E \times \mathbb{R}^d \times F_j$;
(ii)* for all T and all $(s, \lambda) \in E \times \mathbb{R}^d$, the set $X^*(s, \lambda)(\mathbb{Q}, T)$ is covered by at most cT^ε sections $V_j^*(s, \lambda, t)$, $(1 \leqslant j \leqslant N$, $t \in F_j)$.

Now we set for $j = 1, \dots, N$,

$$V_j \ := \ \{\big(s, \lambda, t, \pi(\lambda, a)\big) \in (E \times \mathbb{R}^d \times F_j) \times \mathbb{R}^n : \ (s, \lambda, t, a) \in V_j^*\},$$

so $V_j(s, \lambda, t) = \pi\big(\{\lambda\} \times V_j^*(s, \lambda, t)\big)$ for $(s, \lambda, t) \in E \times \mathbb{R}^d \times F_j$. We now show that $V_1, \dots, V_N$ and $c(X, d, \varepsilon) := c(X^*, \varepsilon)$ have the desired properties. Clause (i) is satisfied using (i)* and $(*)$, and (ii) is satisfied in view of Corollary 8.7. The rest follows from (ii)* and $(*)$. $\square$

Extending the Counting Theorem to Algebraic Points. Throughout this subsection we fix $d \geqslant 1$. Instead of rational points we now count algebraic points whose coordinates are of degree at most d over $\mathbb{Q}$. We define the corresponding height of an algebraic number $\alpha \in \mathbb{R}$ with $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leqslant d$ by

$$\mathrm{H}_d^{\mathrm{poly}}(\alpha) \ := \ \min\{\mathrm{H}(\xi) : \ \xi \in \mathbb{Q}^d, \ \alpha^d + \xi_1 \alpha^{d-1} + \cdots + \xi_d = 0\} \in \mathbb{N}^{\geqslant 1}.$$

(For us this height is notationally more convenient than the height for real algebraic numbers used by Pila in [P2]. The two heights are related as follows, where we use an extra subscript P for the height in [P2]: for $\alpha \in \mathbb{R}$ with $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leqslant d$,

$$\mathrm{H}_{\mathrm{P},d+1}^{\mathrm{poly}}(\alpha) \ \leqslant \ \mathrm{H}_d^{\mathrm{poly}}(\alpha) \ \leqslant \ \mathrm{H}_{\mathrm{P},d+1}^{\mathrm{poly}}(\alpha)^2.$$

Thus the results below for our height also hold for the other height.)

We extend the above height to all $\alpha \in \mathbb{R}$ by $\mathrm{H}_d^{\mathrm{poly}}(\alpha) := \infty$ if $[\mathbb{Q}(\alpha) : \mathbb{Q}] > d$, and to all points $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ by $\mathrm{H}_d^{\mathrm{poly}}(\alpha) := \max\{\mathrm{H}_d^{\mathrm{poly}}(\alpha_1), \dots, \mathrm{H}_d^{\mathrm{poly}}(\alpha_n)\}$. For $Y \subseteq \mathbb{R}^n$ we introduce its finite subsets $Y_d(T)$ and their cardinalities:

$$Y_d(T) \ := \ \{\alpha \in Y : \ \mathrm{H}_d^{\mathrm{poly}}(\alpha) \leqslant T\}, \qquad \mathrm{N}_d(Y, T) \ := \ \# Y_d(T).$$

Theorem 8.9. *Let $Y \subseteq \mathbb{R}^n$ be definable, and let ε be given. Then there is a constant $c = c(Y, d, \varepsilon)$ such that for all T,*

$$N_d(Y^{\mathrm{tr}}, T) \ \leqslant \ cT^{\varepsilon}.$$

We shall use the following easy consequence of semialgebraic cell decomposition:

Lemma 8.10. *Let $A_{n,d} \subseteq \mathbb{R}^{n \times d} \times \mathbb{R}^n$ be the semialgebraic set*

$$\{(\xi, \alpha) \in \mathbb{R}^{n \times d} \times \mathbb{R}^n : \ \alpha_i^d + \xi_{i1} \alpha_i^{d-1} + \cdots + \xi_{id} = 0 \ \textit{for} \ i = 1, \dots, n\}.$$

Then we have a natural number $L = L(n, d) \geqslant 1$, a semialgebraic set $D_l \subseteq \mathbb{R}^{n \times d}$ with a semialgebraic continuous map $\phi_l : D_l \to \mathbb{R}^n$, for $l = 1, \dots, L$, such that $A_{n,d} = \bigcup_{l=1}^{L} \mathrm{graph}(\phi_l)$. It follows that for all $\alpha \in \mathbb{R}^n$ with $\mathrm{H}_d^{\mathrm{poly}}(\alpha) < \infty$ there is an $l \in \{1, \dots, L\}$ and a $\xi \in D_l$ such that $\phi_l(\xi) = \alpha$ and $\mathrm{H}(\xi) = \mathrm{H}_d^{\mathrm{poly}}(\alpha)$.

Towards Theorem 8.9 we first prove something stronger:

Theorem 8.11. *Let ε be given. Then there are $N = N(X, d, \varepsilon) \in \mathbb{N}^{\geqslant 1}$, a definable set $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X, d, \varepsilon)$, such that for $j = 1, \dots, N$ and all $(s, t) \in E \times F_j$:*

(i) *$V_j(s, t) \subseteq X(s)$ and $V_j(s, t)$ is connected;*
(ii) *if $\dim V_j(s, t) \geqslant 1$, then $V_j(s, t) \subseteq X(s)^{\mathrm{alg}}$,*

and such that for all T and $s \in E$, the set $X(s)_d(T)$ is covered by at most cT^{ε} sections $V_j(s, t)$, $(1 \leqslant j \leqslant N, \ t \in F_j)$.

Proof. Let $\pi : \mathbb{R}^{n\times d} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be the obvious projection map. Take L and $\phi_1 : D_1 \to \mathbb{R}^n, \dots, \phi_L : D_L \to \mathbb{R}^n$ as in Lemma 8.10. Let $l \in \{1,\dots,L\}$. We set

$$\begin{aligned} X_l &:= \{(s,\xi,\alpha) \in E \times D_l \times \mathbb{R}^n :\ \alpha \in X(s),\ \phi_l(\xi) = \alpha\}, \\ Y_l &:= \{(s,\xi) \in E \times D_l :\ \xi \in \pi\big(X_l(s)\big)\} \ = \ \{(s,\xi) \in E \times D_l :\ \phi_l(\xi) \in X(s)\}, \end{aligned}$$

so for $s \in E$ we have $\phi_l\big(Y_l(s)\big) \subseteq X(s)$, and by Lemma 8.10, for all T,

$$X(s)_d(T) \ = \ \bigcup_{l=1}^{L} \phi_l\big(Y_l(s)(\mathbb{Q},T)\big).$$

We now apply Theorem 8.4 to Y_l in the role of X, and get $N_l = N_l(Y_l,\varepsilon) \in \mathbb{N}^{\geqslant 1}$, a block family $V_{l,i} \subseteq (E \times F_{l,i}) \times \mathbb{R}^{n\times d}$ in $\mathbb{R}^{n\times d}$ with definable $F_{l,i} \subseteq \mathbb{R}^{m_{l,i}}$, for $i = 1,\dots,N_l$, and a constant $c_l = c_l(Y_l,\varepsilon) \in \mathbb{R}^{>}$ such that:

(i) $V_{l,i}(s,t) \subseteq Y_l(s)$ for $i = 1,\dots,N_l$ and (s,t) in $E \times F_{l,i}$;
(ii) for all T and all $s \in E$, the set $Y_l(s)(\mathbb{Q},T)$ is covered by at most $c_l T^{\varepsilon}$ blocks $V_{l,i}(s,t)$, $(1 \leqslant i \leqslant N_l,\ t \in F_{l,i})$.

Set $N := N_1 + \cdots + N_L$, and for $1 \leqslant i \leqslant N_l$ and $j = N_1 + \cdots + N_{l-1} + i$, set $F_j := F_{l,i}$, and let $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ be the definable set given by

$$V_j(s,t) \ := \ \phi_l\big(V_{l,i}(s,t)\big), \qquad (s \in E,\ t \in F_j).$$

It is easily verified using Lemma 8.7 that $V_1,\dots,V_N$ and $c(X,d,\varepsilon) := c_1 + \cdots + c_L$ have the properties stated in the Theorem. □

Just as with Theorem 8.8 this leads to a family version of Theorem 8.9 as follows. Let $V_1,\dots,V_N$ and c be as in Theorem 8.11. Take the definable set $V \subseteq E \times \mathbb{R}^n$ such that for all $s \in E$,

$$V(s) \ := \ \bigcup \{V_j(s,t) :\ 1 \leqslant j \leqslant N,\ t \in F_j,\ \dim V_j(s,t) \geqslant 1\}.$$

Then for all $s \in E$ and all T we have

$$V(s) \ \subseteq \ X(s)^{\mathrm{alg}} \quad \text{and} \quad \mathrm{N}_d(X(s) \setminus V(s), T) \ \leqslant \ cT^{\varepsilon}.$$

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